Homework #1

3.1.4 Let $f(x_1, x_2) = x_1/(x_1 + x_2) + \ln(x_1 + x_2)$ on $\mathbb{R}^2_{++}$. Show that f is increasing and has derivative that is homogeneous of degree minus one.

Answer: The derivative is

$$\begin{aligned} df &= \left(\frac{x_2}{(x_1+x_2)^2}, -\frac{x_1}{(x_1+x_2)^2}\right) + \left(\frac{1}{x_1+x_2}, \frac{1}{x_1+x_2}\right) \\ &= \frac{1}{(x_1+x_2)^2}\big(x_1 + 2x_2, x_2\big) \gg \mathbf{0} \end{aligned}$$

on $\mathbb{R}^2_{++}$. Now

$$\begin{aligned} df(t\mathbf{x}) &= \frac{1}{t^2(x_1+x_2)^2}\big(tx_1 + 2tx_2, tx_2\big) \\ &= t^{-1}\frac{1}{(x_1+x_2)^2}\big(x_1 + 2x_2, x_2\big) \\ &= t^{-1} df(\mathbf{x}), \end{aligned}$$

so df is homogeneous of degree minus one.

3.1.5 Let u and v be equivalent utility functions on $\mathbb{R}^L_+$.

a) Suppose u and v are both homogeneous of degree one. Show that $v = Cu$ for some $C > 0$.

b) Suppose u and v are homogeneous of degree β and γ, respectively. Show that $u = Cv^{(\beta/\gamma)}$ for some $C > 0$.

Answer:

a) **Version 1:** Let $t > 0$ and $\mathbf{x} \in \mathbb{R}^L_+$ be arbitrary. Since the utility functions are equivalent, there is an increasing function φ with $v(\mathbf{x}) = \varphi\big(u(\mathbf{x})\big)$. Now

$$\begin{aligned} \varphi\big(u(\mathbf{x})\big) &= tv(\mathbf{x}) \\ &= v(t\mathbf{x}) \\ &= \varphi\big(tu(\mathbf{x})\big) \\ &= \varphi\big(u(t\mathbf{x})\big). \end{aligned}$$

This implies that φ itself is homogeneous of degree 1. Since $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(z) = Cz$ for some C. Moreover, since φ is increasing, $C > 0$.

Version 2: When the functions are differentiable, an alternative method is to use Euler's formula. Since the utilities are equivalent, there is an increasing function φ with $v(\mathbf{x}) = \varphi\big(u(\mathbf{x})\big)$. Differentiate to obtain $dv = \varphi' du$. Dotting with $\mathbf{x}$, Euler's formula yields $v(\mathbf{x}) = dv{\cdot}\mathbf{x} = \varphi' du{\cdot}\mathbf{x} = \varphi'\big(u(\mathbf{x})\big)u(\mathbf{x})$. Since u and v are homogeneous of degree one, we can conclude that φ' is homogeneous of degree zero in u. This implies φ' is some constant C and that $u(\mathbf{x}) = cv(\mathbf{x})$.

b) Consider $\psi(\mathbf{x}) = [u(\mathbf{x})]^{1/\beta}$ and $\phi(\mathbf{x}) = [v(\mathbf{x})]^{1/\gamma}$. Apply part (a) to find ζ so that $\psi = \zeta\phi$. Then raise to the β power to get $u = \zeta^\beta v^{(\beta/\gamma)}$. Set $C = \zeta^\beta$ to obtain the result. There is also an alternative method as in part (a).

3.2.4 Consider the CES utility function on $\mathbb{R}^2_+$ defined by $u(x_1, x_2) = [\delta x_1^{-\rho} + (1-\delta)x_2^{-\rho}]^{-\nu/\rho}$ where $0 < \delta < 1$, $\nu > 0$, $\rho > -1$ and $\rho \neq 0$. For what values of ρ is u equivalent to an additive separable utility function?

Answer: If $0 > \rho > -1$, we take the increasing transformation $\varphi(u) = u^{-\rho/\nu}$ to obtain $\varphi\big(u(\mathbf{x})\big) = \delta x_1^{-\rho} + (1-\delta)x_2^{-\rho}$, which is in additive separable form.

If $\rho > -1$, we apply the increasing transformation $\varphi(u) = -u^{-\nu/\rho}$. Then $\varphi\big(u(\mathbf{x})\big) = -\delta x_1^{-\rho} - (1-\delta)x_2^{-\rho}]$, which is again in additive separable form.

3.3.2 Let $\succsim_{lex}$ be the lexicographic preference order on $\mathbb{R}^L_+$.

a) Does $\succsim_{lex}$ induce an order on every singleton?

b) Is $\succsim_{lex}$ completely separable?

c) We know that $\succsim_{lex}$ does not have an additive separable utility representation. Explain why this does not contradict Corollary 3.3.14 for $L \geq 3$.

