Recursive Utility and Optimal Capital Accumulation, I: Existence*

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This paper demonstrates existence of optimal capital accumulation paths when the planner's preferences are represented by a recursive objective functional. Time preference is flexible. We employ a general multiple capital good reduced-form model. Existence of optimal paths is addressed via the classical Weierstrass theorem. The topology is uniform convergence of capital stocks on compact subsets, which is equivalent to weak convergence of investment flows under our maintained hypotheses. An improved version of a lemma due to Varaiya proves compactness of the feasible set. A monotonicity argument is combined with a powerful theorem of Cesari to demonstrate upper semicontinuity. *Journal of Economic Literature* Classification Numbers: 022, 111, 213.

1. Introduction

This paper demonstrates existence of optimal capital accumulation paths when the planner's preferences are represented by a recursive objective functional. Time preference is flexible. The question of existence of optimal paths is addressed via the classical Weierstrass method. By carefully choosing an appropriate topology, we can ensure the objective is upper semicontinuous and the feasible set is compact. Optimal paths exist.

Following Ramsey [54], optimal growth theory has generally relied on models incorporating a fixed pure rate of time preference (possibly equal to zero). In this context, the preference order of the planner is represented by a time-additive utility functional. This type of objective has been criticized by various authors, dating back to Fisher [29], on the grounds that the pure rate of time preference should not be independent of the size and shape of the consumption profile. When preferences are time-additive, this independence is a direct consequence of the strong separability property by which the marginal rate of substitution for consumption at any two dates is independent of the rest of the consumption stream (Hicks [36] gives an extended critique along these lines). Lucas and Stokey [44]

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argued that the only basis for studying the time-additive case is its analytic tractability. Koopmans [39] laid the foundation for eliminating this deficiency of optimal growth theory by introducing "time-stationary" or "recursive" preferences.

Uzawa [67] extended Koopmans' discrete-time concept of recursive utility to continuous-time. A rigorous treatment of the existence problem for the Uzawa model was finally given by Nairay [51]. He drew upon a general existence theorem presented by Magill [45]. Both Uzawa and Nairay operated in the single-good setting of Ramsey's original model, but with the additional limitation of an affine production function.

Following Uzawa's lead, Epstein and Hynes [28] proposed another formulation of continuous-time recursive utility.[1] They took an *indirect* approach to the existence question that was based upon solving equations associated with Pontryagin's necessary conditions for optimality. This may be possible only under very strong assumptions.[2]

Our existence theorem proceeds via a *direct* method. Moreover, we consider a general multiple capital good model motivated by the Uzawa, Epstein, and Hynes formulation of continuous-time recursive utility. We cast our problem in terms of a *reduced form* model. This framework focuses on the vector of capital stocks and the corresponding net investment flows. These stock and flow variables jointly determine the consumption flow at each point in time. This formulation has much in common with the traditional fixed discount rate reduced-form model, which helps to clarify both the similarities and differences between the fixed discount rate and recursive models. The continuity properties of the objective functional are a key difference. Our existence theorems also accommodate technologies exhibiting stock, but not flow nonconvexities, such as models with increasing returns to scale.

The first step in the existence proof is to choose an appropriate topology. Ours is that of uniform convergence of capital stocks on compact subsets. On feasible sets it is equivalent to weak convergence of investment flows under our maintained hypotheses. We prove compactness using an improved version of a theorem originally due to Varaiya [68]. We show that stock convergence in this topology does not imply the existence of a subsequence where the flow variables converge almost everywhere.[3] Therefore, a Fatou's Lemma argument cannot be used to show a limit point of a maximizing sequence is an optimum.

The failure of almost everywhere convergence poses severe technical difficulties for the existence theory. One way around it, used by Romer [56], is to impose conditions on the flows that ensure pointwise convergence.[4] We deal directly with the problem of weak flow convergence. The key to our approach is a powerful theorem of Cesari [17].[5] Cesari's theorem

[1] Epstein [25, 26, 27] has subsequently generalized the Epstein and Hynes form of recursive utility. He did not address the existence question in those papers.

[2] Specifically, they assume there is a twice continuously differentiable solution to a generalized Hamilton-Jacobi equation. The existence of such a solution is an open problem.

[3] Brock and Haurie [13] thought stock convergence implied flow convergence almost everywhere. Yano [72] pointed out their mistake. He then claimed there would be a subsequence where the flows converged pointwise. This is only a more sophisticated version of the same error.

[4] Romer's result admits two interpretations. The first uses weak convergence of the derivatives of the flows. As this implies pointwise convergence of the flows, the objective need not be concave in the flows. The second requires the objective be concave in the flows.

