Homework #3

11.1.1 Show that indirect utility has the Gorman form $v_i(\mathbf{p}, m_i) = a_i(\mathbf{p}) + b(\mathbf{p})m_i$ for some functions a_i and b if and only if the expenditure function has the Gorman form $e_i(\mathbf{p}, \bar{u}) = c_i(\mathbf{p}) + d(\mathbf{p})\bar{u}$ for some functions c_i and d.

Answer: There are two cases. First, the "only if" case. Suppose $v_i(\mathbf{p}, m_i) = a_i(\mathbf{p}) + b(\mathbf{p})m_i$. Then by duality, $\bar{u} = v_i\big(\mathbf{p}, e(\mathbf{p}, u)\big) = a_i(\mathbf{p}) + b(\mathbf{p})e(\mathbf{p}, \bar{u})$. Then $e(\mathbf{p}, \bar{u}) = \bar{u}/b(\mathbf{p}) - a_i(\mathbf{p})/b(\mathbf{p})$, which is the required form with $c_i(\mathbf{p}) = -a_i(\mathbf{p})/b(\mathbf{p})$ and $d(\mathbf{p}) = 1/b(\mathbf{p})$.

For the "if" case, suppose $e_i(\mathbf{p}, \bar{u}) = c_i(\mathbf{p}) + d(\mathbf{p})\bar{u}$. By duality, $m = e_i(\mathbf{p}, v(\mathbf{p}, m) = c_i(\mathbf{p}) + d(\mathbf{p})v(\mathbf{p}, m)$. Solving for v yields $v(\mathbf{p}, m) = -c_i(\mathbf{p})/d(\mathbf{p}) + m/d(\mathbf{p})$, which is the required form.

11.2.1 Suppose $\boldsymbol{\omega} = (1, 0)$, $\mathbf{p} = (1, 0)$, $\mathbf{p}' = (0, 1)$ and utility has the Leontief form $u(\mathbf{x}) = \min\{x_1, 2x_2\}$.

a) Does the Law of Demand hold for Walrasian demand $\boldsymbol{\xi}(\mathbf{p}, \boldsymbol{\omega})$?

b) Does the Law of Demand hold when the prices are $\mathbf{p} = (1, p)$ and $\mathbf{p}' = (p, 1)$ for $0 < p < 1$?

Answer:

a) **No.** The Law of Demand does not hold here. Income in the two cases is $m = \mathbf{p}\cdot\boldsymbol{\omega} = 1$ and $m' = \mathbf{p}'\cdot\boldsymbol{\omega} = 0$. Thus demand obeys $\boldsymbol{\xi} = (1, 1/2) \in \boldsymbol{\xi}(\mathbf{p}, 1)$ and $\boldsymbol{\xi}' = (0, 0) \in \boldsymbol{\xi}(\mathbf{p}', 0)$. But $(\mathbf{p}' - \mathbf{p})\cdot(\boldsymbol{\xi}' - \boldsymbol{\xi}) = (-1, +1)\cdot(-1, -1/2) = 1/2 > 0$, violating the Law of Demand.

b) **No.** The Law of Demand still does not hold even though demand is now a function. Income is now $m = 1$ and $m' = p$ It follows that demand is now

$$\xi = \frac{1}{2 + p}(2, 1)$$
$$\xi' = \frac{p}{2p + 1}(2, 1).$$

Now $\mathbf{p}' - \mathbf{p} = (p - 1)(1, -1)$ and $\xi' - \xi = [(2 + p)^{-1} - p(2p + 1)^{-1}](2, 1)$. It follows that

$$(\mathbf{p}'-\mathbf{p})\cdot(\boldsymbol{\xi}'-\boldsymbol{\xi}) = (p-1)\left(\frac{p}{2p+1}-\frac{1}{2+p}\right)\Big[(1,-1)\cdot(2,1)\Big]$$
$$= (p-1)\frac{p^2-1}{(2+p)(2p+1)}$$
$$= \frac{(p+1)(p-1)^2}{(2+p)(2p+1)}.$$

This is positive when $p \neq 1$, showing that the Law of Demand is violated.

12.1.2 Find sets obeying the following conditions:

a) The set fails the no free lunch condition.

b) The set is a production set that is additive, but not divisible.

c) The set is a production set that is not convex.

Answer: Many such examples are possible.

a) The set $Y = \{\mathbf{y} \in \mathbb{R}^2 : y_1 + y_2 \leq 2\}$ is a set that fails the no free lunch condition (T3) because $(1,1) \in Y$. Notice that it obeys the other conditions for a production set: It is non-empty, closed, and obeys inaction and free disposal.

b) Consider the production function $f(z) = z^2$. This increasing returns to scale production function yields a production set that fails convexity. Let $Y = \{(-z,q) : q \leq z^2, z \leq 0\}$. Then $(-1,1) \in Y$, but $\frac{1}{2}(-1,1) + \frac{1}{2}(0,0) = (-\frac{1}{2},\frac{1}{2}) \notin Y$ because $\frac{1}{2} \not\leq (\frac{1}{2})^2 = \frac{1}{4}$. Since f is continuous and $f(0) = 0$, we showed in Example 12.1.1 that Y is a production set.

Now $1 \leq (-1)^2$, so $(-1,1) \in Y$. But $1/2 \not\leq (-1/2)^2 = 1/4$, so $(-1/2,1/2) \notin Y$. Since $(-1,1) \in Y$, but $(-1/2,1/2) \notin Y$, this set is not divisible.

It is additive because if $(-z_1,q_1) \in Y$ and $(-z_2,q_2) \in Y$, $q_1+q_2 \leq z_1^2+z_2^2 \leq (z_1+z_2)^2$. From this it follows that $(q_1+q_2, z_1+z_2) \in Y$.

