# Homework #7

21.3.1 Suppose there are 5 states, $s = 1, \ldots, 5$. Lottery $L_1$ has probabilities $(1/5, 1/10, 3/10, 1/5, 1/5)$ while lottery $L_2$ has probabilities $(1/5, 3/10, 1/10, 2/5, 0)$. Suppose $L_3 = (1/5, 1/5, 1/5, 3/10, 1/10)$.

*a)* Write $L_3$ as a compound lottery over $L_1$ and $L_2$.

*b)* Suppose $u$ is an expected utility function with $u(L_1) = 1$ and $u(L_2) = 3$. Compute $u(L_3)$.

**Answer:**

*a)* Here $L_3 = \frac{1}{2}L_1 \oplus \frac{1}{2}L_2$.

*b)* Since $L_3 = \frac{1}{2}L_1 \oplus \frac{1}{2}L_2$, $Eu(L_3) = \frac{1}{2}Eu(L_1) + \frac{1}{2}Eu(L_2) = 2$.

21.3.5 Let $\mathbf{S}$ be a finite set of states and $s^*$ and $s_*$ be best and worst states in $\mathbf{S}$. Assume that a preference order defined over pairs of lotteries in $\mathfrak{L}(\mathbf{S})$ obeys independence.

*a)* Let $s \in \mathbf{S}$. Show that $s \succsim (1-\alpha)s \oplus \alpha s_*$.

*b)* Now show that $s^* \succsim L \succsim s_*$ for all $L \in \mathfrak{L}(\mathbf{S})$.

**Answer:**

*a)* Since $s_*$ is a worst state, $s \succsim s_*$. We apply independence using $s$ as the third lottery to find $\alpha s \oplus (1-\alpha)s \succsim \alpha s_* \oplus (1-\alpha)s$. Then $\alpha s \oplus (1-\alpha)s = s$ yields the desired result.

*b)* Let $L_s$ denote the lottery that assigns 100% probability to state $s$. Then any lottery $L$ can be written as $L = \bigoplus_{s=1}^{S} p_s L_s$ where $p_s$ is the probability that $L$ reaches state $s$. If only a single state $s$ occurs with positive probability in $L$, it must have probability 1, so $L = L_s$. But then $s^* \succsim L_s \succsim s_*$ by definition of $s^*$ and $s_*$.

We will proceed by induction. Suppose we know that $s^* \succsim L \succsim s_*$ whenever there are $N < S$ states that have positive probability in $L$. We have already done the case $N = 1$. We will show that $s^* \succsim L \succsim s_*$ when there are $N+1$ states with positive probability.

Suppose $L$ is a lottery with $N+1$ states with positive probability. Let $t$ be the first state with positive probability. Note $0 < p_t < 1$ since at least one other state must have positive probability. We can write

$$L = p_t L_t + (1-p_t)\left(\bigoplus_{s>t} \frac{p_s}{1-p_t} L_s\right).$$

Then since $s^* \succsim L_1 \succsim s_*$, independence tells us that

$$p_t s^* + (1-p_t)\left(\bigoplus_{s>t} \frac{p_s}{1-p_t} L_s\right) \succsim p_t L_t + (1-p_1)\left(\bigoplus_{s>t} \frac{p_s}{1-p_t} L_s\right)$$
$$\succsim p_t s_* + (1-p_1)\left(\bigoplus_{s>t} \frac{p_s}{1-p_t} L_s\right).$$

Now $L' = \left(\bigoplus_{s>t} p_s/(1-p_t)L_s\right)$ has only $N$ states with positive probability, so

$$s^* \succsim \left(\bigoplus_{s>t} \frac{p_s}{1-p_t} L_s\right) \succsim s_*.$$

By independence, $s^* \succsim L \succsim s_*$.

21.4.4 Suppose the random variable $X$ has distribution $F$ is described by a probability density function $f(x) = \beta e^{-\alpha x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$ where $\alpha, \beta > 0$.

*a)* What must $\beta$ be (in terms of $\alpha$) for $f$ to be a probability density function?

*b)* What is the mean of $X$?

*c)* Compute the variance of $X$.

*d)* Suppose $u(x) = x^2$. Find $Eu(X)$.

**Answer:**

*a)* This must obey $\beta \int_0^\infty e^{-\alpha x}\,dx = 1$ to be a probability density function. Evaluating the integral, we obtain $\beta/\alpha$, so $\beta = \alpha$.

*b)* The mean is $\mu = \int_0^\infty \alpha x e^{-\alpha x}\,dx$. This may be integrated by parts to find $\mu = \alpha^{-1}[-ue^{-u} - e^{-u}]_0^{+\infty} = 1/\alpha$.

*c)* The variance is $\text{var}(X) = E(X^2) - \mu^2 = E(X^2) - \alpha^{-2}$. Now

$$\begin{aligned} E(X^2) &= \alpha \int_0^\infty x^2 e^{-\alpha x}\,dx \\ &= \alpha^{-2} \int_0^\infty u^2 e^{-u}\,du \\ &= \alpha^{-2}\left[-u^2e^{-u} - 2ue^{-u} - 2e^{-u}\right]_0^\infty = 2\alpha^{-2}. \end{aligned}$$

So $\text{var}(X) = E(X^2) - \mu^2 = 1/\alpha^2$.