[5] His Lower Semicontinuity Theorem (10.8.i) is the first that is general enough for our purposes. Earlier results of this type (see Cesari [15, 16], Ekeland and Temam [23], Olech [52], Rockafellar [55], and Ioffe [37])

is a type of Fatou's Lemma for weakly convergent sequences. It is the foundation of our major result, the Upper Semicontinuity Theorem for Objectives.

Cesari's theorem involves joint conditions on preferences and technology. A growth condition on feasible stocks is intertwined with the properties of the objective on growing paths. We show how the classical Tonelli necessary condition for upper semicontinuity of integral functionals implies the joint conditions cannot be separated into independent conditions on preferences and technology. A version of the joint condition had already arisen in one-sector time-additive models involving a linear technology. Its general importance is revealed by the recursive framework. The joint condition also emphasizes Koopmans' [40, 41] view that optimal growth theory can screen proposed welfare criteria. Depending on the technology, an otherwise attractive welfare criterion may prove nonsensical due to its inability to determine an optimum.

Similar joint conditions appear in the discrete-time existence theory for recursive preferences (Boyd [11]). Boyd also combined compactness and continuity conditions on the economically relevant subset of the commodity space to obtain existence of an optimum.[6] The same idea is implicitly exploited here by the requirement that felicity conditions hold only on the set of feasible programs. What happens off the feasible set is economically irrelevant.

Of course, our existence theorem also applies to time-additive utility as a special case.[7] Balder's [3] existence theorem extends concepts used in the classical case of a bounded time domain to infinite-horizon models. Our technology and felicity conditions play a role similar to his uniform integrability condition for flow variables and strong uniform integrability condition for cost functions.[8] In contrast with Magill [45], we use a single commodity space and topology for the entire analysis. Further, his methods are not really suitable for extension to recursive utility.[9] Gaines and Peterson [30] permit the same type of nonconvexities as we do. They actually prove a simple form of Cesari's theorem. Their results apply to stationary technologies subject to diminishing returns. In contrast, even in the time-additive case, we include non-stationary and constant returns to scale technologies.

The paper is organized as follows: The model is introduced in Section 2. Section 3 investigates the topological structure of the feasible sets. Section 4 proves the existence theorems and shows applications to specific examples. Final comments are in Section 5. The Appendix further clarifies the relation between the topologies used on feasible sets.

were not powerful enough to handle the general recursive case. Cesari's theorem makes use of a weaker hypothesis than the various "normality" conditions of Rockafellar and Ekeland and Temam (see Cesari [17, pp. 351, 365]).

[6] See Beals and Koopmans [6] and Majumdar [46] for earlier treatments of existence questions in discrete-time models with general utility functions.

[7] See Baum [5], Brock and Haurie [13], Bates [4], Takekuma [65], Magill [45], Yano [72], Ekeland and Turnbull [24], Leizarowitz [42], Gaines and Peterson [30], Carlson [14], Eirola and Kaitala [22], and Romer [56].

[8] Balder's work was built on the results cited in footnote 4. He was also indirectly influenced by Poljak [53].

[9] Nairay [51] applied Magill's result to a subclass of the Uzawa utility functionals. However, his method is not suitable for generalization.

2. The Model

There are m capital goods in the general model economy. The *technology* is a set $\Omega \subset \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$. We assume that the t-sections of Ω, $\{(k, y) : (t, k, y) \in \Omega\}$, are non-empty for every $t \geq 0$. Define $D = \{(t, k) : (t, k, y) \in \Omega \text{ for some } y\}$ and $G(t, k) = \{y : (t, k, y) \in \Omega\}$. The *investment correspondence* G thus has domain D. Given capital stock k at time t, any $y \in G(t, k)$ can be accumulated as additional capital. The investment correspondence is allowed to vary with time. The programming problem is defined for a given felicity function L and discounting function R, both mapping Ω to $\mathbb{R}$, and an assigned initial capital stock $x \in \mathbf{X}$ where $\mathbf{X} \subset \mathbb{R}^m_+$ is the set of potential initial capital stocks. Formally, an *economy* is a quadruple (Ω, L, R, x) with $x \geq 0$ that satisfies the Technology and Felicity Conditions given below.

A capital accumulation *program*, k, is an absolutely continuous function from $\mathbb{R}_+$ to $\mathbb{R}^m_+$, written $k \in \mathcal{A}$.[10] It may thus be represented as the integral of its derivative $\dot{k}$ by $k(t) = k(0) + \int_0^t \dot{k}(s)\, ds$. Further, the derivative of k not only exists almost everywhere, but it is *locally integrable*—its integral over any compact set is finite. We denote this by $k \in L^1_{loc}$. Define the set of *attainable (admissible) programs from initial stock* x, $\mathbf{A}(x)$ by

$$\mathbf{A}(x) = \{k \in \mathcal{A} : \dot{k} \in G(t, k) a.e., 0 \leq k(0) \leq x\}.$$

Let $D(t) = \{k : (t, k) \in D \text{ for some } t\}$. We call the technology *convex* if the section $\{(k, y) : y \in G(t, k)\}$ is convex for each t. In this case, $\mathbf{A}(x)$ is convex for each $x \in \mathbf{X}$.

