Homework #5

14.3.3 Consider a two-person, two-good exchange economy where the consumers have identical *convex* utility functions on $\mathbb{R}^2_+$ given by $u_i(\mathbf{x}) = \max\{x_1, x_2\}$. Endowments are $\boldsymbol{\omega}^1 = (3, 0)$ and $\boldsymbol{\omega}^2 = (0, 3)$. Find *all* equilibria for this exchange economy in $\mathbb{R}^2_+$, or show that there are no equilibria.

Answer: Preferences are strictly monotonic, so any equilibrium prices have to be strictly positive. We take good one as numéraire and write $\mathbf{p} = (1, p)$. Incomes are then $m^1 = 3$ and $m^2 = p$. If the prices of the two goods are different, each consumer will only demand the cheapest one. Thus if $p < 1$ market demand is $(3 + 3p, 0)$ and if $p > 1$ demand is $(0, 3+3/p)$. In either case, there is excess demand, so the only possible equilibrium price is $p = 1$. Due to the convex utility functions, demand for each consumer is either $(3, 0)$ or $(0, 3)$. It follows that there are two possible equilibria, both with prices $\mathbf{p} = (1, 1)$ but with different individual consumption vectors. The two equilibrium allocations are $(\mathbf{x}^1, \mathbf{x}^2) = \big((3, 0), (0, 3)\big)$ and $(\mathbf{x}^1, \mathbf{x}^2) = \big((0, 3), (3, 0)\big)$.

14.4.1 Consider a two-agent, two-good, one-firm production economy where utility is $u_1(\mathbf{x}^1) = (x^1_1)^{1/2}(x^1_2)^{1/2}$ and $u_2(\mathbf{x}^2) = (x^2_1)^{1/3}(x^2_2)^{2/3}$, and endowments are $\boldsymbol{\omega}^1 = (3, 0)$ and $\boldsymbol{\omega}^2 = (6, 0)$. There is one firm with production set $Y = \{\mathbf{y} : y_1 \le 0, y_2 \le -y_1\}$. Find the equilibrium prices, equilibrium demands by individuals, and the firm's equilibrium net output.

Answer: If the firm does not produce anything, the economy will not have any of good two. Since demand for good two will be positive, this cannot be an equilibrium.

If the firm does produce ($y_2 > 0$), constant returns to scale implies that profits will be zero. This requires $p_1 = p_2$. We can now normalize prices so $\mathbf{p} = (1, 1)$, which are the equilibrium prices. Incomes are then $m^1 = 3$ and $m^2 = 6$. The Cobb-Douglas utilities yield demands $\mathbf{x}^1 = (3/2, 3/2)$ and $\mathbf{x}^2 = (2, 4)$. All that is left is to find the production vector. Now $\mathbf{x}^1 + \mathbf{x}^2 = \boldsymbol{\omega} + \mathbf{y}$ by market clearing. This can be rewritten $(7/2, 11/2) = (9, 0) + (y_1, y_2)$. It follows that the production vector is $\mathbf{y} = (-11/2, +11/2)$.

14.4.4 Consider a two-agent, two-good, one-firm production economy where utility is $u_1(\mathbf{x}^1) = (x^1_1)^{1/2}(x^1_2)^{1/2}$ and $u_2(\mathbf{x}^2) = (x^2_1)^{1/2}(x^2_2)^{1/2}$, and endowments are $\boldsymbol{\omega}^1 = (4, 0)$ and $\boldsymbol{\omega}^2 = (4, 0)$. Each agent will receive half of the profits of the firm. There is one firm with production set $Y = \{\mathbf{y} : y_1 \le 0, y_2 \le \sqrt{-y_1}\}$. Find the equilibrium prices, equilibrium demands by individuals, and the firm's equilibrium net output.

