Recursive Utility and Optimal Capital Accumulation II: Sensitivity and Duality Theory*

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Summary. This paper provides sensitivity and duality results for continuous-time optimal capital accumulation models where preferences belong to a class of recursive objectives. We combine the topology used by Becker, Boyd and Sung (1989) with a controllability condition to demonstrate that optimal paths are continuous with respect to changes in both the initial capital stock, and the rate of time preference. Under convexity and an interiority condition, we find the value function is differentiable, and derive a multiplier equation for the supporting prices. Finally, under some mild additional conditions, we show that supporting prices obeying the transversality and multiplier equations are both necessary and sufficient for an optimum.

1. Introduction

This paper provides sensitivity and duality results for continuous-time optimal capital accumulation models where the planner's preferences are represented by a recursive objective functional. Time preference is flexible. A previous paper (Becker, Boyd and Sung, 1989) established the existence of optimal paths by choice of an appropriate topology. In this paper, we combine the same topology with a controllability condition to demonstrate the sensitivity of the optimal path with respect to changes in the initial endowment of capital, and changes in the planner's rate of time preference. Under convexity and an interiority condition, we find the value function is differentiable, and derive a multiplier equation for the supporting prices. Finally, under some mild additional conditions, we show that supporting prices obeying the transversality and multiplier equations are necessary and sufficient for an optimum.

Following Ramsey (1928), optimal growth theory has generally relied on models incorporating a fixed pure rate of time preference (possibly equal to zero). In this context, the preference order of the planner is represented by a time-additive utility functional. This type of objective has been criticized by various authors, dating back to Fisher (1930), on the grounds that the pure rate of time preference should not be independent of the size and shape of the consumption profile. When preferences are time-additive, this independence is a direct consequence of a strong separability property. The marginal rate of substitution for consumption at any two dates is independent of the rest of the consumption stream (Hicks, 1965, gives an extended critique along these lines). Lucas and Stokey (1984) argued that the only basis for studying the time-additive case is its analytic tractability.

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Koopmans (1960) laid the foundation for ameliorating this deficiency of optimal growth theory by introducing "time-stationary" or "recursive" preferences. Uzawa (1968) extended Koopmans' discrete-time concept of recursive utility to continuous-time. Following Uzawa's lead, Epstein and Hynes (1983) proposed another formulation of continuous-time recursive utility. We consider a general multiple capital good model motivated by the Uzawa, Epstein and Hynes formulation of continuous-time recursive utility. We cast our problem in terms of a *reduced-form* model. This framework focuses on the vector of capital stocks and the corresponding net investment flows. These stock and flow variables jointly determine the consumption flow at each point in time. This formulation has much in common with the traditional fixed discount rate reduced-form model, which helps to clarify both the similarities and differences between the fixed discount rate and recursive models.

Brock (1971) initiated studies of the sensitivity of the optimal path to changes in the initial capital stock in a discrete-time one-sector setup. Nermuth (1978) considered the problem in a multi-commodity discrete-time model. Takekuma (1980) and Balder (1983) have provided continuous-time analogs of Nermuth's results. In their models, an optimal path varies continuously with respect to the initial stock. All of these authors stressed that their results might have implications for proving turnpike theorems. Therefore, sensitivity results can not be taken for granted.

Known turnpike results require additional assumptions. Epstein (1987a, b) has found both local and global turnpike results for the continuous time model under an appropriate normality condition. Benhabib, Jafarey and Nishimura (1988) and Benhabib, Majumdar and Nishimura (1987) have done the same for discrete time, again under a normality hypothesis. In contrast, the sensitivity analysis for discrete time is an immediate consequence of the existence theory (Boyd, 1990).

We study two types of sensitivity: Sensitivity to changes in initial stocks, and sensitivity to changes in the discount rate. We show that the feasible correspondence is closed and upper semicontinuous in the initial capital stock. Combined with a controllability condition, this yields continuity of the value function, even in non-convex, multisector models. As in Takekuma (1980), this implies the maximizer correspondence is closed and upper semicontinuous. Even when restricted to the additively separable case, our results strengthen those of Balder to permit non-convexities in both the reduced-form felicity and the technology. We also prove a sensitivity theorem for parametric changes in the discounting function entering the recursive objective.

Duality theory for recursive utility has remained virtually unexplored. Typically, a twice continuously differentiable value function is posited (Epstein and Hynes, 1983; Chang, 1987).[1] This is not required for the duality theory. Concavity and interiority suffice for the existence of supporting prices. Under additional assumptions, related to those of Benveniste and Scheinkman (1982), supporting prices that satisfy the transversality condition are necessary for a path to be optimal. As usual, the price equations are derived via a perturbation argument. The sufficiency argument requires an additional mild assumption. The demonstration of sufficiency is a bit unusual. Unlike the additively separable case, direct calculation of the change in utility on an alternative path cannot easily be carried through.

[1]Santos (1990) reports sufficient conditions for a twice continuously differentiable value function in the traditional time additive separable utility framework.

A major difficulty is that concavity of utility is not defined pointwise when preferences are not additively separable. However, the transversality condition can still be used to show that all directional derivatives are non-positive, which implies optimality.

The paper is organized as follows: Section Two sets up the model, Section Three investigates the properties of feasible paths, Section Four examines the sensitivity of the value function and optimal paths to changes in the initial stock, Section Five examines changes in the discount rate, Section Six contains the duality theory. Final comments are in Section Seven.

2. The Model

There are m capital goods in the general model economy. The *technology* is a measurable set $\Omega \subset \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$. We assume that the $t-$sections of Ω, $\{(k,y) : (t,k,y) \in \Omega\}$, are non-empty for every $t \geq 0$, and that $\{k : (t,k,y) \in \Omega \text{ for some } y \in \mathbb{R}^m\}$ is closed for every $t \geq 0$.[2] Define $\mathbf{D} = \{(t,k) : (t,k,y) \in \Omega \text{ for some } y\}$ and $G(t,k) = \{y : (t,k,y) \in \Omega\}$. The *investment correspondence* G thus has domain $\mathbf{D}$. Given capital stock k at time t, any $y \in G(t,k)$ can be accumulated as additional capital. The investment correspondence is allowed to vary with time. The programming problem is defined for a given felicity function L and discounting function R, both mapping Ω to $\mathbb{R}$, and an assigned initial capital stock $x \in \mathbf{X}$ where $\mathbf{X} \subset \mathbb{R}^m_+$ is the set of potential initial capital stocks. Formally, an *economy* is a quadruple (Ω, L, R, x) with $x \geq 0$ that satisfies the Technology and Felicity Conditions given below.

