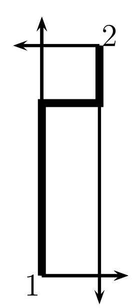
Homework #6

18.1.2 Suppose there are three goods and utility has the Cobb-Douglas forms $u_1(\mathbf{x}^1) = (x_1^1)^{1/3}(x_2^1)^{1/3}(x_3^1)^{1/3}$, $u_2(\mathbf{x}^2) = (x_1^2)^{1/3}(x_2^2)^{1/3}(x_3^2)^{1/3}$, and $\boldsymbol{\omega} = (5, 2, 3)$. Find all Pareto optimal allocations.

Answer: We appeal to Example 18.1.5. Each consumer receives a share of the endowment. The Pareto set is $\{(\mathbf{x}^1, \mathbf{x}^2) : \mathbf{x}^1 = t(5, 2, 3), \mathbf{x}^2 = (1-t)(5, 2, 3), 0 \leq t \leq 1\}$.

18.1.7 Suppose consumer one has the linear utility function $u_1(\mathbf{x}^1) = 2x_1^1 + x_2^1$, while consumer two has quasi-linear utility $u_2(\mathbf{x}^2) = x_1^2 + \sqrt{x_2^2}$. The endowment is $\boldsymbol{\omega} = (1, 4)$. Find all Pareto optimal allocations. **Answer:** Here $\text{MRS}_{12}^1 = 2$ and $\text{MRS}_{12}^2 = 2\sqrt{x_2^2}$. The interior solutions have $x_2^2 = 1$, so $x_2^1 = 3$. Good one can be allocated in any fashion. There are two kinds of boundary solutions. One where $x_2^2 > 1$. In that case, all of good one must be allocated to consumer two because of the larger MRS. Good two can be allocated in any fashion, as long as $x_2^2 > 1$. The other boundary case has $x_2^2 < 1$. Then MRS^1 is larger, so consumer 1 gets all of good one while good two can be allocated in any fashion, so long as $x_2^2 < 1$. The Pareto set is illustrated by the heavy lines in the diagram below.



19.1.2 An economy has two goods and two identical Cobb-Douglas consumers with $u_i(\mathbf{x}^i) = \sqrt{x_1^i x_2^i}$. The total endowment is $(0, 6)$. There is one constant returns to scale firm that produces good 1 and uses good 2 as its only input. The production function is $f(z) = 5z$.

a) Find all Pareto optimal allocations of goods and the corresponding net output vector.

b) Find prices and wealth levels that make each Pareto optima a quasi-equilibrium with taxes and transfers.

Answer:

a) The net output vector will have the form $\mathbf{y} = (-5y_2, y_2)$ with $y_2 \leq 0$. Here the marginal rate of transformation is $\text{MRT}_{12} = 1/5$. This must also be the marginal rate of substitution at any interior Pareto optimum. Note that since utility is zero if there is no production, the production technology must be used.

Now $\text{MRS}^i_{12} = x^i_2/x^i_1 = 1/5$, so $5x^i_2 = x^i_1$. Summing over both consumers, aggregate consumption obeys $5x_2 = x_1$. Since good 1 can only be obtained from the production sector, $x_1 = -5y_2$ and $x_2 = 6 + y_2$. Thus $x_2 = -y_2$ and $x_2 = 6 + y_2$. It follows that $x_2 = 3$ and $y_2 = -3$, so $x_1 = 15$.

Since both consumers will consume in the same proportions, $\mathbf{x}_1 = \alpha(15, 3)$ and $\mathbf{x}_2 = (1 - \alpha)(15, 3)$ for some α between 0 and 1. Also, $\mathbf{y} = (15, -3)$. These are the Pareto optimal allocations. The cases $\alpha = 0$ and $\alpha = 1$ are the Pareto optima on the border of the Edgeworth box.

b) We have already derived implicitly the prices in part (a), $\mathbf{p} = (1, 5)$. The wealth levels that yield the goods allocation $\mathbf{x}^1 = \alpha(15, 3)$ and $\mathbf{x}^2 = (1 - \alpha)(15, 3)$ are $m^1 = 30\alpha$ and $m^2 = 30(1 - \alpha)$.

19.2.3 Suppose there are two goods and two consumers, both with consumption set $\mathfrak{X} = \mathbb{R}^2_+$. The aggregate endowment is $\boldsymbol{\omega} = (100, 200)$. Consumer one has utility function $u_1(x_1, x_2) = (x_1x_2)^{1/3}$ and consumer two has utility $u_2(x_1, x_2) = x_1 + x_2$. Find the utility possibility set.

Answer: We first look for interior Pareto optima. These have $x^1_1/x^1_2 = \text{MRS}^1_{12} = \text{MRS}^2_{12} = 1$, so $x^1_1 = x^1_2$. Since $x^1_1 \le 100$, these optima require $x^1_2 \le 100$. We set $z = x^1_2$. The interior Pareto optima have $0 \le z \le 100$ with $u_1 = z^{2/3}$ and $u_2 = 300 - 2z$. Thus $u_2 = 300 - 2u_1^{3/2}$ for $0 \le u_1 \le 10^{4/3}$ or $u_1 = (150 - u_2/2)^{2/3}$ for $200 \le u_2 \le 300$.

If $100 < z \le 200$, we are no longer in an interior solution, and consumer one will get all of good one. Then $u_1 = (100z)^{1/3}$ and $u_2 = 200 - z$, so $u_2 = 200 - (u_1)^3/100$ or $u_1 = \big(100(200 - u_2)\big)^{1/3}$ for $0 \le u_2 \le 100$.

Summing up, the Pareto frontier is given by

$$u_1 = \begin{cases} (20000 - 100u_2)^{1/3} & \text{when } 0 \le u_2 \le 100 \\ \left(\frac{300-u_2}{2}\right)^{2/3} & \text{when } 100 \le u_2 \le 300 \end{cases}$$

The utility possibility set is all points to the left and below the Pareto frontier.

19.2.4 Suppose there are two goods, aggregate endowment is $\boldsymbol{\omega} = (2000, 0)$, and good 1 is used in production, with production function $f(z) = 75z^{1/3}$. Utility is $u_i(\mathbf{x}^i) = x^i_1 + 4x^i_2$. The social welfare function is $W(u) = u_1u_2$. Find the social welfare maximizing allocation.

Answer: The solution must be a Pareto optimum and good two must be produced. The marginal rate of substitution for both consumers is $\text{MRS}_{12} = 1/4$. The marginal rate of transformation is $25z^{-2/3} = 1/4$. It follows that $z = 1000$ and $\mathbf{y} = (-1000, 750)$. This means that $\mathbf{y} + \boldsymbol{\omega} = (1000, 750)$ is available for consumption.

There is a 1-1 utility trade-off as we move goods from one consumer to another, so $u_1 + u_2 = u(1000, 750) = 4000$ with $u_1, u_2 \geq 0$. We must maximize $u_1 u_2$ under this constraint. This is the same as maximizing Cobb-Douglas utility over a budget set with equal prices, thus $u_2/u_1 = 1$. It follows that $u_1 = u_2 = 2000$ at the optimum.

There are a number of ways to allocate goods to obtain the optimal utility levels. They all have the form $\mathbf{x}^1 = (x_1^1, 500 - x_1^1/4)$, $\mathbf{x}^2 = (1000 - x_1^1, 250 + x_1^1/4)$ with $0 \leq x_1^1 \leq 1000$.