Homework #9

24.5.2 Consider the following Ramsey problem. A consumer has utility $U(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t u(c_t)$ where $0 < \delta < 1$ and the felicity function is $u(c) = c^{1/2}$. The production function is $f(a) = \beta a$ where $\beta > 1$. The initial endowment is $b > 0$. Consider an optimal path with $c_t > 0$ for every t (in fact, all optimal paths obey this).

a) Use the Euler equations to find c_t in terms of initial consumption c_0.

b) Does consumption grow over time? If so, what is the growth factor?

c) What are the corresponding time-zero prices p_t? The interest rate at time t is given by $1 + r_t = p_t/p_{t+1}$. Calculate r_t.

Answer:

a) The Euler equations are

$$\delta f'(a_t) u'(c_{t+1}) = u'(c_t)$$

yielding

$$\delta \beta c_{t+1}^{-1/2} = c_t^{-1/2}.$$

It follows that $c_{t+1} = (\delta\beta)^2 c_t$. Then $c_t = (\delta\beta)^{2t} c_0$ by induction.

b) Consumption grows by the growth factor $(\delta\beta)^2$ when $\delta\beta > 1$, is constant if $\delta\beta = 1$, and shrinks if $\delta\beta < 1$, all of which are possible with $\delta < 1$ and $\beta > 1$.

c) The time-zero prices are given by $p_t = \partial U/\partial c_t = \delta^t u'(c_t) = \delta^t / 2c_t^{1/2}$.

Substituting $c_t = (\delta\beta)^{2t} c_0$, we obtain $p_t = 1/2\beta^t c_0^{1/2}$.

The equilibrium interest rate at time t is $r_t = p_t/p_{t+1} - 1 = \beta - 1 > 0$.

26.2.2 Consider a contingent goods exchange economy with two consumers, one good and two states. Endowments are $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 2)$. Both consumers have identical utility function $u(\mathbf{x}) = \pi \ln x_1 + (1 - \pi) \ln x_2$ where $0 < \pi < 1$ is the probability of state one.

a) Find all Arrow-Debreu equilibria.

b) How does the equilibrium price of good two relative to good one relate to the probability π?

Answer:

a) Let $\mathbf{p} \gg \mathbf{0}$ be the price vector. Note that zero price is not allowed in equilibrium due to the Cobb-Douglas preferences. Since preferences are Cobb-Douglas and identical for both consumers, so demand is $\mathbf{x}^i = m^i\big(\pi/p_1, (1-\pi)/p_2\big)$ where $m^i = \mathbf{p}\cdot\boldsymbol{\omega}^i$ is the income of consumer i. It follows that market demand is $m\big(\pi/p_1, (1-\pi)/p_2\big)$ where $m = m^1 + m^2$.

Market supply is $\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2 = (2,2)$. Setting demand equal to supply we find $\pi m/p_1 = (1-\pi)m/p_2 = 2$. Using good one as numéraire, $m = 2/\pi$, the equilibrium prices are $\mathbf{p} = \big(1, (1-\pi)/\pi\big)$. Individual incomes are $m^1 = 2$ and $m^2 = 2(1-\pi)/\pi$ and the corresponding allocation is $\mathbf{x}^1 = \big(2\pi, 2\pi\big)$ and $\mathbf{x}^2 = \big(2(1-\pi), 2(1-\pi)\big)$.

Any positive scalar multiple of $\mathbf{p}$ is also an equilibrium price vector with the same allocation.

b) From part (a), the relative price of good two is $p_2/p_1 = (1-\pi)/\pi$.

26.3.2 Consider a contingent goods exchange economy with one good and two states. Endowments are $\boldsymbol{\omega}^1 = (2,0)$ and $\boldsymbol{\omega}^2 = (0,5)$. Both consumers have utility $u(\mathbf{x}) = \ln x_1 + \ln x_2$.

a) Find the Arrow-Debreu equilibrium.

b) Are the consumers fully insured?