The *recursive objective functional*, I, is given by

$$I(k) = -\int_0^\infty L(t, k, \dot{k}) \exp\Big(\int_0^t R(s, k, \dot{k})\, ds\Big)\, dt.$$

The *programming problem*, $P(x)$, is defined by

$$P(x) : J(x) = \sup\{I(k) : k \in \mathbf{A}(x)\}.$$

Here J is the *value function*. An admissible program k^* is an *optimal solution to*$P(x)$ if $k^* \in \mathbf{A}(x)$ and $I(k^*) = J(x)$.

Define a map $k \to \mathcal{M}k$ by the formula

$$\mathcal{M}k(t) = -L(t, k(t), \dot{k}(t)) \exp \int_0^t R(s, k, \dot{k})\, ds.$$

This map is *concave*if for each $\alpha \in [0, 1]$, for any k, k' with $k_\alpha = \alpha k + (1 - \alpha)k'$, it follows that

$$\mathcal{M}k_\alpha(t) \geq \alpha \mathcal{M}k(t) + (1 - \alpha)\mathcal{M}k'(t)$$

[10] More formally, a function k is *absolutely continuous* if for every T and $\epsilon > 0$, there is a δ with $\sum_{i=1}^m |k(t_i) - k(s_i)| < \epsilon$ whenever $0 \leq t_1 \leq s_1 \leq \cdots \leq t_m \leq s_m \leq T$ with $\sum_{i=1}^m |t_i - s_i| < \delta$. This implies it may be represented as the integral of its derivative. Conversely, any locally integrable function f has an absolutely continuous integral $k(t) = \int_0^t f(s)\, ds$.

holds for almost every t. The objective functional is *concave* when the map $\mathcal{M}$ is concave. If the technology is also convex, we say the *problem* or the *economy is concave*. A concave problem yields a value function that is concave and thus continuous on the relative interior of its domain $\mathbf{X}$.

Technology Conditions.

(i) *$G(t,k)$ is compact and convex for each (t,k) and $k \to G(t,k)$ is upper semicontinuous for each t.*
(ii) *For all $x \in \mathbf{X}$, there is a $\mu_x \in L^1_{loc}$ such that $|\dot{k}| \leq \mu_x$ a.e. whenever $k \in \mathbf{A}(x)$.*
(iii) *$\mathbf{A}(0) \neq \varnothing$.*

Note that the non-triviality condition (iii) implies $\mathbf{A}(x) \neq \varnothing$ for all $x \in \mathbf{X}$. In many economic applications inaction is always possible by virtue of free disposal, thus $0 \in G(t,k)$. Condition (iii) immediately follows since $k(t) = x$ is feasible.

Felicity Conditions.

(i) *$L\colon \Omega \to \mathbb{R}$ and $R\colon \Omega \to \mathbb{R}$ are continuous on Ω and convex in $y \in G(t,k)$ for each fixed $(t,k) \in D$.*
(ii) *$L \geq 0$ and there exists a non-positive function $\rho_x \in L^1_{loc}$ such that $R(t,k(t),\dot{k}(t)) \geq \rho_x(t)$ for all $k \in \mathbf{A}(x)$.*
(iii) *There is a program $k \in \mathbf{A}(x)$ such that $I(k) > -\infty$.*

We will refer to L as the *felicity function* and R as the *discounting function*. The second and third Felicity Conditions both involve a type of joint condition on preferences and technology. Koopmans [40] observed that technological possibilities may restrict the type of utility functions that can be sensibly used. These joint conditions should be taken in this spirit. Condition (F.ii) will be satisfied regardless of the technology whenever R is bounded below by a constant. At first glance, the implied upper bound on the discount rate $(-R)$ may seem counter-intuitive. Note, however, that the bound only applies on the feasible set, and need not be uniform over time. If (F.ii) fails, we could construct a maximizing sequence k_n with arbitrarily large discounting at time $1/n$. Utility would approach 0, but this value could not be attained if $L < 0$ since any feasible path would always have a finite discount rate. As when chattering is possible, an extended, non-economic, solution concept, such as Young's [73] generalized curves may be required (for additively separable examples see Avgerinos and Papageorgiou [2]).

