Homework #2

3.3.13 Let u be a continuous utility function and $\mathcal{P}$ be a partition of goods. Suppose there is a continuous function v that is increasing in each argument and continuous subutility functions u_P defined on each $\mathbf{x}_P$ with $u(\mathbf{x}) = v\big[\big(u_P(\mathbf{x}_P)\big)_{P\in\mathcal{P}}\big]$. Show that u is weakly separable relative to $\mathcal{P}$.

Answer: We must show that u induces an order on each commodity group $P \in \mathcal{P}$. Let $\mathcal{P} = \{P_i\}_{i=1}^K$. Now suppose $\mathbf{x} = (\mathbf{x}_{P_1}, \mathbf{x}_{\sim P_1}) \succsim \mathbf{x} = (\mathbf{y}_{P_1}, \mathbf{x}_{\sim P_1})$. Let $u_i = u_{P_i}(\mathbf{x}_{P_i})$. We can then write this preference in utility terms as

$$v(u_1, \ldots, u_K) \geq v(u_{P_1}(\mathbf{y}_{P_1}), u_2, \ldots, u_K)$$

Since v is increasing in argument 1, this is equivalent to $u_1 = u_{P_1}(\mathbf{x}_{P_1}) \geq u_{P_1}(\mathbf{y}_{P_1})$. The actual values of $u_2, \ldots, u_K$ don't matter here, so v induces an order on P_1. In fact, $\mathbf{x} \succsim_{P_1} \mathbf{y}$ if and only if $u_1 = u_{P_1}(\mathbf{x}_{P_1}) \geq u_{P_1}(\mathbf{y}_{P_1})$.

By repeating for every P_i, we find that v induces an order on each P_i. In other words, it is weakly separable relative to $\mathcal{P} = \{P_i\}$.

6.1.4 Show that when utility is homothetic, every cross-price elasticity $\varepsilon_{k\ell}$ is independent of income.

Answer: Proposition 2.3 shows $\mathbf{x}(\mathbf{p}, m) = m\mathbf{x}(\mathbf{p}, 1)$. Let $\boldsymbol{\varphi}(\mathbf{p}) = \mathbf{x}(\mathbf{p}, 1)$, so $\mathbf{x}(\mathbf{p}, m) = m\boldsymbol{\varphi}(\mathbf{p})$. Then $\partial x_k/\partial p_\ell = m\partial\varphi_k/\partial p_\ell$. The cross-price elasticity is then $\varepsilon_{k\ell} = \big(p_\ell/\varphi_k(\mathbf{p})\big)\partial\varphi_k/\partial p_\ell$, which is independent of m.

6.1.6 For the Stone-Geary utility function:

a) Compute the price elasticities of demand and determine their sign.

b) Compute the income elasticity of demand.

c) Under what conditions (if any) will good ℓ be a normal good? Under what conditions will it be a luxury good?

Answer: The Marshallian demands for Stone-Geary utility

$$u(\mathbf{x}) = \prod_\ell (x_\ell - a_\ell)^{\gamma_\ell}$$

with $\gamma_\ell > 0$ and $\sum_\ell \gamma_\ell = 1$ are $x_\ell(\mathbf{p}, m) = a_\ell + (\gamma_\ell/p_\ell)(m - \mathbf{p}{\cdot}\mathbf{a})$ when $m \geq \mathbf{p}{\cdot}\mathbf{a}$. They are undefined otherwise.

a) The own-price elasticity is

$$\varepsilon_{\ell\ell} = -\gamma_\ell \frac{p_\ell a_\ell + m - \mathbf{p}{\cdot}\mathbf{a}}{p_\ell a_\ell + \gamma_\ell(m - \mathbf{p}{\cdot}\mathbf{a})},$$

which is negative.