A capital accumulation *program*, k, is an absolutely continuous function from $\mathbb{R}_+$ to $\mathbb{R}^m_+$, denoted $k \in \mathbf{A}$.[3] It may be represented as the integral of its derivative $\dot{k}$ by $k(t) = k(0) + \int_0^t \dot{k}(s)\, ds$. Further, the derivative of k not only exists almost everywhere, but it is *locally integrable* – its integral over any compact set is finite. We denote this by $k \in L^1_{loc}$. Define the set of *attainable (admissible) programs from initial stock* x, $\mathbf{A}(x)$ by

$$\mathbf{A}(x) = \{k \in \mathbf{A} : \dot{k} \in G(t,k) \ a.e., 0 \leq k(0) \leq x\}.$$

Let $\mathbf{D}(t) = \{k : (t,k) \in \mathbf{D} \text{ for some } t\}$. We call the technology *convex* if $\mathbf{X}$ is convex and the section $\{(k,y) : y \in G(t,k)\}$ is convex for each t. In this case, $\mathbf{A}(x)$ is convex for each $x \in \mathbf{X}$, and the correspondence $x \to \mathbf{A}(x)$ is a concave process in the sense of Rockafellar (1967), i.e. $\lambda\mathbf{A}(x) + (1-\lambda)\mathbf{A}(x') \subset \mathbf{A}(\lambda x + (1-\lambda)x')$ for $0 \leq \lambda \leq 1$ whenever x, $x' \in \mathbf{X}$.

The *recursive objective functional*, I, is given by

$$I(k) = -\int_0^\infty L(t,k,\dot{k}) \exp\left(\int_0^t R(s,k,\dot{k})\, ds\right) dt,$$

[2]This assumption, as well as measurability of Ω should also be assumed in Becker, Boyd and Sung (1989). Balder (1990) gives a counter-example when they are omitted.

[3]More formally, a function k is *absolutely continuous* if for every T and $\epsilon > 0$, there is a δ with $\sum_{i=1}^m |k(t_i) - k(s_i)| < \epsilon$ whenever $0 \leq t_1 \leq s_1 \leq \cdots \leq t_m \leq s_m \leq T$ with $\sum_{i=1}^m |t_i - s_i| < \delta$. Any such function may be represented as the integral of its derivative. Conversely, any locally integrable function f has an absolutely continuous integral $k(t) = \int_0^t f(s)\, ds$.

and the *programming problem*, $P(x)$, is defined by

$$P(x) : J(x) = \sup\{I(k) : k \in \mathbf{A}(x)\}.$$

Here J is the *value function*. An admissible program k^* is an *optimal solution to* $P(x)$ if $k^* \in \mathbf{A}(x)$ and $I(k^*) = J(x)$. Similarly, we define $J(x,T)$ as the supremum of $-\int_T^\infty L(t,k,\dot{k}) \exp(\int_T^t R(s,k,\dot{k})\,ds)\,dt$ over the set of feasible paths with initial condition $k(T) = x$.

Define a map $k \to \mathcal{M}k$ by the formula

$$\mathcal{M}k(t) = -L(t,k(t),\dot{k}(t)) \exp \int_0^t R(s,k(s),\dot{k}(s))\,ds.$$

This map is *concave* if for each $\alpha \in [0,1]$, for any k, k' with $k^\alpha = \alpha k + (1-\alpha)k'$, it follows that

$$\mathcal{M}k^\alpha(t) \geq \alpha \mathcal{M}k(t) + (1-\alpha)\mathcal{M}k'(t)$$

holds for almost every t. The objective functional is *concave* when the map $\mathcal{M}$ is concave. If the technology is also convex, we say the *problem* or the *economy is concave*. A concave problem yields a value function that is concave and thus continuous on the relative interior of its domain $\mathbf{X}$.

Technology Conditions.

i) $G(t,k)$ is compact and convex for each (t,k) and $k \to G(t,k)$ is upper semicontinuous for each t.
ii) For all $x \in \mathbf{X}$, there is a $\mu_x \in L^1_{loc}$ such that $|\dot{k}| \leq \mu_x$ a.e. whenever $k \in \mathbf{A}(x)$.
iii) $\mathbf{A}(0) \neq \varnothing$.

Note that the non-triviality condition (iii) implies $\mathbf{A}(x) \neq \varnothing$ for all $x \in \mathbf{X}$. In many economic applications inaction is always possible by virtue of free disposal, thus $0 \in G(t,k)$. Condition (iii) immediately follows since $k(t) = x$ is feasible.

Felicity Conditions.

i) $L\colon \Omega \to \mathbb{R}$ and $R\colon \Omega \to \mathbb{R}$ are continuous on Ω and convex in $y \in G(t,k)$ for each fixed $(t,k) \in \mathbf{D}$.
ii) $L \geq 0$ and there exists a non-positive function $\rho_x \in L^1_{loc}$ such that $R(t,k(t),\dot{k}(t)) \geq \rho_x(t)$ for all $k \in \mathbf{A}(x)$.
iii) There is a program $k \in \mathbf{A}(x)$ such that $I(k) > -\infty$.

We will refer to L as the *felicity function* and R as the *discounting function*. Becker, Boyd and Sung (1989) found that optimal programs exist whenever the felicity and technology conditions are satisfied. Carlson (1990) contains an existence result for a different class of similar problems. Both results appear as special cases in Balder (1990).

Henceforth, we denote the amount of discounting along a path k from time s to time t by $D(t,s) = \exp(\int_s^t R(\tau,k(\tau),\dot{k}(\tau))\,d\tau)$ and discounting from time 0 by $D(t) = D(t,0)$. Thus $I(k) = -\int_0^\infty LD\,dt$.