Answer:

a) With Cobb-Douglas utility, both prices must be strictly positive. We take good one as numéraire and set $\mathbf{p} = (1,p)$. Then demand is

$$\mathbf{x}^1 = (1, 1/p) \quad \text{and} \quad \mathbf{x}^2 = \frac{5p}{2}(1, 1/p).$$

Setting supply equal to demand for good one we obtain $1 + 5p/2 = 2$, so $p = 2/5$. Equilibrium prices are $\mathbf{p} = (1, 2/5)$. Then $\mathbf{x}^1 = (1, 5/2)$ and $\mathbf{x}^2 = (1, 5/2)$.

b) Here there is aggregate uncertainty and full insurance for both consumers is not possible. In fact, neither consumers is fully insured since $x_1^i \neq x_2^i$.

26.3.5 Suppose a contingent goods exchange economy has two goods, two states, and two consumers. Each consumer has utility function $U_i(\mathbf{x}) = \frac{1}{2}\ln(x_{11} + x_{21}) + \ln(x_{12} + x_{22})$. The endowments are $\big((1,2),(2,1)\big)$ and $\big((2,1),(1,2)\big)$.

a) Find all Arrow-Debreu equilibria.

b) Show that there is an Arrow-Debreu equilibrium with $\hat{\mathbf{x}}^1 = \boldsymbol{\omega}^1$ and $\hat{\mathbf{x}}^2 = \boldsymbol{\omega}^2$.

c) Why does neither consumer care that they are not fully insured?

Answer:

a) The marginal rate of substitution between goods one and two is one in either states. It follows that $p_{1,s} = p_{2,s}$ must be the same in equilibrium. We can then write

$\mathbf{p} = \big((p_1, p_1), (p_2, p_2)\big)$. It's clear that each consumer's utility depends only on the total amount consumed in each state, not how consumption is distributed between the goods. Let y_s^i denote the total consumed by consumer i in state s. We must maximize $v(\mathbf{y}^i) = \frac{1}{2}\ln y_1^i + \frac{\ln^i}{y\,2}$ under the budget constraint $p_1 y_1^i + p_2 y_2^i = 3(p_1 + p_2)$. Here $y_1^i = 3(p_1+p_2)/2p_1$ and $y_2^i = 3(p_1+p_2)/2p_2$. Market clearing requires $y_s^1 + y_s^2 = 6$.

Thus $\hat{\mathbf{p}} = \big((1,1),(1,1)\big)$, or a positive scalar multiple thereof. The consumption goods must be distributed so that $y_s^i = 3$. It follows that any $\mathbf{x}^i$ with $x_{11}^1 + x_{21}^1 = 3 = x_{12}^1 + x_{22}^1$, $0 \le x_{\ell s}^1 \le 3$ and $x_{\ell s}^2 = 3 - x_{\ell s}^1$ will be an equilibrium allocation.

b) Consumption by each consumer in each state is feasible (adds up to 3), so by part (a), it is an Arrow-Debreu equilibrium for $\hat{\mathbf{p}} = (1,1,1,1)$.

c) Although the consumers are not fully insured in the sense that the consumption vector is certain, they are fully insured in the weaker sense that the utility is the same in each state. This happens even though the consumers are risk averse (logarithmic utility).

26.4.4 Consider a contingent goods exchange economy with one good and two states. Endowments are $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 5)$. Both consumers have utility $u(\mathbf{x}) = \ln x_1 + \ln x_2$. (See also Exercise 26.3.2).

a) Find the Arrovian securities equilibrium.

b) Are the consumers fully insured?

c) Is the equilibrium Pareto optimal?

Answer:

a) Because we can normalize each spot market separately, we set $p_s = 1$ for both states. Then $\mathbf{p} = (1,1)$ and indirect utility is

$$v_i(\mathbf{z}^i) = \ln(\omega_1^i + z_1^i) + \ln(\omega_2^i + z_2^i).$$

We take asset one as numéraire for the asset market as set $\mathbf{q} = (1, q)$. Then $z_1^i + q z_2^i = 0$. We substitute $-z_1^i/q$ for z_2^i to obtain

$$v_i(\mathbf{z}_1^i) = \ln(\omega_1^i + z_1^i) + \ln(\omega_2^i - z_1^i/q).$$

The first-order conditions are

$$\frac{1}{\omega_1^i + z_1^i} = \frac{1}{q\omega_2^i - z_1^i}.$$

This yields asset demand $z_1^i = (q\omega_2^i - \omega_1^i)/2$. By market clearing, $0 = z_1^1 + z_1^2 = (q\omega_2 - \omega_1)/2 = (5q-2)/2$ It follows that $q = 2/5$, so $\mathbf{q} = (1, 2/5)$.