Answer:

a) Yes, lexicographic preferences are weakly monotonic, so Proposition 3.3.4 shows that $\succsim_{lex}$ induces an order on every singleton.

b) If $L = 1$ or $L = 2$, $\succsim_{lex}$ is completely separable. When $L = 1$, we need only consider the commodity group $\{1\}$, where $\succsim_{lex}$ induces an order by definition. When $L = 2$, we need only consider the commodity groups $\{1\}$, $\{2\}$, and $\{1, 2\}$. The last group includes all commodities, so the induced order is $\succsim_{lex}$ itself. For the other two groups, part (a) shows that $\succsim_{lex}$ induces an order.

Now consider the case $L > 2$. If P is a commodity group, $(\mathbf{x}_P, \mathbf{x}_{\sim P}) \succsim_{lex} (\mathbf{y}_P, \mathbf{x}_{\sim P})$ if and only if $\mathbf{x}_P$ is lexicographicly weakly preferred to $\mathbf{y}_P$ on $\mathbb{R}^P_+$ because all of the other coordinates remain fixed.

To make this clear, let $L = 3$, and consider the commodity group $P = \{1, 3\}$. Then

$(\mathbf{x}_P, \mathbf{x}_{\sim P}) \succsim_{lex} (\mathbf{y}_P, \mathbf{x}_{\sim P})$ if and only if $(x_1, x_2, x_3) \succsim_{lex} (y_1, x_2, y_3)$, which happens if and only if either $x_1 > y_1$ or $x_1 = y_1$ and $x_3 \geq y_3$. But that is equivalent to $(x_1, z_2, x_3) \succsim_{lex} (y_1, z_2, y_3)$, or in other words, $(\mathbf{x}_P, \mathbf{z}_{\sim P}) \succsim_{lex} (\mathbf{y}_P, \mathbf{z}_{\sim P})$.

c) Since $L \geq 3$, we have at least three essential goods. Corollary 3.3.14 tells us that a completely separable preference order (such as $\succsim_{lex}$) has a continuous additive separable utility representation if and only if it is continuous. But $\succsim_{lex}$ is not continuous, and so the corollary does not apply. Indeed, we found in Example 2.1.3 that $\succsim_{lex}$ has no utility representation of any kind.

3.3.3 Suppose $u(\mathbf{x}) = x_1 + x_2(x_3 + x_4)$ on $\mathbb{R}^L_{++}$.

a) Find all commodity groups A in $\{1, 2, 3, 4\}$ where u induces an order on A.

b) For each commodity group A in part (a), find a subutility function that represents $\succsim_A$.

c) Rewrite u in terms of the non-trivial subutilities you found in (b). If there is more than one way to do this, demonstrate all of them.

Answer:

a) For simplicity, denote $\partial u/\partial x_k$ by u_k. The marginal utilities are $u_1 = 1$, $u_2 = x_3 + x_4$, $u_3 = x_2$ and $u_4 = x_2$. Since each marginal utility is strictly positive, u is strictly increasing and induces an order on each singleton.

We also find the following marginal rates of substitution involving x_1: $u_1/u_2 = 1/(x_3 + x_4)$, $u_1/u_3 = 1/x_2$, $u_1/u_4 = 1/x_2$. This means that any subutility that includes x_1 as an argument must include all other goods.

For $A = \{2, 3, 4\}$, we also have $\text{MRS}_{23} = \text{MRS}_{24} = (x_3 + x_4)/2$ and $\text{MRS}_{34} = 1$. These are all independent of x_1, meaning that u induces an order on $\{2, 3, 4\}$. Moreover, any group including good 2, must include both goods 3 and 4.

Finally, for $A = \{3, 4\}$, $u_3/u_4 = 1$, which is independent of x_1 and x_2, so u induces an order on $\{3, 4\}$.

b) For the singleton groups, the trivial subutility $v_k(x_k) = x_k$ will do. For $A = \{3, 4\}$, the subutility $\phi(x_3, x_4) = x_3 + x_4$ works. For $B = \{2, 3, 4\}$, the subutility $\psi(x_2, x_3, x_4) = x_2(x_3 + x_4)$ does the job.

c) Using the singletons, we can write $u(\mathbf{x}) = v_1(x_1) + v_2(x_2)\big(v_3(x_3) + v_4(x_4)\big)$. We can also write $u(\mathbf{x}) = v_1(x_1) + \psi(x_2, x_3, x_4) = v_1(x_1) + v_2(x_2)\phi(x_3, x_4)$.