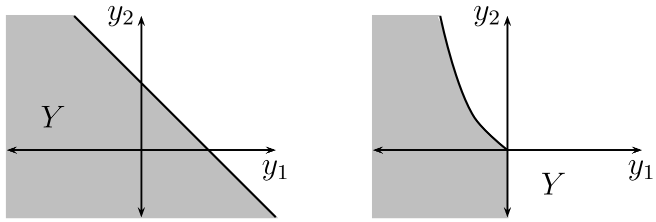
c) Using the production set from (b), we find that $(0,0),(-1,1) \in Y$, but $(-1/2,1/2) \notin Y$, so Y is a production set that is not convex.

These are illustrated in the figures below.

12.3.1 Suppose there are two inputs and one output with the Leontief production function $f(z_1,z_2) = \min(z_1, 3z_2)$. The output price is $p > 0$ and the input prices are $w_\ell > 0$.

a) Find all profit-maximizing net output vectors.

b) Calculate the profit function.



(a) **(b, c)**

c) Show directly that the Law of Supply holds.

Answer: We will follow the convention of Example 12.1.1 and write $\mathbf{y} = (-z_1, -z_2, q)$ where q is the output and z_i are inputs. We write price as (w_1, w_2, p) and net output as $\mathbf{y} = (-z_1, -z_2, q)$.

a) Since both inputs are costly, cost is minimized when there is no excess of either input. Thus $z_1 = 3z_2 = q$ and cost is $w_1 q + w_2(q/3)$. Profit is then $(w_1, w_2, p)\cdot(-q, -q/3, q) = (p - w_1 - w_2/3)q$. There is no maximum when $p > w_1 + w_2/3$, and profit is maximized at $q = 0$ when $p < w_1 + w_2/3$. The net output correspondence is:

$$\mathbf{y}(\mathbf{p}) = \begin{cases} \{(-q, -q/3, q) : q \geq 0\} & \text{when } p = w_1 + w_2/3 \\ \{\mathbf{0}\} & \text{when } p < w_1 + w_2/3 \\ \text{undefined} & \text{when } p > w_1 + w_2/3. \end{cases}$$

b) Using part (a), we find the profit function is

$$\pi(p, \mathbf{w}) = \begin{cases} 0 & \text{for } p \leq w_1 + w_2/3 \\ +\infty & \text{otherwise.} \end{cases}$$

c) For $p < w_1 + w_2/3$, the net supply vector $\mathbf{y}$ is zero. Then $\Delta\mathbf{p}\cdot\Delta\mathbf{y} = \mathbf{p}'\cdot\mathbf{y}' - \mathbf{p}\cdot\mathbf{y}'$. Now $\mathbf{p}'\cdot\mathbf{y}' = 0$ due to constant returns to scale, and $\mathbf{p}\cdot\mathbf{y}' \leq \mathbf{p}\cdot\mathbf{y} = 0$. It follows that $\Delta\mathbf{p}\cdot\Delta\mathbf{y} \geq 0$.

Similarly, if $p' < w_1' + w_2'/3$, $\mathbf{y}' = \mathbf{0}$ and $\Delta\mathbf{p}\cdot\Delta\mathbf{y} = -\mathbf{p}'\cdot\mathbf{y} \geq 0$.

Now suppose $p = w_1 + w_2/3$ and $p' = w_1' + w_2'/3$. Then $\Delta\mathbf{p} = (w_1' - w_1, w_2' - w_2, p' - p)$. Net outputs are $\mathbf{y} = (-q, -q/3, q)$ and $\mathbf{y}' = (-q', -q'/3, q')$, so $\Delta\mathbf{y} = \Delta q(-1, -1/3, 1)$. Now $\Delta\mathbf{p}\cdot\Delta\mathbf{y} = (\Delta p - \Delta w_1 - \Delta w_2/3)\Delta q$. Then $\Delta p - \Delta w_1 - \Delta w_2/3 = 0$, so $\Delta\mathbf{p}\cdot\Delta\mathbf{y} = 0$.

Either way, the Law of Supply holds.

12.4.3 Suppose production is described by a $\mathcal{C}^1$ transformation function $T\colon \mathbb{R}^L_+ \to \mathbb{R}_+$ with $\partial T/\partial y_1 \neq 0$. Use the marginal rate of transformation to describe the efficient net output vectors for this technology.

Answer: The technology set is $Y = \{\mathbf{y} : T(\mathbf{y}) \leq 0\}$. Translating the definition into a maximization problem, if a vector $\mathbf{y}^*$ is production efficient it maximizes y_1 over $\{\mathbf{y} \in Y : y_j \geq y_j^*, j \neq 1\}$. The Lagrangian for this problem is

$$y_1 - \lambda T(\mathbf{y}) + \sum_{j \neq 1} \mu_j (y_j - y_j^*).$$

The first-order conditions are $1 - \lambda\, \partial T / \partial y_1 = 0$ and $\mu_j - \lambda\, \partial T / \partial y_j = 0$ for $j \neq 1$. Since $\partial T / \partial y_1 > 0$, we can substituting the first equation in the second, finding $\mu_j = (\partial T / \partial y_j)/(\partial T / \partial y_1)$ for all $j \neq 1$, or $\mu_j = \mathrm{MRT}_{j1}$ for $j \neq 1$. We must also satisfy the constraint $T(\mathbf{y}) = 0$ since $\lambda = 1/\partial T / \partial y_1 > 0$.

This amounts to saying that $\mathbf{y}$ maximizes profit with price vector $(1, \mu_2, \mu_3, \ldots, \mu_L)$.