2.1. The One-Sector Model

One technology satisfying the technology conditions is the one-sector growth model.[4] Denote consumption by $c(t)$, capital by $k(t)$ and net investment by $\dot{k}(t)$. Assume the gross production function $f\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ is continuous in (k, y) and increasing in k. Further, there is a continuous function $\tau(t)$ with $0 \leq f(k,t) \leq \tau(t)(1+k)$ for $k \geq 0$. This growth condition is automatically satisfied if f is concave in k. Capital depreciates at rate $\beta \in [0,1]$, yielding a net production function $g(k,t) = f(k,t) - \beta k$. Consumption c is given by $c(t) = g(k,t) - \dot{k}(t) = f(k(t),t) - \beta k(t) - \dot{k}(t)$. Define $G(t,k) = [-\beta k, g(k,t)]$ and $\Omega = \{(t,k,y) \in \mathbb{R}_+ \times \mathbb{R}^m_+ \times \mathbb{R}^m : y \in G(t,k)\}$. This defines the *standard technology*.

Without loss of generality, we may take $\beta = 0$. This is accomplished by using undepreciated capital defined by $K(t) = e^{\beta t}k(t)$. Undepreciated capital obeys $0 \leq K \leq F(K,t)$ where $F(K,t) = e^{\beta t}f(e^{-\beta t}K, t)$ is undepreciated gross output. Since F obeys the growth condition whenever f does, and since $K \to e^{-\beta t}K$ is continuous in $\mathcal{C}$, it is enough to consider the no-depreciation case ($\beta = 0$).

The one-sector framework encompasses numerous technologies. These range from affine production functions $f(k) = rk + w$ with $r > \beta$ and $w \geq 0$ (Uzawa, 1968; Nairay, 1984) to non-convex technologies. One such example would be $f(k) = \pi/4 + \arctan(k-1)$. Technologies with stock non-convexities are admissible; flow non-convexities are inadmissible.

Epstein (1983, 1987b) has introduced generalizations of Uzawa's (1968) recursive utility function. A felicity function, u, and a discounting function, v, are defined in terms of consumption c. We assume the felicity function is negative, continuous, concave and increasing with $u(0) > -\infty$ and the discounting function is continuous, concave and increasing with $v(c) \geq v(0) = \rho > 0$ for all $c \geq 0$. Now take $L(t,k,y) = -u(f(k,t) - y - \beta k)$ and $R(t,k,y) = -v(f(k,t) - y - \beta k)$. These objectives are a continuous-time version of Koopmans' (1960) recursive preferences. These objectives are bounded below by $I(0) = u(0)/\rho > -\infty$.

Examples of admissible discounting functions include: $v(c) = 1 + \arctan c^\alpha$ for $0 < \alpha \leq 1$; $v(c) = 1 + \operatorname{arcsec}(1+c)$, and $v(c) = 2 - e^{-c}$. The first two are bounded from above by $1 + \pi/2$, the last by 2. In all of these cases $\rho = 1$. Other examples of discounting functions may be unbounded above, for example, $v(c) = 1 + \log(1+c)$ and $v(c) = 1 + c^\alpha$, $0 < \alpha \leq 1$. When u is a constant, we obtain the Epstein-Hynes (1983) utility function. We will restrict our attention to felicity and discounting functions that satisfy the Felicity Conditions. However, Nairay (1984) introduces a transformation that can recast certain other types of felicity functions, in particular, those considered by Uzawa (1968), into our framework.

Production functions and objectives meeting the above conditions define the *standard recursive one sector model* (f, β, u, v, x).

3. Properties of Feasible Sets

Two equivalent modes of convergence are particularly important for our feasible set. Both are defined on the space $\mathbf{A}$ of absolutely continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^m$. The $\mathcal{C}$ topology is the topology of uniform convergence on compact subsets (the compact-open topology) and is defined by the family of norms $\|f\|_{\infty,T} = \sup\{|f(t)| : 0 \leq t \leq T\}$ for

[4]See Becker, Boyd and Sung (1989) for details on this.

$T = 1, 2, \ldots$, where $|\cdot|$ is any of the equivalent Euclidean norms on $\mathbb{R}^m$. Under these norms, $\mathcal{C}$ is a complete, countably-normed space and thus a Fréchet space.

The second topology is obtained by focusing on the flows rather than stocks. When f is absolutely continuous, its derivative is locally integrable. Consider $(L^1_{loc})' = \{f \in L^\infty : f$ has compact support$\}$. For $(\varphi_0, \varphi_1) \in \mathbb{R} \oplus (L^1_{loc})'$, define $\langle \varphi, f \rangle = \varphi_0 f(0) + \int_0^\infty \varphi_1(s) f(s)\, ds$ recalling that φ_1 has compact support. This duality defines a weak topology on $\mathbf{A}$. Curiously, although any sequence that converges in the weak topology on $\mathbf{A}$ converges in the $\mathcal{C}$ topology, weakly convergent *nets* need not converge in the $\mathcal{C}$ topology. We will be working with a subspace $\mathbf{F}(\mu)$, defined by a growth condition, where the two topologies actually are equivalent. For each $\mu \in L^1_{loc}$ with $\mu \geq 0$, define the subspace $\mathbf{F}(\mu)$ by $\mathbf{F}(\mu) = \{k \in \mathbf{A} : |\dot{k}| \leq \mu\}$. Becker, Boyd and Sung (1989) investigated these topologies in detail. They found:

Proposition 1. *On $\mathbf{F}(\mu)$, the $\mathcal{C}$ and weak $\mathbf{A}$ topologies are equivalent. Further, $\mathbf{F}(\mu)$ is compact and this topology is metrizable.*

Proposition 2. *Suppose the technology and felicity conditions are satisfied and $k_n \to k$ with $\dot{k}_n \in G(k_n, t)$ a.e. Then $\dot{k} \in G(k, t)$ a.e.*

3.1. Sensitivity of the Feasible Set

One easy, but important, application of these techniques is to establish upper semicontinuity of the feasible correspondence.

