Homework #4

13.2.2 The production set is $Y = \{(y_1, y_2) : y_1 \le 0, y_2 \le -(2y_1 + 1/y_1)\}$. Find all prices (p_1, p_2) where profit maximization is possible.

Answer: The production set allows free disposal, so profit-maximizing prices must be non-negative. Profit is then $p_1y_1+p_2y_2$. To maximize profit, we must set $y_2 = -2y_1-1/y_1$, yielding profit $p_1y_1 - 2p_2y_1 - p_2/y_1$. The first deriviative is $(p_1 - 2p_2) + p_2/y_1^2$ and the second deriviate is negative. If the first derivative is always positive, profit cannot be maximized. Thus $p_1 < 2p_2$ for profit maximization. If $p_1 < 2p_2$, profit is maximized at $y_1 = \big(p_2/(2p_2 - p_1)\big)^{1/2}$.

It follows that profit may be maximized whenever $\mathbf{p} \ge \mathbf{0}$ and $p_1 < 2p_2$.

13.3.4 Recall the production function of Example 13.2.1, $f(z) = 1 + z - 1/(1 + z)$. Find the homogenized production function F.

Answer: We first homogenize the production function. For $z_2 > 0$, $F(z_1, z_2) = z_2f(z_1/z_2) = z_2 + z_1 - z_2/(1 + z_1/z_2) = z_1 + z_2 - z_2^2/(z_1 + z_2)$. For $z_1 > 0$, $F(z_1, 0) = \lim_{z_2\to0} F(z_1, z_2) = z_1$. For $z_1 = 0$, $F(0, 0) = \lim_{z_2\to0} F(0, z_2) = \lim(z_2 - z_2) = 0$. Thus

$$F(z_1, z_2) = \begin{cases} z_1 + z_2 - \frac{z_2^2}{z_1+z_2} & \text{when } z_2 > 0 \\ z_1 & \text{when } z_2 = 0. \end{cases}$$

The production set corresponding to f is $Y = \{(q, -z, -z_2) : z_1, z_2 \ge 0, q \le F(z_1, z_2)\}$. Now $F(z_1, 0) = z_1$, so $F(\lambda z_1, 0) = \lambda z_1 = \lambda F(z_1, 0)$. It follows that anything of the form $(q, -z_1, 0)$ with $q \le z_1$ is in the production set. Such net outputs are CRS elements of Y.

13.3.7 Suppose a convex production set Y is also a constant returns production set. Find the augmented production set $\hat{Y}$. What price do you expect the entrepreneurial factor to have? Why?

Answer: The augmented production set is given by $\hat{Y} = \text{cl}\{(z\mathbf{y}, -z) : \mathbf{y} \in Y,\ z > 0\}$. Note that if $\mathbf{y} \in zY$ for any $z > 0$, then $\mathbf{y} \in z'Y$ for all $z' > 0$. Then $\hat{Y} = \overline{(Y \times (-\mathbb{R}_{++})} = Y \times (-\mathbb{R}_+)$.

Since the entrepreneurial factor is not needed to allow a firm using $\hat{Y}$ to operate at any scale, it will not be demanded at any positive price, and we expect the price of the entrepreneurial factor to be zero.

14.3.2 Consider a two-person, two-good exchange economy. Endowments are $\boldsymbol{\omega}^1 = (1, 1)$ and $\boldsymbol{\omega}^2 = (2, 1)$. Utility is $u_1(\mathbf{x}^1) = \min\{x_1^1, 2x_2^1\}$ and $u_2(\mathbf{x}^2) = 2x_1^2 + x_2^2$. Find all Walrasian equilibrium prices and allocations.

Answer: If either price is zero, consumer two will be unable to maximize utility. We can safely take good 1 as numéraireand set $p_2 = p$. Since both prices are positive, consumer one chooses a point where $x_1^1 = 2x_2^1$. Consumer one's income is $1 + p$. Using the budget constraint, we find $\mathbf{x}^1(p) = \big((1+p)/(2+p)\big)$. Consumer two will buy only good 1 if $p < 1/2$, only good 2 of $p > 1/2$ and will be happy with any point on the budget constraint if $p = 1/2$.

If $p < 1/2$, demand for good 2 is $(1+p)/(2+p)$ which must equal the endowment of good 2. But then $p = -3$, which is impossible. This cannot be an equilibrium.

If $p > 1/2$, demand for good 1 is $2(1+p)/(2+p)$ which must equal the endowment of good 1. But then $p = -4$, which is impossible. This is also not an equilibrium.

That leaves $p = 1/2$. Then $\mathbf{x}^1 = (6/5, 3/5)$. Market clearing yields $\mathbf{x}^2 = (9/5, 7/5)$. As that is on the budget constraint for consumer two, it maximizes utility and we have found the equilibrium.