*d)* This is just $E(X^2)$ from part (c), which is $Eu(F) = 2/\alpha^2$.

21.4.5 Suppose the distribution $F$ is described by the probability density function $f(x) = (2\pi)^{-1/2}e^{-(x-\mu)^2/2}$ defined on all of $\mathbb{R}$.

*a*) Compute the mean of $F$. You may use the fact that $\int_{\mathbb{R}} f(x)\,dx = 1$.

*b*) Compute the variance of $F$.

*c*) Let utility be $u(x) = -e^{-ax}$. Compute $Eu(F)$.

**Answer:**

*a*) We start with the mean $EF$.

$$\begin{aligned} EF &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} xe^{-(x-\mu)^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (x-\mu)e^{-(x-\mu)^2/2}\,dx + \frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-(x-\mu)^2/2}\,dx \end{aligned}$$

Since the last term is just the integral of $f$ multiplied by $\mu$, it evaluates to $\mu$. The substitution $u = x - \mu$ converts the first term to $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} ue^{u^2/2}\,du$. As the integral of an odd function ($f(-u) = -f(u)$), it must be zero. Thus $EF = \mu$.

*b*) We turn our attention the variance, $\text{var}(F) = E\big((X-\mu)^2\big)$. Thus

$$\begin{aligned} \text{var}(F) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (x-\mu)^2 e^{-(x-\mu)^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} z^2 e^{-z^2/2}\,dz \\ &= \frac{1}{\sqrt{2\pi}}\Big[-ze^{-z^2/2}\Big]_{-\infty}^{+\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-z^2/2}\,dz \\ &= 1. \end{aligned}$$

Where we have integrated by parts using $v = z$ and $du = ze^{-z^2/2}\,dz$, so that $u = -e^{-z^2/2}$ and $dv = dz$.

*c*) We compute

$$\begin{aligned} Eu(F) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\alpha x}e^{-(x-\mu)^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-(x^2-2(\mu-\alpha)x+\mu^2)/2}\,dx \\ &= \left(e^{\alpha^2/2-\alpha\mu}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-(x-(\mu-\alpha))^2/2}\,dx \\ &= e^{\alpha^2/2-\alpha\mu}. \end{aligned}$$

22.1.3 Calculate the probability premium $\pi(x, \epsilon, u)$ and describe how it is affected by changes in $x$ and $\epsilon$ is the following cases.

*a)* $u(x) = x$.

*b)* $u(x) = x^2$.

*c)* $u(x) = 20x - x^2$.

**Answer:**

*a)* The probability premium obeys $u(x) = (\frac{1}{2} + \pi)u(x + \epsilon) + (\frac{1}{2} - \pi)u(x - \epsilon)$. Thus $x = (\frac{1}{2} + \pi)(x + \epsilon) + (\frac{1}{2} - \pi)(x - \epsilon) = x + 2\pi\epsilon$. This means $\pi = 0$. The probability premium for this risk neutral consumer is unaffected by changes in $x$ or $\epsilon$.

*b)* Here $x^2 = (\frac{1}{2} + \pi)(x + \epsilon)^2 + (\frac{1}{2} - \pi)(x - \epsilon)^2 = x^2 + 4\pi x\epsilon + \epsilon^2$. It follows that $\pi = -\epsilon/4x$. The negative sign indicates a risk seeker. The probability premium is proportional to $\epsilon$ and inversely proportional to $x$.

*c)* Here $20x - x^2 = 20x - x^2 - \epsilon^2 + \pi(40\epsilon - 4x\epsilon)$. Then $\epsilon^2 = \pi\epsilon(40 - 4x)$, so $\pi = \epsilon/(40 - 4x)$. This individual is risk averse for $x < 10$ and risk seeking for $x > 10$. The probability premium is proportional to $\epsilon$ and inversely proportional to $40 - 4x$.

22.1.6 Suppose the distribution $F$ is described by a probability density function $f(x) = (2\pi)^{-1/2}e^{-(x-\mu)^2/2}$ and utility is $u(x) = -e^{-\alpha x}$. Calculate the risk premium for $F$. How do changes in $\alpha$ affect the risk premium?

**Answer:** We use the results of Problem 21.4.5 to find $EF = \mu$ and $Eu = -e^{\alpha^2/2 - \alpha\mu}$. We find the certainty equivalent by setting $-e^{-\alpha c} = -e^{\alpha^2/2 - \alpha\mu}$, so $-\alpha c = \alpha^2/2 - \alpha\mu$. This yields the certainty equivalent $c(\alpha) = \mu - \alpha/2$. The risk premium is then $EF - c(\alpha) = \alpha/2$. Increases in $\alpha$ increase the risk premium.