As for the cross-price elasticities, for $k \neq \ell$,

$$\varepsilon_{\ell k} = -\frac{\gamma_k p_k a_k}{p_\ell a_\ell + \gamma_k(m - \mathbf{p}{\cdot}\mathbf{a})} < 0.$$

The cross-price elasticity is also negative.

b) The income elasticity is

$$\eta_k = \frac{m\gamma_k}{p_k a_k + \gamma_k(m - \mathbf{p}{\cdot}\mathbf{a})}.$$

c) Now $\eta_k < 1$ when $p_k a_k > \gamma_k \mathbf{p}{\cdot}\mathbf{a}$ and $\eta_k > 1$ when $p_k a_k < \gamma_k \mathbf{p}{\cdot}\mathbf{a}$. The income elasticity depends on whether the **a**-cost share of k is greater or smaller than its utility share (γ_k).

6.3.2 Suppose utility has the Cobb-Douglas form $u(x) = \sum_k \gamma_k \ln x_k$ where each $\gamma_k > 0$ and $\sum_k \gamma_k = 1$. Show that indirect utility is additive separable.

Answer: The Cobb-Douglas demands are $x_k = \gamma_k m/p_k$. Then indirect utility is

$$v(\mathbf{p}, m) = \left(\sum_k \gamma_k \ln \gamma_k\right) + \sum_k \gamma_k \ln\left(\frac{m}{p_k}\right),$$

which is also in Cobb-Douglas form, albeit with a different constant term.

To have the same constant term would require $\sum_k \gamma_k \ln \gamma_k = 0$, or in other words $\prod_k \gamma_k^{\gamma_k} = 1$, which cannot happen as all γ_k obey $0 < \gamma_k < 1$.

6.3.3 Suppose utility is given by $u(\mathbf{x}) = \sum_{\ell=1}^L \alpha_\ell x_\ell^\gamma$ where each $\alpha_\ell > 0$ and $0 < \gamma < 1$.

Compute the indirect utility function and show that it fails to be additive separable as defined in section 6.3.1. Then explain what equivalent indirect utility is additive separable.

Answer: The first-order conditions are $\lambda p_\ell = \alpha_\ell \gamma x_\ell^{\gamma-1}$. As usual, divide the first-order condition for k by that for ℓ to eliminate λ. This yields

$$\frac{p_k}{p_\ell} = \frac{\alpha_k}{\alpha_\ell}\left(\frac{x_k}{x_\ell}\right)^{\gamma-1}.$$

We solve for x_k in terms of x_ℓ.

$$x_k = \left(\frac{\alpha_\ell p_k}{\alpha_k p_\ell}\right)^{\frac{1}{\gamma-1}} x_\ell.$$

Then we multiply by p_k and sum over all $k = 1, \dots, L$ and use the budget constraint.

$$m = \left(\frac{p_\ell}{\alpha_\ell}\right)^{\frac{1}{1-\gamma}} \left(\sum_k \left(\frac{p_k^\gamma}{\alpha_k}\right)^{\frac{1}{\gamma-1}}\right) x_\ell.$$

Rearranging, we find

$$x_\ell = m\left(\frac{\alpha_\ell}{p_\ell}\right)^{\frac{1}{1-\gamma}} \Bigg/ \sum_k \left(\frac{\alpha_k}{p_k^\gamma}\right)^{\frac{1}{1-\gamma}}$$

Finally, we substitute back in u to obtain indirect utility v.

$$\begin{aligned} v(\mathbf{p}, m) &= m^\gamma \left(\sum_\ell \left(\frac{\alpha_\ell}{p_\ell^\gamma}\right)^{\frac{1}{1-\gamma}}\right)\left(\sum_k \left(\frac{\alpha_k}{p_k^\gamma}\right)^{\frac{1}{1-\gamma}}\right)^{-\gamma} \\ &= m^\gamma \left(\sum_\ell \left(\frac{\alpha_\ell}{p_\ell^\gamma}\right)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}. \end{aligned}$$

When $m = 1$ this is the $(1-\gamma)$ power of an additive separable function with $0 < \gamma < 1$. It is not additive separable. However, if we had used $f(\mathbf{x}) = \left(\sum_{\ell=1}^L \alpha_\ell x_\ell^\gamma\right)^{\frac{1}{1-\gamma}}$ as our utility function, we would have obtained the additive separable indirect utility

$$v(\mathbf{p}, m) = \sum_\ell \left(\frac{\alpha_\ell m^\gamma}{p_\ell^\gamma}\right)^{\frac{1}{1-\gamma}}.$$