Lemma 1. *When the technology conditions are satisfied, $x \to \mathbf{A}(x)$ is a closed, upper semicontinuous correspondence.*

Proof. Let $x_n \to x$. For n large enough, all of the $\mathbf{A}(x_n)$ are contained in the compact set $\mathbf{A}(x + 1)$. It is enough to show the correspondences are closed. Let $k_n \in \mathbf{A}(x_n)$ with $k_n \to k$. As $k_n(0) \to k(0)$, $0 \leq k(0) \leq x$. Further, since $k_n(t) \to k(t)$ and $\dot{k}_n \in G(t, k_n)$ a.e., $k \in G(t, k)$ a.e. by Proposition 2. Thus $k \in \mathbf{A}(x)$. □

In general, we shall also need the following strong controllability (reachability) condition. This condition says that if we start near a path k, we can catch up to it.

Strong Controllability at k. *Let $x = k(0)$. There exists a T such that for all $\epsilon > 0$ there is a δ such that whenever $|x - y| < \delta$, there is a $k' \in \mathbf{A}(y)$ with $|(k(t), \dot{k}(t)) - (k'(t), \dot{k}'(t))| < \epsilon$ for all t and $k(t) = k'(t)$ for $t \geq T$.*

Generally speaking, we expect the optimal path to be strongly controllable. Intuitively, some consumption must occur on the optimal path. Thus by sacrificing consumption, we can catch up to it even if our initial stock is lower. We can easily see this when the path is interior at the start. Consider Benveniste and Scheinkman's (1979) interiority condition.

Interiority at k. *There are $\epsilon, T > 0$ such that $\dot{z}(t) \in G(t, z(t))$ whenever* $\max\{\|k - z\|_{\infty,T}, \|\dot{k} - \dot{z}\|_{\infty,T}\} < \epsilon$.

Lemma 2. *Whenever interiority holds at k, k is strongly controllable.*

Proof. Let $\epsilon > 0$. Perturb the path k as follows. Define $k(t|y) = k(t) + (1 - t/T)(y - x)$ for $0 \leq t \leq T$ and $k(t|y) = k(t)$ otherwise. Then $\dot{k}(t|y) = \dot{k}(t) - (y - x)/T$ and $k(0|y) = k(0) + (y - x) = y$. Choose any $\delta < \epsilon, \epsilon T$. Then $k(t|y)$ is feasible from x by the interiority condition and is the desired k'. □

4. Changes in Initial Stocks

When the value function is continuous, these techniques can be used for sensitivity analysis. There are various sensitivity questions to consider. In this section, we study the effect of changes in the initial stock on the the value function and optimal paths.

We first consider the case where the value function is continuous. This analysis is reminiscent of Takekuma's (1980) and Balder's (1983) treatment of the additively separable case. Although concavity is the easiest way to obtain continuity of the value function, it is not required. Various controllability conditions yield a continuous value function. In fact, as Balder demonstrates (1983, Appendix B), convexity of the technology is a type of controllability condition.

If the economy is concave, the value function is concave and so continuous. Thus the maximizer correspondence is upper semicontinuous. When the objective is strictly concave, the maximizer is unique, and so a continuous function.

Before proceeding, we require information about continuity of the objective. Becker, Boyd and Sung (1989) prove the recursive objective functional is upper semicontinuous.

Upper Semicontinuity Theorem. *Suppose the Felicity and Technology Conditions are satisfied and $k_n \to k$ weakly in $\mathbf{A}(x)$. Then $\limsup_{n\to\infty} I(k_n) \leq I(k)$.*

4.1. Continuity of the Value Function

There are two ways to obtain continuity of the value function. The first requires the problem be concave. The value function is then concave, and hence continuous. The second does not require a concave economy, but uses strong controllability instead.

Theorem (Continuity of the Value Function). *Suppose (Ω, L, R, x) is an economy and either the optimal path from x is strongly controllable, or the economy is concave with x in the relative interior of $\mathbf{X}$. Then the value function is continuous at x.*

Proof. First suppose the economy is concave. As I is concave and the technology is convex, standard arguments show the value function is a concave function on its domain $\mathbf{X}$. It immediately follows that the value function is continuous on the relative interior of its domain.

Suppose strong controllability holds and consider $y_n = x - u/n$ and $z_n = x + u/n$ where $u = (1, \ldots, 1)$. For n large enough, $k(t|y_n)$ is feasible from y_n. Further, $I(k(t|y_n)) \leq J(y_n) \leq J(x) \leq J(z_n)$ by the monotonicity of the feasible set. It suffices to show $\lim_{n\to\infty} J(z_n) = \lim_{n\to\infty} I(k(t|y_n)) = J(x)$.

First we consider convergence from above. Let k_n be optimal from z_n. Take any subsequence that converges. Use k_n to denote the subsequence and let its limit be k_0. We know $k_0 \in \mathbf{A}(x)$ since the correspondence $\mathbf{A}(x)$ is closed (Lemma 1). By the Upper Semicontinuity Theorem, $\lim_{n\to\infty} J(z_n) = \limsup_{n\to\infty} I(k_n) \leq I(k_0) \leq J(x)$. But as J is clearly increasing, $J(z_n) \geq J(x)$, and so $\lim_{n\to\infty} J(z_n) = J(x)$.

Now consider convergence from below. For n large let k_n be given by the strong controllability condition. Define $L_n = L(t, k_n(t), \dot{k}_n(t))$, $L = L(t, k(t), \dot{k}(t))$ and define D_n and D similarly.

Since $k_n(t) = k(t)$ for $t \geq T$,

$$|I(k_n) - I(k)| = \int_0^T [L_n D_n - LD]\, dt + [D_n(T) - D(T)]J(T, k(T)).$$

For any ϵ, $|(k, \dot{k}) - (k_n, \dot{k}_n)| < \epsilon$ for n sufficiently large. As R and L are continuous in k, both terms converge to zero as $n \to \infty$. Thus $I(k_n) \to I(k) = J(k)$, as required. □

4.2. Sensitivity of Maximizers

We can now combine the continuity of the value function with the upper semicontinuity of the feasible correspondence to show that the maximizer correspondence is upper semicontinuous.