Any program that satisfies the third Felicity Condition is called a *good program*. Many capital accumulation models have good programs. When R is bounded below zero, condition (F.iii) clearly holds in two polar cases. If L is bounded, good programs exist regardless of the technology. At the opposite extreme, when a maximum sustainable stock exists, this condition is also trivially satisfied regardless of L. In particular, this occurs in the usual additively separable model (R is a negative constant). Further, (F.ii) is also automatically satisfied in the additively separable model with a maximum sustainable stock. When we consider additively separable utility functions in more detail, the second Felicity Condition will be modified to include a joint condition on preferences and technology. Applications given in Section 4 will further illustrate the role of the Technology and Felicity Conditions.

3. The Topology of Feasible Sets

Two equivalent modes of convergence are particularly important for our feasible set. The first (the compact-open topology) will be used to establish compactness of the feasible set. The second (the weak topology) is more appropriate for demonstrating upper semicontinuity of the objective. Both are defined on the space $\mathcal{A}$. The $\mathcal{C}$ topology is the topology of uniform convergence on compact subsets (the compact-open topology) and is defined by the family of norms $\|f\|_{\infty,T} = \sup\{|f(t) : 0 \le t \le T\}$ for $T = 1, 2, \ldots$, where $|\cdot|$ is any of the equivalent ℓ^p-norms on $\mathbb{R}^m$. Under these norms, $\mathcal{C}$ is a complete, countably-normed space and thus a Fréchet space.

The second topology is obtained by focusing on the flows rather than stocks. When f is absolutely continuous, its derivative is locally integrable. The space of locally integrable functions, denoted L^1_{loc}, is a Fréchet space under the topology given by the norms $\|f\|_{1,T} = \int_0^T |f(s)|\, ds$. The space $\mathcal{A}$ can be topologized by the norms $\|f\|_{A,T} = \|f\|_{\infty,T} + \|\dot{f}\|_{1,T}$. These norms turn $\mathcal{A}$ into a Fréchet space. Since $\|f\|_{\infty,T} = \sup |f(0) + \int_0^t \dot{f}(s)ds| \le \|f\|_{\infty,T} + \|\dot{f}\|_{1,T} = \|f\|_{A,T}$, this topology is equivalent to that given by the natural metric on the direct sum $\mathbb{R} \oplus L^1_{loc}$. Thus $\mathcal{A} = \mathbb{R} \oplus L^1_{loc}$. As a Fréchet space, it has a dual and associated weak topology.

Since the dual of L^1_{loc} is given by $(L^1_{loc})' = \{f \in L^\infty : f \text{ has compact support}\}$, φ is in the dual of $\mathcal{A}$ if and only if $\langle \varphi, f \rangle = \alpha f(0) + \int_0^T \beta(s) f(s)\, ds$ for some $\alpha \in \mathbb{R}$, $T > 0$ and $\beta \in L^\infty$. This duality defines the weak topology on $\mathcal{A}$. Curiously, any sequence that converges in the weak topology on $\mathcal{A}$ converges in the $\mathcal{C}$ topology. (Use the Dunford-Pettis Theorem and mimic the proof of Lemma 2 below.) However, weakly convergent *nets* need not converge in the $\mathcal{C}$ topology.[11] We will be working with a subspace $\mathbf{F}(\mu)$, defined by a growth condition, where the two topologies actually are equivalent. For each $\mu \in L^1_{loc}$ with $\mu \ge 0$, define the subspace $\mathbf{F}(\mu)$ by $\mathbf{F}(\mu) = \{k \in \mathcal{A} : |\dot{k}| \le \mu\}$. As the following lemmas show, the two topologies are equivalent on $\mathbf{F}(\mu)$. Moreover, given this equivalence, we can easily establish that $\mathbf{F}(\mu)$ is compact in this topology.

Lemma 1. *Suppose a net $\{k_\alpha : \alpha \in A\}$ converges to k in $\mathcal{C}$, then $k \in \mathbf{F}(\mu)$ and k_α converges weakly to k in $\mathcal{A}$.*

Proof. Let $T > 0$ be given. For $|E| > 0$, choose δ such that $\int_E \mu < \epsilon$ whenever $E \subset [0, T]$

[11] Balder ([3, Appendix A]) confuses the equivalence of sequential convergence with the equivalence of the topologies. Sequences are not sufficient to characterize the topologies. See the Appendix for details.

with $|E| < \delta$. Suppose $0 \le t_1 \le s_1 \le \cdots \le t_m \le s_m \le T$ with $\sum_{i=1}^m |t_i - s_i| < \delta$. Then

$$\begin{aligned}
&\sum_{i=1}^m |k(t_i) - k(s_i)| \\
&\qquad \le \sum_{i=1}^m \left(|k(t_i) - k_\alpha(t_i)| + |k_\alpha(s_i) - k(s_i)| + \int_{s_i}^{t_i} |\dot{k}_\alpha|\, dt \right) \\
&\qquad \le 2m\|k - k_\alpha\|_{\infty,T} + \sum_{i=1}^m \int_{s_s}^{t_s} \mu\, dt \\
&\qquad < 2m\|k - k_\alpha\|_{\infty,T} + \epsilon < 2\epsilon
\end{aligned}$$

for large α. Thus $k \in \mathcal{A}$.