Answer: We start by considering profit, $p_2\sqrt{-y_1} + p_1 y_1$. The first-order conditions for profit maximization are $(p_2/2)(1/\sqrt{-y_1}) = p_1$, so the net output is

$$\mathbf{y} = \begin{pmatrix} -p_2^2/4p_1^2 \\ p_2/2p_1 \end{pmatrix}.$$

The resulting profit function is $\pi(\mathbf{p}) = p_2^2/4p_1$.

Each consumer receives half of the profit and has endowment income $4p_1$, so each consumer has income $m = 4p_1 + p_2^2/8p_1$. Since consumers have identical preferences, and income, they will consume the same amount. Utility is equal-weighted Cobb-Douglas, and market demand is $2m(1/2p_1, 1/2p_2) = m(1/p_1, 1/p_2)$.

Adding the endowment to the firm's supply yields $(8 - p_2^2/4p_1^2, p_2/2p_1)$. Both goods are demanded in equilibrium, and prices must be strictly positive. We normalize prices so that $p_1 = 1$ and $p_2 = p$. Then market clearing for good two says $p/2 = m/p = (4 + p^2/8)/p$. Thus $4p^2 = 32 + p^2$ implying $p = \pm\sqrt{32/3}$. Only the positive root makes economic sense, and the equilibrium has $\mathbf{p} = (1, 4\sqrt{2/3})$, $\mathbf{y} = (-8/3, \sqrt{8/3})$, $m = 16/3$, and $\mathbf{x}^1 = \mathbf{x}^2 = (8/3, \sqrt{2/3})$.

14.4.6 An economy has two consumers. Consumer one has Cobb-Douglas utility $u_1(\mathbf{x}) = \sqrt{x_1 x_2}$. Consumer two has lexicographic preferences. That is $\mathbf{x} \succsim_2 \mathbf{x}'$ if and only if either $x_1 > x_1'$ or $x_1 = x_1'$ and $x_2 \ge x_2'$. Good one can be used to produce good two. This is described by the production set $Y = \{(y_1, y_2) : y_1 \le 0 \text{ and } y_2 \le -y_1\}$. Endowments of goods are $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (3, 0)$. Consumer one owns 100% of the firm, while consumer two owns none of the firm. Find the equilibrium prices $\hat{\mathbf{p}}$ and all equilibrium allocations $(\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2, \hat{\mathbf{y}})$.

Answer: If the price of either good is zero, consumer one will not be able to maximize utility, so both prices must be positive. Moreover, there will be excess demand for good two unless it is produced. This means that good two must be produced in equilibrium. As the marginal rate of transformation is 1, $p_1 = p_2$. We can normalize prices so that $\hat{\mathbf{p}} = (1, 1)$.

The income earned by the firm will be zero, so the only income is from the endowments. Using $\hat{\mathbf{p}} = (1, 1)$, we calculate incomes $m_1 = 2$ and $m_3 = 3$. Consumer one has Cobb-Douglas utility, so demand is $\mathbf{x}^1 = (m_1/2)(1/p_1, 1/p_2) = (1, 1)$.

Consumer two, with lexicographic utility, will not waste income on good two. Everything is spent on good 1 and demand is $\mathbf{x}^2 = (m_2/p_1, 0) = (3, 0)$.

Thus $\hat{\mathbf{x}}^1 = (1, 1)$ and $\hat{\mathbf{x}}^2 = (3, 0)$. Market clearing requires $(5, 0) + \hat{\mathbf{y}} = \boldsymbol{\omega} + \hat{\mathbf{y}} = \hat{\mathbf{x}}^1 + \hat{\mathbf{x}}^2 = (4, 1)$. It follows that $\hat{\mathbf{y}} = (-1, 1)$, which is in Y.

To sum up, in equilibrium, $\hat{\mathbf{p}} = (1, 1)$ or any positive multiple thereof, $\hat{\mathbf{x}}^1 = (1, 1)$, $\hat{\mathbf{x}}^2 = (3, 0)$ and $\hat{\mathbf{y}} = (-1, 1)$. This is the only equilibrium allocation.