Stock Sensitivity Theorem. *Suppose (Ω, L, R, x) is an economy with J continuous. Then the maximizer correspondence $x \to \mathbf{m}(x) = \{k \in \mathbf{A}(x) : I(k) = J(x)\}$ is closed and upper semicontinuous.*

Proof. Once again, it suffices to show $\mathbf{m}$ is closed since compactness will yield upper semicontinuity. Let $x_n \to x$ and $k_n \to k$ with $k_n \in \mathbf{m}(x_n)$. By Lemma 2, $k \in \mathbf{A}(x)$. The Upper Semicontinuity Theorem shows $J(x) = \limsup_{n\to\infty} J(x_n) = \limsup_{n\to\infty} I(k_n) \leq I(k)$. But $I(k) \leq J(x)$, thus $I(k) = J(x)$ and $k \in \mathbf{m}(x)$. □

Corollary 1. *Suppose (Ω, L, R, x) is an economy and either the optimal path from x is strongly controllable, or the economy is concave with x in the relative interior of $\mathbf{X}$. Then the maximizer correspondence is closed and upper semicontinuous.*

Of course, if the objective is strictly concave, the maximizer will be unique, and the maximizer correspondence a continuous function.

4.3. Application: The One-Sector Model

Continuity of the value function and closedness of the maximizer correspondence always obtain in the standard one-sector model. Any path with non-zero consumption is strongly controllable. Here the intuition carries through exactly. A small deficiency in capital can be made up by remaining on the path of pure accumulation a bit longer.

Controllability Lemma. *Suppose $c = f(k, t) - k$ is a feasible path in the standard model (f, β, u, v, x). Then k is strongly controllable.*

Proof. If $y > x$, k is feasible, so $k' = k$ will do. Thus we assume $y < x$. Set $T = 1 + \inf(\text{ess-}$

$\operatorname{supp} c$)[5] and let $\epsilon > 0$ be arbitrary. Since $c \in L^1_{loc}$, we may use the Carathéodory Existence Theorem to obtain an absolutely continuous solution to $\dot{k} = f(k,t) - c(t)$, $k(0) = y$ for any y (Coddington and Levinson, 1955, pg. 43). Denote this solution by $k(t|y)$. Since f is increasing, the solution is unique and increasing in y as well as t. Thus $k(t|y) \leq k(t|x)$ for all t. Of course, $k(t|x) = k(t)$.

Next, we modify $k(t|y)$ so that it catches up to k by converting some of the available consumption (up to ϵ) to investment. In this way, we can catch up by an amount $\varphi(t) = \int_{T-1}^{t} \max\{\epsilon, c(t)\}\, dt$ by time t. Choose δ_0 so that $|f(k,t) - f(k^*,t)| < \epsilon$ for $0 \leq t \leq T$ whenever $|k - k^*| < \delta_0$. Then choose δ with $k(T|x) - k(T|y) \leq \min\{\varphi(T), \delta_0\}$ whenever $|y - x| < \delta$ (Coddington and Levinson, pg. 58). For $0 \leq t \leq T-1$, consumption is zero and $|\dot{k}(t) - \dot{k}(t|y)| = |f(k(t)) - f(k(t|y))| \leq \epsilon$. Now fix y with $|y - x| < \delta$. Since φ is continuous with $0 = \varphi(T-1) \leq k(T-1|x) - k(T-1|y)$ and $\varphi(T) \geq k(T|x) - k(T|y)$, there is a largest δ_1, $0 \leq \delta_1 \leq 1$ with $\varphi(T - \delta_1) = k(T - \delta_1|x) - k(T - \delta_1|y)$. Define $k'(t)$ by $k'(0) = y$ and

$$\dot{k}'(t) = \begin{cases} \dot{k}(t|y) & 0 \leq t \leq T-1 \\ \dot{k}(t|y) + \max\{\epsilon, c(t)\} & T-1 \leq t \leq T - \delta_1 \\ \dot{k}(t) & T - \delta_1 < t. \end{cases}$$

Since $k'(T - \delta_1) = k(T - \delta_1|y) + \varphi(T - \delta_1) = k(T - \delta_1)$, $k'(t) = k(t)$ for $t \geq T \geq T - \delta_1$. For $0 \leq t \leq T - \delta_1$, $k(t|y) \leq k'(t) \leq k(t)$ on $[0, T - \delta_1]$. Combining these shows $|k(t) - k'(t)| \leq \epsilon$ for all t. Now $|\dot{k}' - k| \leq |f(k,t) - f(k',t)| < \epsilon$ on $[0, T-1]$, and $|\dot{k} - \dot{k}'| = |\dot{k}(t) - \max\{c, \epsilon\} - \dot{k}(t|y)| \leq 2\epsilon$ on $[T-1, T-\delta_1]$. All that remains is to note that k' is feasible since $\dot{k}'(t) = \dot{k}(t|y) + \max\{\epsilon, c\} \leq f(k(t|y), t) \leq f(k'(t), t)$ on $[T-1, T-\delta_1]$. □

Since consumption is not indentically zero on the optimal path for any $x > 0$, we have:

Corollary 2. *The value function for the standard one-sector model is continuous, and the maximizer correspondence is closed and upper semicontinuous.*

5. Parametric Changes in Discounting

This section addresses the effect of parametric changes of the felicity function on the value function and optimal paths. The theorem examines continuous parametric changes in the discounting function R for both convex and non-convex technologies. Consider a family R_α of discounting functions with α in some parameter space. One important family of this type is given by $R_\alpha = -(\alpha + v(c))$ where v is an Epstein-Hynes discounting function. This parametric change adjusts the rate of impatience and is analogous to varying the discount factor in the additively separable model.

Discounting Sensitivity Theorem. *Given parametrized economies (Ω, L, R_α, x), suppose $|R_\alpha(t, k(t), \dot{k}(t)) - R_0(t, k(t), \dot{k}(t))| \to 0$ uniformly in $(t,k) \in \mathbb{R}_+ \times \mathbf{A}(x)$ as $\alpha \to \alpha_0$ and $R_\alpha \leq \rho < 0$ for α near α_0 with $Le^{\rho t}$ uniformly integrable. Then $J_\alpha(x) \to J_0(x)$.*

[5]The *essential support* of c is defined by ess-supp $c = \{t$:every neighborhood of t has a subset of positive measure where $c \neq 0\}$.