Now $\mathbf{z}^1 = (-1, +5/2)$ and $\mathbf{z}^2 = (+1, -5/2)$, so $\mathbf{x}^1 = (1, 5/2)$ and $\mathbf{x}^2 = (1, 5/2)$.

b) No, neither consumer is fully insured. They consume the different amounts in different states.

c) Yes, the equilibrium allocation is Pareto optimal because it is an interior allocation with $\mathrm{MRS}_{12}^1 = 5/2 = \mathrm{MRS}_{12}^2$.

26.4.6 Consider a contingent goods exchange economy with two goods and two states. Endowments are $\boldsymbol{\omega}^1 = (4, 2)$ and $\boldsymbol{\omega}^2 = (2, 4)$. Preferences are described by the utility functions $u_1(\mathbf{x}) = \frac{3}{8}\ln x_1 + \frac{5}{8}\ln x_2$ and $u_2(\mathbf{x}) = \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2$. Note that the consumers have different beliefs about the probabilities of the two states. Consumer one thinks that state two is more likely, consumer two thinks both states are equally likely.

a) Find the Arrovian securities equilibrium.

b) Find the market probabilities.

c) Are the consumers fully insured?

d) Is the equilibrium Pareto optimal?

Answer: This is an Arrovian securities version of Exercise 26.3.4

a) Because we can normalize each spot market separately, we set $p_s = 1$ for both states. Then $\mathbf{p} = (1, 1)$ and indirect utility is

$$v_1(\mathbf{z}^1) = \frac{3}{8}\ln(4 + z_1^1) + \frac{5}{8}\ln(2 + z_2^1)$$

and

$$v_2(\mathbf{z}^2) = \frac{1}{2}\ln(2 + z_1^1) + \frac{1}{2}\ln(4 + z_2^1)$$

Since both assets are valuable to the consumers ($\mathrm{MU}_1^i > 0$), asset prices must be positive. That allows us to use asset two as numéraire and ensures that the asset budget constraint will bind. We can write asset prices as $\mathbf{q} = (q, 1)$. The budget constraint can then be written $z_2^i = -qz_1^i$. This allows us to rewrite indirect utility in terms of z_1^i.

$$v_1(z_1^1) = \frac{3}{8}\ln(4 + z_1^1) + \frac{5}{8}\ln(2 - qz_1^1)$$

and

$$v_2(z_1^2) = \frac{1}{2}\ln(2 + z_1^1) + \frac{1}{2}\ln(4 - qz_1^2).$$

The first-order conditions are

$$\frac{3}{4+z_1^1} = \frac{5q}{2-qz_1^1}$$

and

$$\frac{1}{2+z_2^1} = \frac{q}{4-qz_1^2}.$$

Then asset demands are

$$z_1^1(q) = \frac{3-10q}{4q} \text{ and } z_1^2(q) = \frac{2-q}{q}.$$

Asset market clearing requires $11 - 14q = 0$, so $q = 11/14$. Thus $\mathbf{q} = (11/14, 1)$. It then follows that equilibrium portfolios are $\mathbf{z}^1 = (-17/11, 17/14)$ and $\mathbf{z}^2 = (17/11, -17/14)$. The equilibrium goods allocation is $\mathbf{x}^1 = (27/11, 45/14)$, $\mathbf{x}^2 = (39/11, 39/14)$.

b) The asset prices are $(11/14, 1)$. We divide by the sum to get market probabilities $\boldsymbol{\pi} = (11/25, 14/25)$.

c) The consumers are not fully insured since they consume different amounts in the two states.

d) The equilibrium is Pareto optimal. The marginal rates of substitution are $\text{MRS}_{12}^1 = \text{MRS}_{12}^2 = 11/14$.