Since $k_\alpha(0) \to k(0)$ and $\|\dot{k}_\alpha\|_{1,T} \le \|\mu\|_{1,T}$, k_α will converge weakly in $\mathcal{A}$ to k provided, for each T, $\int_E \dot{k}_\alpha \to \int_E \dot{k}$ for all measurable $E \subset [0,T]$.[12] Let $\epsilon > 0$ and choose δ such that $\int_F \mu < \epsilon$ and $|\int_F \dot{k}| < \epsilon$ whenever $F \subset [0,T]$ with $|F| < \delta$. When $|E| > 0$, take a finite disjoint union of intervals, $G = \cup_{i=1}^m (t_i, s_i)$ with $|E \triangle G| < \delta$. Set $F = E \triangle G$. Then

$$\begin{aligned}
\left| \int_E \dot{k}_\alpha - \int_E \dot{k} \right| &\le \left| \int_G (\dot{k}_\alpha - \dot{k}) \right| + \int_F |\dot{k}_\alpha| + \left| \int_F \dot{k} \right| \\
&\le \sum_{i=1}^m \{ |k_\alpha(t_i) - k(t_i)| + |k_\alpha(s_i) - k(s_i)| \} + 2\epsilon \\
&\le 2m\|k_\alpha - k\|_{\infty,T} + 2\epsilon.
\end{aligned}$$

As $n \to \infty$, we see that $k_\alpha \to k$ weakly. Clearly $|\dot{k}| \le \mu$ a.e., so $k \in \mathbf{F}(\mu)$. ∎

LEMMA 2. *If a net, $\{k_\alpha : \alpha \in A\}$, converges weakly in $\mathcal{A}$ to k and $k_\alpha \in \mathbf{F}(\mu)$, then $k_\alpha \to k$ in $\mathcal{C}$.*

Proof. Let $\epsilon, T > 0$. Since μ and $\dot{k}$ are locally integrable, we can choose δ with $|\int_s^t \mu| < \epsilon$ and $|\int_s^t \dot{k}| < \epsilon$ whenever $0 \le s, t \le T$ and $|s - t| < \delta$. Since k_α converges weakly to k, there is a $\alpha' \in A$ with $|k_\alpha(\delta n) - k(\delta n)| = |k_\alpha(0) - k(0) + \int_0^{\delta n} (\dot{k}_\alpha - \dot{k})| < \epsilon$ for all $\alpha > \alpha'$ and all integers n, $0 \le n \le T/\delta$. Let $t \in [0,T]$. Take an integer n, $0 \le n \le T/\delta$ with $|t - \delta n| < \delta$. Then

$$|k_\alpha(t) - k(t)| \le |k_\alpha(\delta n) - k(\delta n)| + |\int_{\delta n}^t (\dot{k}_\alpha - \dot{k})| < 3\epsilon$$

for all $\alpha > \alpha'$ and so $k_\alpha \to k$ uniformly on $[0,T]$ for all T. ∎

Combining the preceding lemmas with the fact that the $\mathcal{C}$ topology is metrizable yields the following proposition.

[12] It immediately follows that $\int_0^T \dot{k}_\alpha \varphi \to \int_0^T \dot{k}\varphi$ for any simple function φ in $L^\infty(0,T)$. As the simple functions are dense in L^∞ and the $\dot{k}_\alpha$ are norm-bounded, the integrals will converge for any φ in L^∞. Thus $\dot{k}_\alpha$ converges weakly to k.

PROPOSITION 1. *On $\mathbf{F}(\mu)$, the $\mathcal{C}$ and weak $\mathcal{A}$ topologies are equivalent. Further, $\mathbf{F}(\mu)$ is closed and this topology is metrizable.*

LEMMA 3. *$\mathbf{F}(\mu)$ is compact in the $\mathcal{C}$ topology.*

Proof. Since $\mathbf{F}(\mu)$ is closed, it suffices to show $\mathbf{F}(\mu)$ is equicontinuous on $[0,T]$ for each T. Ascoli's Theorem will then establish compactness. Let $\epsilon > 0$ and choose δ such that $\int_E \mu < \epsilon$ whenever $E \subset [0,T]$ and $|E| < \delta$. Then, for $|s-t| < \delta$, $|k(s) - k(t)| = |\int_s^t \dot{k}| \leq |\int_s^t \mu| < \epsilon$. Therefore $\mathbf{F}(\mu)$ is equicontinuous on $[0,T]$ for each T. ∎