Proof. For $k \in \mathbf{A}(x)$, $R_0 \leq \rho$, thus $\int_0^t R_0(t,k,\dot{k}) \leq \rho t$. Hence

$$\begin{aligned} |I_\alpha(k) - I_0(k)| &\leq \int_0^\infty Le^{\rho t}|1 - \exp(\textstyle\int_0^t (R_\alpha - R_0)\,ds)|\,dt \\ &\leq \int_0^\infty Le^{\rho t}(e^{\epsilon t} - e^{-\epsilon t})\,dt \end{aligned}$$

for α near α_0. Thus $I_\alpha(k) \to I(k)$ uniformly on $\mathbf{A}(x)$.

Now take k_α and k_0 such that $I_\alpha(k_\alpha) = J_\alpha(x)$ and $I_0(k_0) = J_0(x)$. Note $I_\alpha(k_0) \leq J_\alpha(x)$. Thus $\lim I_\alpha(k_0) = I_0(k_0) \leq \liminf J_\alpha(x)$. Also, $I_0(k_\alpha) \leq I_0(k_0) = J_0(x)$ and $|I_\alpha(k_\alpha) - I_0(k_\alpha)| \to 0$ uniformly, so $\limsup J_\alpha(x) = \limsup I_\alpha(k_\alpha) \leq J_0(x)$. Therefore $\lim J_\alpha(x) = J_0(x)$. □

The proof of the Stock Sensitivity Theorem now yields:

Corollary 3. *Under the conditions of the Stock Sensitivity Theorem, $\alpha \to m_\alpha(x)$ is closed. Further, if I is strictly concave and the technology is convex, $\alpha \to m_\alpha(x)$ is a continuous function.*

Proof. Let $\alpha_n \to \alpha_0$ and let k_n be optimal given α_n with $k_n \to k_0$. As $\mathbf{A}(x)$ is closed, k_0 is feasible. But as $I_n(k_n) = J_n(x) \to J_0(x)$ and $I_n(k) \to I(k_0)$, k_0 is a maximizer for I_0. Thus $\alpha \to m_\alpha(x)$ is closed.

If I is strictly concave and the technology is convex, the maximizer is unique. Therefore $\alpha \to m_\alpha(x)$ is a continuous function. □

5.1. Application: Uncertain Lifetimes

One situation where the Discounting Sensitivity Theorem applies is under an uncertain lifetime. Consider an Epstein-Hynes discounting function and suppose the probability of death is given by a continuous density $p(t)$. The probability of dying by time s is $P(s) = \int_0^s p(\tau)\,d\tau$. Treating death as a zero consumption state (Yaari, 1965; Blanchard, 1985; Chang, 1986; Boukas, Haurie and Michel, 1990), expected utility $EU(C)$ is then

$$-\int_0^\infty p(s)\,ds\left[\int_0^s \exp\left(-\textstyle\int_0^t v(c)\,d\tau\right)\,dt + \int_0^\infty \exp\left(-\textstyle\int_0^s v(c)\,d\tau - \int_s^t v(0)\,d\tau\right)\,dt\right]$$

Integrating the first term by parts and integrating the second term yields, after some rearrangement,

$$EU(C) = -\int_0^\infty [1 - P(s) + p(s)/v(0)]\exp(-\textstyle\int_0^s v(c)\,d\tau)\,ds$$

where $P(s) = \int_0^t p(\tau)\,d\tau$ is the probability of dying by time s. Letting $Q(s) = -\log[1 - P(s) + p(s)/v(0)]$ and $q = \dot{Q}$ yields the time-varying felicity function $q(s) + v(c)$. Thus

$$EU(C) = -[1 + p(0)/v(0)]\int_0^\infty \exp\left(-\textstyle\int_0^t v(c) + q(s)\,ds\right)\,dt.$$

If the probability of death is given by the Poisson density $p(s) = \lambda e^{-\lambda s}$, $q(s) = \lambda$ and $R_\lambda = -(\lambda + v(c))$. The Discounting Sensitivity Theorem then applies.

6. The Duality Theory

When concavity of the value function is combined with interiority, more can be said. The value function is differentiable, and its derivative is an absolutely continuous function that obeys a type of multiplier equation.

6.1. Differentiability of the Value Function

As before, we consider a perturbation of the optimal path (Benveniste and Scheinkman, 1979). Given $k(t)$ optimal from x_0, define $k(t|x) = k(t) + (1 - t/h)(x - x_0)$ for $0 \leq t \leq h$ and $k(t|x) = k(t)$ otherwise. Note that $k(0|x) = k(0) + (x - x_0) = x$.

Differentiability Theorem. *Suppose the Technology, Felicity and Interiority Conditions are satisfied for a concave economy with L and R continuously differentiable. Then the value function is differentiable at x_0 with $J'(x_0) = L_3(0, k(0), \dot{k}(0)) - R_3(0, k(0), \dot{k}(0))J(k)$ where k is the optimal path from x_0.*

Proof. The technology and felicity conditions insure k exists. Let $h \leq T$ with $h \leq 1$. By the interiority condition, $k(t|x)$ is feasible when $|x - x_0| < \epsilon h \leq \epsilon$. Consider $I_h(x) = I(k(t|x))$. Clearly $I_h(x_0) = J(x_0)$ and $I_h(x) \leq J(x)$. Since both J and I_h are concave, they have the same derivative at x_0.

$$dI_h(x_0)/dx = -\int_0^h [L_2(1 - t/h) - L_3/h]D(t)\,dt \\ -\int_0^\infty L\left[\int_0^m [R_2(1 - t/h) - R_3/h]\,ds\right]\,D(t)\,dt.$$

Where $m = \min(t, h)$ and all of the R and L terms are evaluated at $k(t)$. All of the action takes place on $[0, h]$ since $k(t) = k(t|x)$ for $t > h$. As R and L are C^1, we obtain $J'(x_0) = L_3(0, k(0), \dot{k}(0)) - R_3(0, k(0), \dot{k}(0))J(k)$ by letting h approach zero. □

When utility is additively separable with $R = -\rho$ and $L(t, k, k^{\cdot}) = -u(f(k) - \beta k - \dot{k})$, this reduces to $J'(x_0) = u'(c(0))$ as found by Benveniste and Scheinkman (1979). Epstein and Hynes (1983) investigate the case, $L = 1$ and $R = -v(c) = -v(f(k) - \beta k - \dot{k})$ in detail. Using the Volterra derivative at $T = 0$ (Volterra, 1959; Wan, 1970; Ryder and Heal, 1973), they obtain $J'(x_0) = -v'(c(0))J(k)$. Chang (1987) uses a simpler dynamic programming argument to show $J'(k(t)) = u'(c(t)) - v'(c(t))J(k(t))$ for the standard one-sector model $(f, 0, u, v, x)$.

