Homework #8

22.2.1 Consider the problem of saving under uncertainty. Let $u(c) = c^{1-\sigma}/(1-\sigma)$ with $\sigma > 0$ (use $\ln c$ in the $\sigma = 1$ case).

a) First solve the saving problem without uncertainty.

b) Let F be uniformly distributed over the interval $[\bar{r}/2, 3\bar{r}/2]$. Find the optimal c_1.

c) Under F, does consumption in period 1 increase or decrease compared with a certain interest rate of $\bar{r}$ when $\sigma = 1$? What if $\sigma = 2$?

Answer:

a) Here utility is $u(c_1) + \delta u\big((1+r)(w-c_1)\big)$. We choose c_1 to maximize utility, obtaining $u'(c_1) = \delta(1+r)u'\big(((1+r)(w-c_1)\big)$. Now $u' = c^{-\sigma}$, so $c_1^{-\sigma} = (w-c_1)^{-\sigma}[\delta(1+r)^{1-\sigma}]$. Note that this formula works even if $\sigma = 1$. Let $\gamma_0 = [\delta(1+r)^{1-\sigma}]^{1/\sigma}$. Then $c_1 = (w-c_1)/\gamma_0$ and so $c_1 = w/[1+\gamma_0] = w/[1+\big(\delta(1+r)^{1-\sigma}\big)^{1/\sigma}]$.

b) Here $c_1^{-\sigma} = (w-c_1)^{-\sigma}\delta E[(1+r)^{1-\sigma}]$. We set $\gamma = [\delta E[(1+r)^{1-\sigma}]^{1/\sigma}$. Once again $c_1 = (w-c_1)/\gamma$ and so $c_1 = w/[1+\gamma]$.

Next we compute γ. Now

$$\begin{aligned}\gamma = \delta E[(1+r)^{1-\sigma}] &= \frac{\delta}{\bar{r}}\int_{\bar{r}/2}^{3\bar{r}/2}(1+r)^{1-\sigma}\,dr \\ &= \frac{\delta}{\bar{r}(2-\sigma)}\left[\big(1+\frac{3\bar{r}}{2}\big)^{2-\sigma} - \big(1+\frac{\bar{r}}{2}\big)^{2-\sigma}\right].\end{aligned}$$

for $\sigma \neq 2$ and

$$\gamma = \frac{\delta}{\bar{r}}\ln\frac{1+3\bar{r}/2}{1+\bar{r}/2}$$

for $\sigma = 2$.

c) When $\sigma = 1$, $\gamma_0 = \delta$ and $\gamma = \delta$, so there is no difference in consumption. When $\sigma = 2$, $\gamma_0 = (\delta/(1+\bar{r})^{1/2}$ while γ is given in part (b).

22.2.4 A firm with cost function $C(q) = 2q$ faces an uncertain price p. The firm chooses the production level q before the price is revealed.

a) Write expected profit in terms of expected price Ep and the chosen quantity of output q.

b) Suppose the firm maximizes expected profit. At what expected prices can the firm maximize expected profit?

c) When expected profit can be maximized, what quantity q maximizes expected profit?

d) Suppose the owner of the firm maximizes *expected utility* of income where $u(m) = \ln m$. The owner has \$10 of other income in addition to profit income. There is 50% chance that the price is \$1 and 50% chance that the price is \$3. Find the profit maximizing quantity q.

Answer:

a) Profit is $\pi(q) = pq - C(q) = pq - 2q$. Expected profit is $E\big(\pi(q)\big) = E(pq - 2q) = q(Ep - 2)$.

b) If $Ep > 2$, expected profit cannot be maximized because $\lim_{q\to\infty} E\pi = +\infty$. If $Ep = 2$, profit is also zero, which is the maximum. If $Ep < 2$, profit is negative for $q > 0$, and maximum profit is zero (at $q = 0$).

c) When $Ep = 2$, any $q \geq 0$ maximizes profit. When $Ep < 2$, $q = 0$ maximizes profit.

d) The owner's income is $10 + (p - 2)q$. The expected utility of income is

$$Eu = \frac{1}{2}\ln(10 + q) + \frac{1}{2}\ln(10 - q).$$

We differentiate with respect to q to find the first-order conditions.

$$\frac{1}{10 + q} - \frac{1}{10 - q} = 0.$$

It follows that expected profit is maximized when $q = 0$.