Various authors have assumed that weak convergence in $\mathcal{A}$ implies the derivatives either converge pointwise (this is implicit in Lemma 3.2 of Brock and Haurie [13]) or have a subsequence that converges pointwise (Yano [72, Lemma 3]).[13] As the following example shows, neither of these need happen.[14]

Consider the Rademacher functions. Define

$$R(t) = \begin{cases} 1 & \text{for } 0 \leq t - [\![t]\!] < 1/2 \\ -1 & \text{for } 1/2 \leq t - [\![t]\!] < 1 \end{cases}$$

where $[\![t]\!]$ is the greatest integer function. The n-th *Rademacher function*, $r_n(t)$ is defined on $[0,1]$ by $r_n(t) = R(2^n t)$. The Rademacher functions form an orthonormal set in $L^2(0,1) \subset L^1(0,1)$ and are elements of $L^1(0,1)$. By Bessel's Inequality (Halmos [32, pg. 18]), any $f \in L^\infty$ has $\sum_{n=1}^\infty |\int_0^1 f(s) r_n(s)\, ds|^2 \leq \|f\|_2 \leq \|f\|_\infty$, thus $\int_0^1 f(s) r_n(s)\, ds \to 0$. The Rademacher functions converge weakly to 0, but no subsequence can converge pointwise. If it did, the L^2 distance between the Rademacher functions in the subsequence would converge to zero by the Lebesgue Bounded Convergence Theorem. This cannot happen since the L^2 distance between two Rademacher functions is always $\sqrt{2}$.[15]

The strongest statement that can be made about pointwise limits is expressed in the following lemma.

LEMMA 4. *Suppose k and k_n are real-valued and $k_n \to k$ weakly in L^1_{loc}. Then $\limsup_{n\to\infty} k_n(t) \geq k(t) \geq \liminf_{n\to\infty} k_n(t)$ a.e.*

Proof. Suppose the first half of the inequality is violated. Then there are $T, \delta > 0$ such that $E = \{t \leq T : x(t) > \limsup_{n\to\infty} x_n(t) + \delta\}$ has positive measure. Let $g(t) = \limsup_{n\to\infty} x_n(t)$ and $g_n(t) = \sup\{x_m(t) : m \geq n\}$. Note $g_n \to g$ pointwise.

By Egoroff's Theorem there is a measurable set F with $|F| < |E|/2$ such that $g_n \to g$ uniformly on $G = E \setminus F$. Now take n large enough that $\delta/2 + g(t) > g_n(t)$ for $t \in G$. Then $x(t) + \delta/2 > g_n(t)$ on G. But then $x(t) + \delta/2 > x_m(t)$ on G for all $m \geq n$. Integrating over G and letting $m \to \infty$ now yields the contradiction.

[13] Eirola and Kaitala [22] uncritically used Yano's claim in their existence article.

[14] The example is based on a mild form of chattering. The chattering only involves the flows, not the state. Romer [56] gives an economic example where even the state variable suffers from chattering.

[15] The same argument applies to any orthonormal set in L^2. The classical orthonormal functions—trigonometric, Hermite, Legendre, etc.—all share this property. These examples are commonly used to show weak convergence does not imply norm convergence (Ekeland and Turnbull [24, pp. 76-78]).

Consideration of $-k$ now shows that the second half of the inequality is also satisfied. ∎

When the functions take values in $\mathbb{R}^m$, and $p \in \mathbb{R}^m$, this lemma applies to the inner product $\langle p, k(t)\rangle$.

4. Existence of Optimal Paths

The main existence result for the problem $P(x)$ is stated below.

Existence Theorem. *For each economy (Ω, L, R, x), there is an optimal solution to $P(x)$.*

The proof follows once we establish compactness of the feasible set and upper semicontinuity of the objective in the weak topology. Compactness of the feasible set is the subject of Lemma 6, while the Lower Semicontinuity Theorem will ensure upper semicontinuity of the objective. We conclude the section by examining the theorem's application to one-sector and multi-sector models.

4.1. *Existence Theory*

Lemmas 1 and 6 together constitute a variation on a theorem due to Varaiya ([68, Theorem 2.1]) that is more suited to our setting. We relax Varaiya's conditions somewhat by establishing compactness of a larger set, $\mathbf{F}(\mu)$, via a bound on $\dot{k}$. This bound, which is naturally related to the technology, plays a crucial role in showing the Felicity Conditions hold in applications. Varaiya works under more restrictive conditions that imply such a bound. Another point of note is the role of a compact-valued investment correspondence.