Essentially, our method uses the Volterra derivative. The perturbation has only one sign, magnitude less than ϵ and takes place over a time interval of length h. Thus, it is precisely the type of perturbation used to obtain the Volterra derivative, and our limiting technique yields the Volterra derivative along the optimal path.

Define present value prices by $p(s) = D(s)[L_3(s, k(s), \dot{k}(s)) - R_3(s, k(s), \dot{k}(s))J(k(s), s)]$ and current value prices $q(s) = L_3(s, k(s), \dot{k}(s)) - R_3(s, k(s), \dot{k}(s))J(k(s), s)$. Note that $J(k(s), s) = -\int_s^\infty LD(t, s)\,dt$ with D and L evaluated along the optimal path from $(s, k(s))$. Clearly k is optimal from $(s, k(s))$ by the Principle of Optimality. By the Differentiability Theorem, $p(s) = D(s)J_1(k(s), s)$

Apart from the discount factor, two terms contribute to present value prices. The first (L_3) gives the marginal increase in current felicity. The second ($R_3 J$) gives the marginal change in future utility, due to the effect of current consumption on future discounting.

6.2. The Price Equation

By examining another perturbation, we can derive expressions for $\dot{p}$ and $\dot{q}$. Consider the path $x(t) = k(t) + x - k(\sigma)$ for $\sigma \leq t \leq \tau$ and $x(t)$ optimal from $k(\tau) + x - k(\sigma)$ thereafter.

Theorem (Multiplier Equation). *Suppose the Technology, Felicity and Interiority Conditions are satisfied for a concave economy with L and R continuously differentiable. Then the associated prices $p(t)$ and $q(t)$ are absolutely continuous with $\dot{p}(t) = D(t)[L_2 - R_2 J(k(t), t)]$.*

Proof. Let $g(x) = -\int_\sigma^t LD(s,\sigma)\,ds + D(t,\sigma)J(x(t),t)$. Note that g is concave, $g(x) \leq J(x,\sigma)$ and $g(k(\sigma)) = J(k(\sigma),\sigma)$. Thus $g'(k(\sigma)) = J_1(k(\sigma),\sigma)$. We can rewrite this as

$$J_1(k(\sigma),\sigma) = -\int_\sigma^t \left[L_2 + L\left(\int_\sigma^s R_2\,d\tau\right)\right] D(s,\sigma)\,ds + \left(\int_\sigma^t R_2\,d\tau\right) D(t,\sigma)J(k(t),t) + D(t,\sigma)J_1(k(t),t).$$

Multiplying by $D(\sigma)$ and using $D(t)J_1(k(t),t) = p(t)$ gives

$$p(t) = p(k(\sigma)) + \int_\sigma^t \left[L_2 + L\left(\int_\sigma^s R_2\,d\tau\right)\right] D(s)\,ds - \left(\int_\sigma^t R_2\,d\tau\right) D(t)J(k(t),t).$$

It follows that p is absolutely continuous. Furthermore, since $D(t)J(k(t),t)$ has derivative LD, $\dot{p}(t) = D(t)[L_2 - R_2 J(k(t),t)]$. Thus current value prices obey $\dot{q}(t) = -Rq(t) + L_2 - R_2 J(k(t),t)$. □

6.3. The Transversality Condition

The final result of this section examines when the existence of supporting prices obeying the *transversality condition* (TVC) that $\lim_{t\to\infty} p(t)k(t) = 0$ is necessary and sufficient for an optimum. As in Benveniste and Scheinkman (1982), this requires somewhat more stringent conditions on felicity and technology.

Theorem 1. *Suppose the Technology, Felicity and Interiority Conditions are satisfied for a concave economy with L and R continuously differentiable. Further, suppose $0 = \mathbf{A}(0)$, $I(0)$ is finite and $D(t)J(0,t) \to 0$. If k is optimal, the supporting prices $p(t) = D(t)[L_3 - R_3 J(k(t),t)]$ satisfy the multiplier equation and transversality condition.*

Proof. By the Multiplier Theorem, we need only show that the transversality condition is satisfied. Since J is concave,

$$q(t)(z - k(t)) \geq J(z,t) - J(k(t),t). \tag{1}$$

As $0 \in \mathbf{A}(0)$, $k/2 \in \mathbf{A}(x/2)$ by concavity of the correspondence $x \to \mathbf{A}(x)$. Further, $I(k/2) > -\infty$ by concavity of the objective. Let $z = k(t)/2$ and multiply equation (1) by

$D(t)$ to obtain

$$\begin{aligned} 0 \geq -p(t)k(t)/2 &\geq D(t)[J(k(t)/2,t) - J(k(t),t)] \\ &\geq D(t)[J(0,t) - J(k(t)/2,t)] \geq D(t)J(0,t) \end{aligned}$$

where the last inequality follows since $J \leq 0$. By assumption, $D(t)J(0,t)$ converges to zero, establishing the transversality condition. □

The requirement that $D(t)J(0,t) \to 0$ is actually quite mild. For example, consider a stationary problem with $R \leq \rho < 0$. Then $|D(t)J(0,t)| \leq |J(0)|e^{\rho t}$ which converges to zero as $\rho < 0$.