24.2.2 An infinitely-lived consumer has an endowment of $W = 100$ and preferences described by the utility function $U(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t c_t^{1/2}$ where $0 < \delta < 1$. The (present-value) price at time t is $p_t = 1/2^t$.

Consider the consumer's problem of maximizing utility $U(\mathbf{c})$ by choosing a sequence $\{c_t\}$ with $c_t \geq 0$ for all t and $\sum_{t=0}^{\infty} p_t c_t \leq 100$.

Suppose $2\delta^2 < 1$. Find the consumption stream that maximizes utility.

Answer: Letting $u(c) = c^{1/2}$, the first-order conditions are

$$\frac{\delta u'(c_{t+1})}{u'(c_t)} = \frac{\delta c_t^{1/2}}{c_{t+1}^{1/2}} = \frac{p_{t+1}}{p_t} = \frac{1}{2}.$$

Then $c_{t+1} = 4\delta^2 c_t$. Iterating, we find $c_t = (4\delta^2)^t c_0$.

The budget constraint is

$$100 \geq \sum_{t=0}^{\infty} p_t c_t = c_0 \sum_{t=0}^{\infty} (4\delta^2)^t / 2^t = c_0 \sum_{t=0}^{\infty} (2\delta^2)^t = \frac{c_0}{1 - 2\delta^2}.$$

Here the sum converges because $2\delta^2 < 1$. Utility is maximized by setting $c_0 = 100(1 - 2\delta^2)$, yielding the optimal path $c_t = 100(1 - 2\delta^2)(4\delta^2)^t$.

24.2.6 Suppose a consumer discounts at rate $\rho > 0$, so the discount factor is $\delta = (1 + \rho)^{-1}$. There is one good and its price at time t is $p_t = (1 + r)^{-t}$ where $r > 0$ is the interest rate. The consumer has felicity function $u(c_t) = c_t^{1-\sigma}/(1 - \sigma)$ where $\sigma > 0$ and $\sigma \neq 1$. The consumer has wealth W.

a) How fast does an optimal consumption path grow?

b) Let $\beta = 1 + g$ be the growth factor with g the growth rate. How must g and r be related in order for the sum $\sum_t p_t c_t$ to converge.

c) Are there cases where $\rho < 0$ makes sense?

Answer:

a) The first-order conditions are $p_{t+1}/p_t = \delta u'(c_{t+1})/u'(c_t)$. Using $\delta = (1 + \rho)^{-1}$, $u'(c) = c^{-\sigma}$, and $p_t = (1+r)^{-t}$, we obtain $c_{t+1}/c_t = \big((1+r)/(1+\rho)\big)^{1/\sigma}$. Consumption grows (or shrinks) by the factor $\beta = \big((1 + \rho)/(1 + r)\big)^{-1/\sigma}$ Since $\sigma > 0$, consumption grows when $r > \rho$ and shrinks when $r < \rho$.

b) Here $p_t c_t = c_0 \beta^t (1+r)^{-t}$. The sum $\sum_t p_t c_t$ converges if and only if $1+g = \beta < 1+r$, that is, if $g < r$.

c) The condition $\beta < 1+r$ means $(1+r)^{\sigma-1} < 1+\rho$. Negative values of ρ will obey this for certain values of r and σ. E.g., if $\sigma = 1/2$ and $r = 0.1$, we find any $\rho > 1-1/\sqrt{1.1}$ works, such as $\rho = -0.46$.

24.3.4 Consider the case of one-sector production with exogenous technical progress. The production function at time t is $f_t(k) = k^\gamma$ for $0 < \gamma < 1$. Suppose prices are $p_t = p(1+r)^{-t}$ where $r > 0$ is the interest rate. Find k_t for every t. Does k_t increase over time? If so, at what rate does it increase?

Answer: The first-order conditions are $\gamma p_{t+1} k_t^{\gamma-1} = p_t$. Using $p_t = p(1 + r)^{-t}$, this can be rewritten $\gamma k_t^{\gamma-1} = 1 + r$. Thus $k_t = [\gamma/(1 + r)]^{1/(1-\gamma)}$. The capital stock is constant.