Our approach clarifies the role of compactness of the technology set. Varaiya showed that $\dot{k}$ is almost everywhere an element of every closed half-space containing the technology set. It might seem that we just take the intersection of the closed half-spaces containing the (convex) technology set to show that $\dot{k}$ is in the technology set. This fails as $\dot{k}$ need only be in each half-space almost everywhere. When we intersect over arbitrarily many sets, $\dot{k}$ may be in the intersection almost nowhere! When the technology set is compact for each (k, t), the following lemma can be used to solve the problem.

Lemma 5. *Suppose G is a compact, convex, non-empty subset of $\mathbb{R}^m$ and $\mathcal{P}$ is dense in $\mathbb{R}^m$. Then G is the intersection of the half-spaces of the form $\{y : \langle p, y\rangle \leq \alpha\}$ containing it, where $p \in \mathcal{P}$ and α is a real number.*

Proof. Clearly G is contained in the intersection. It is enough to show that any point not in G is not in a half-space containing G. Let $x \notin G$. Then there is a p', not necessarily in P, with $\langle p', x\rangle > \max\{\langle p', y\rangle : y \in G\}$. Let $f(q) = \langle q, x\rangle - \max\{\langle q, y\rangle : y \in G\}$. Since G is compact, f is continuous. As P is dense in $\mathbb{R}^m$, there is a $p \in P$ with $f(p) > 0$. Taking any α with $\langle p, x\rangle > \alpha > \max\{\langle p, y\rangle : y \in G\}$ yields the desired half-space. ∎

Lemma 6. *If the technology conditions are satisfied, then $\mathbf{A}(x)$ is a compact subset of $\mathcal{A}$.*

Proof. By Proposition 1, $\mathbf{A}(x) \subset \mathbf{F}(\mu)$ with $\mathbf{F}(\mu)$ compact. It suffices to show that $\mathbf{A}(x)$ is closed. Let $k_n \to k$ in $\mathcal{C}$ with $k_n \in \mathbf{A}(x)$. Then $k_n(0) \to k(0)$, so $0 \leq k(0) \leq x$. Similarly,

$k(t) \in D(t)$ for all t. By Proposition 1, $k \in \mathcal{A}$ and $\dot{k}_n \to \dot{k}$ weakly in L^1_{loc}. Let $\mathcal{P}$ be a countable dense subset of $\mathbb{R}^m$. Since $\mathcal{P}$ is countable, we can use Lemma 4 to obtain a set F with zero measure such that for each $p \in \mathcal{P}$:

$$\limsup_{n\to\infty} \langle p, \dot{k}_n(t)\rangle \geq \langle p, \dot{k}(t)\rangle \geq \liminf_{n\to\infty} \langle p, \dot{k}_n(t)\rangle \text{ for all} t \in F.$$

Let $\alpha > \sup\{\langle p, y\rangle : y \in G(t, k(t))\}$. As G is upper semicontinuous in k and $k_n(t) \to k(t)$, $G(t, k_n(t)) \subset \{z : \alpha > \langle p, z\rangle\}$ for n large. Hence $\langle p, \dot{k}_n(t)\rangle < \alpha$ for n large and so

$$\begin{aligned}\max\{\langle p, y\rangle : y \in G(t, k(t))\} &\geq \limsup \langle p, \dot{k}_n(t)\rangle \\ &\geq \langle p, \dot{k}(t)\rangle \text{ for } t \notin F.\end{aligned}$$

Now $G(t, k(t))$ is compact and convex. By Lemma 5, it is the intersection of the half-spaces containing it of the form $\{y : \langle p, y\rangle \leq \alpha\}$ where $p \in \mathcal{P}$. Thus $\dot{k}(t) \in G(t, k(t))$ for $t \notin F$. It immediately follows that $k \in \mathbf{A}(x)$. ∎

If $G(t, k)$ is only closed, a countable set $\mathcal{P}$ with the desired properties may not exist. In Lemma 6 we must choose the countable set $\mathcal{P}$ before looking at the limsup. The same $\mathcal{P}$ must be used for almost every time t to get inclusion in $G(t, k(t))$ almost everywhere. A simple illustration of what can go wrong is when $G(t, k) = \{(y_1, y_2) : y_1 + ty_2 \leq 1\}$. Clearly $\mathcal{P}$ must contain a multiple of $(1, t)$ for almost every t to write $G(t, k)$ as the intersection of half-spaces derived from $\mathcal{P}$. Thus $\mathcal{P}$ cannot be countable.

We also need the following theorem, which holds for any $T < \infty$.