Theorem 2. *Suppose the Technology, Felicity and Interiority Conditions are satisfied for a concave economy with L and R continuously differentiable. Suppose further that*

$$\lim_{T\to\infty} J(k(t),t)D(t) \int_0^T [R_2\eta + R_3\dot{\eta}]\, dt \to 0$$

for any $\eta \in \mathbf{A}(x) - k$. If $p(t) = [L_3 - R_3 J(k(t),t)]D(t)$ satisfies $\dot{p}(t) = [L_2 - R_2 J(k(t),t)]D(t)$ and the transversality condition $p(t)k(t) \to 0$ as $t \to \infty$, then k is optimal.

Proof. Suppose k is not optimal. Let $I(k') > I(k)$, $\delta = I(k') - I(k)$ and $\eta = k' - k$. Let L', D', L_ϵ and D_ϵ denote felicity and the discount factor evaluated along k' and $k + \epsilon\eta$. Choose T_0 so that $-\int_0^T [L'D' - LD]\, dt \geq \delta/2$ for all $T > T_0$. By concavity,

$$-\int_0^T [L_\epsilon D_\epsilon - LD]/\epsilon\, dt = -\int_0^T [L_\epsilon - L]D_\epsilon/\epsilon\, dt - \int_0^T L[D_\epsilon - D]/\epsilon\, dt \geq \delta/2. \tag{2}$$

Since $LD = d(JD)/dt$ and $d(D_\epsilon/D)/dt = (R_\epsilon - R)D_\epsilon/D$, we can integrate the second term by parts. Thus $-\int_0^T L[D_\epsilon - D]/\epsilon\, dt = -\int_0^T LD[D_\epsilon/D - 1]/\epsilon\, dt = -J(T)D(T)\int_0^T [R_\epsilon - R]/\epsilon\, dt + \int_0^T JD_\epsilon[R_\epsilon - R]/\epsilon\, dt$. Substituting in (2) yields

$$\int_0^T D_\epsilon[JR_\epsilon - JR - L_\epsilon + L]/\epsilon\, dt - J(T)[D_\epsilon(T) - D(T)]/\epsilon \geq \delta/2.$$

Since $T < \infty$ and L and R are C^1, we can let $\epsilon \to 0$ to obtain

$$\int_0^T [(JR_2 - L_2)\eta + (JR_3 - L_3)\dot{\eta}]D\, dt - J(T)D(T)\int_0^T [R_2\eta + R_3\dot{\eta}]\, dt \geq \delta/2.$$

By assumption, the second term converges to zero while the first term is just $-\int_0^T (\dot{p}\eta + p\dot{\eta})\, dt > 0$. As $\eta(0) = 0$, we have $p(T)\eta(T) = p(T)[k'(T) - k(T)] \leq -\delta/2$. As $p(T)k(T) \to 0$ by the transversality condition, $p(T)k'(T) < 0$ for T large. But $p(T)k'(T) \geq 0$. This contradiction shows k is optimal. □

The requirement that $J(k(T),T)D(T)\int_0^T[R_2\eta + R_3\dot{\eta}]\,dt \to 0$ as $T \to \infty$ may seem problematic, and can even be thought of as a type of transversality condition.[6] However, in many cases little or no information about the optimal path is required to insure it holds. This condition is automatically satisfied in the additively separable case since $R_2 = R_3 = 0$. Now consider a stationary non-additive problem, so $J(k(t),t) = J(k(t))$. If R_2 and R_3 are bounded on k with R bounded below zero, and if a maximum sustainable stock exists, this second transversality condition will always be satisfied. In other cases where feasible paths are bounded, it is enough that the optimal path be bounded above zero. For example, consider a one-sector model with discounting function $v(c) = 3 - e^{-c}$, so $R(t,k,\dot{k}) = -3 + \exp(\dot{k} - f(k))$, and a maximum sustainable stock. Then $|R_2| \leq f'$ and $|R_3| \leq 1$. The upper bound on k insures η and $\dot{\eta}$ are bounded while the lower bound on k bounds R_2. The integral term grows at rate T while $J(k(T),T)D(T)$ converges to zero exponentially in T, so the product converges to zero.

Sorger (1990b) derives similar first order necessary conditions and a pair of transversality conditions for a class of recursive problems based on the stochastic horizon model in Boukas, Haurie and Michel (1990). His control problem requires that the technology be smooth. It also allows for finite horizon problems with a scrap value. He transforms variables to reduce the model to an additive infinite horizon model. Application of a refined version of Halkin's (1974) version of the Maximum Principle yields the result. For this class of problems, Sorger's Hamiltonian approach avoids our interiority assumption and convexity hypotheses. However, even with smooth technology, the question of when his conditions are sufficient for optimality remains open. In contrast, our approach to the multiplier and transversality condition offers an essentially complete characterization of an optimum for many of the concave models typically used in economic applications.

7. Conclusion

One possible extension of our results would be to demonstrate the necessity of the transversality condition for models with $I(0) = -\infty$; this has been attacked for additive models in discrete time by Ekeland and Scheinkman (1986) and Shinotsuka (1990).

Another extension would be to study the existence of support prices from the view of convex analysis by letting supergradients replace smoothness conditions (see Benveniste and Scheinkman (1982) for the additive case). More generally, we would like to find the general form of the dual for the concave recursive optimization problem; this would place the study of shadow prices within the perturbation function approach associated with Ekeland and Temem (1976); see Ponstein (1984) for some economic applications. One reason for seeking the general form of the dual is to derive shadow prices when there are possible jumps in the dual variables arising from the presence of binding state constraints (c.f. Araujo and Scheinkman (1983)). This would be needed for fully characterizing optimal paths of agents in order to discuss the properties for perfect foresight equilibria.

[6]Sorger (1990a) has shown by example that the conventional transversality condition involving prices is not sufficient for optimality in the general non-additive case. Sorger (1990b) interprets $J(k(T),T)D(T) \to 0$, which amounts to convergence of utility along the optimal path, as a second transversality condition.

Epstein (1987a) proposed a foundation for recursive utility functions based on the concept of a generating function, which is the continuous-time analog of the discrete-time aggregator function. Existence, sensitivity, and duality theory for the general case of utility functions derived from Epstein's generating functions are open questions. Our results provide a first step in the study of sensitivity and duality for an important subclass of the functions included in Epstein's framework.

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