Homework #9

23.2.3 Consider the intertemporal consumers problem with one good in each time period and discount factor $0 < \delta < 1$. Suppose the felicity function is $\mathcal{C}^2$ with $u' > 0$ and $u'' < 0$. Now suppose prices obey $p_t = p > 0$ for all times t. Show that $c_{t+1} < c_t$ whenever both consumption levels are positive.

Answer: The first-order conditions are $\delta u'(c_{t+1})/u'(c_t) = p_{t+1}/p_t = p/p = 1$. Thus $u'(c_{t+1}) > \delta u'(c_{t+1}) = u'(c_t)$. Now $c_{t+1} < c_t$ by diminishing marginal utility.

23.2.4 Suppose a consumer has discount factor $0 < \delta < 1$ and period utility function $u(c) = \ln c$. The consumer has wealth $W > 0$ and faces prices $p_t = p > 0$ for all times t. Find the optimal consumption path.

Answer: Since the marginal utility of zero consumption is infinite, consumption will always be positive (unless wealth is zero). The first-order conditions are $\delta u'(c_{t+1})/u'(c_t) = p_{t+1}/p_t$. This becomes $\delta c_t/c_{t+1} = p/p = 1$, so $c_{t+1} = \delta c_t$. It follows that $c_t = \delta^t c_0$. The budget constraint is $W = \sum_t pc_t = \sum_t p\delta^t c_0 = pc_0/(1-\delta)$. Thus $c_0 = (1-\delta)W/p$ and $c_t = (1-\delta)\delta^t W/p$.

If you don't recall how to sum the infinite series, let $S = \sum_{t=0}^{\infty} \delta^t$. Then $1 + \delta S = \delta^0 + \sum_{t=1}^{\infty} \delta^t = S$. It follows that $S = (1-\delta)^{-1}$. This requires $|\delta| < 1$ for the summation to converge.

23.3.2 Consider the one-sector production technology described by $f(k) = 5k$. Suppose profit is maximized at $k_t > 0$ for all t. Find p_{t+1} as a function of p_t.

Answer: Profit is $p_{t+1}f(k_t) - p_t k_t = (5p_{t+1} - p_t)k_t$. If profit is maximized at some $k_t > 0$, we must have $5p_{t+1} = p_t$, so $p_{t+1} = p_t/5$.

23.5.1 There was a typo in the problem. The felicity should be $u(c) = -1/c$.

Consider the following Ramsey problem. Suppose a consumer has utility $\sum_{t=0}^{\infty} \delta^t u(c_t)$ where $0 < \delta < 1$ and the felicity function is $u(c) = -1/c$. The production function is $f(a) = \beta a$ where $\beta > 1$. Suppose that there is an optimal path with $c_t > 0$ for every t.

a) Does consumption grow? If so, what is the growth factor.

b) Is the (consumption) transversality condition satisfied?

Answer:

a) The Euler equations are

$$\delta f'(a_t)u'(c_{t+1}) = u'(c_t)$$

yielding

$$\delta\beta/c_{t+1}^2 = 1/c_t^2.$$

It follows that $c_{t+1} = (\delta\beta)^{1/2} c_t$, implying that $c_t = (\delta\beta)^{t/2} c_0$.

Consumption grows by the growth factor $(\delta\beta)^{1/2}$ when $\delta\beta > 1$, is constant if $\delta\beta = 1$, and shrinks if $\delta\beta < 1$, all of which are possible.

b) The consumption transversality condition is that $p_t c_t \to 0$ where $p_t = \delta^t u'(c_t) = \delta^t / c_t^2$. Thus $p_t c_t = \delta^t / c_t$. Now $c_t = (\delta\beta)^{t/2} c_0$, so $p_t c_t = (\delta/\beta)^{t/2} / c_0$. Since $\delta < 1$ and $\beta > 1$, $\delta/\beta < 1$, implying that the transversality condition is satisfied.