Lower Semicontinuity Theorem. *Suppose $F_0 \colon \mathbb{R} + \times\mathbb{R}^{2m} \to \mathbb{R}_+ \cup \{+\infty\}$ is measurable in $(t, \chi, \xi) \in \mathbb{R} + \times\mathbb{R}^{2m}$, lower semicontinuous in (χ, ξ) for each t, and convex in ξ for each (t, χ). Further, suppose $F_0 \geq -\psi$ on Ω where $\psi \geq 0$, $\psi \in L^1(0, T)$. Now let $k_n \to k$ weakly in $\mathcal{A}$ and $\dot{k}_n \in G(t, k_n)$ a.e. If G obeys the technology conditions, then for each T*

$$\int_0^T F_0(t, k, \dot{k})\, dt \leq \liminf_{n\to\infty} \int_0^T F_0(t, k_n, \dot{k}_n)\, dt$$

Proof. This is a special case of Theorem 10.8.i in Cesari [17]. ∎

Upper Semicontinuity Theorem for Objective Functionals. *Suppose the Felicity and Technology Conditions are satisfied and $k_n \to k$ weakly in $\mathbf{A}(x)$. Then* $\limsup_{n\to\infty} I(k_n) \leq I(k)$.

Proof. Let $\epsilon, T > 0$. Choose δ such that $|\int_E \rho_x| < \epsilon$ whenever $|E| \leq \delta$. Let $g_n(t) = \inf\{\int_0^t R(s, k_m, \dot{k}_m)\, ds : m \geq n\}$ and $g(t) = \lim_{n\to\infty} g_n(t)$. Apply the Lower Semicontinuity Theorem to $F_0(t, \chi, \xi)$ given by R for $\xi \in G(t, \chi)$ and $+\infty$ otherwise with $\psi = -\rho_x$. Thus $g(t) \geq \int_0^t R(s, k, \dot{k})\, ds$ almost everywhere. By Egoroff's Theorem, there is a set E with $|E| \leq \delta$ and $g_n \to g$ uniformly on $F = [0, T] \setminus E$.

For $n \geq N$ with N large this yields

$$\begin{aligned}\int_0^t R(s,k_n,\dot{k}_n)\,ds &\geq -\epsilon + \int_{t0} R(s,k,\dot{k})\,ds \text{for } t \in F \\ &\geq -2\epsilon + \int_0^t R(s,k,\dot{k})\,ds \text{for } t \in [0,T]\end{aligned} \tag{1}$$

Now since $L \geq 0$,cases

$$\begin{aligned}&\int_0^\infty L(t,k_n,\dot{k}_n)\exp\left(\int_0^t R(s,k_n,\dot{k}_n)\,ds\right)\,dt \\ &\qquad \geq \int_0^T L(t,k_n,\dot{k}_n)\exp\left(\int_0^t R(s,k_n,\dot{k}_n)\,ds\right)\,dt \\ &\qquad \geq e^{-2\epsilon}\int^T 0L(t,k_n,\dot{k}_n)\exp\left(\int_0^t R(s,k,\dot{k})\,ds\right)\,dt\end{aligned} \tag{2}$$

for n large enough.

An application of the Lower Semicontinuity Theorem to the function

$$F_0(t,\chi,\xi) = \begin{cases} L(t,\chi,\xi)\exp(\int_0^t R(s,k(s),\dot{k}(s))\,ds) & \text{for } \xi \in G(t,\chi) \\ +\infty & \text{otherwise}\end{cases}$$

and $\psi = 0$ gives

$$\begin{aligned}&\int_0^\infty L(t,k_n,\dot{k}_n)\exp\left(\int_0^t R(s,k_n,\dot{k}_n)\,ds\right)\,dt \\ &\qquad \geq e^{-2\epsilon}\int^T 0L(t,k_n,\dot{k}_n)\exp\left(\int_0^t R(s,k,\dot{k})\,ds\right)\,dt \\ &\qquad \geq -\epsilon + e^{-2\epsilon}\int_0^T L(t,k,\dot{k})\exp\left(\int_0^t R(s,k,\dot{k})\,ds\right)\,dt\end{aligned}$$

Thus $\liminf_{n\to\infty} -I(k_n) \geq -\epsilon + e^{-2\epsilon}\int_0^T L(t,k,\dot{k})\exp(\int_0^t R(s,k,\dot{k})\,ds)\,dt$ for all $\epsilon, T > 0$. Letting $\epsilon \to 0$ and $T \to \infty$ yields the desired inequality. ∎

In the additively separable case the situation is much simpler. In that case $R(t,k,y) = \rho_x(t)$ is independent of k and y. The lower semicontinuity theorem can be applied directly to $L(t,\chi,\xi)\exp\rho_x(t)$. This is essentially the method used by Gaines and Peterson [30]. In contrast, the more general recursive case requires two applications of the lower semicontinuity theorem. In the general case, allowing L to take negative values causes severe problems due to the double use of the lower semicontinuity theorem. The one-sided estimate of equation (2) must be replaced by a two-sided estimate. This would follow if equation (1) gave a two-sided estimate. However, this would require the integral of R to be both upper and lower semicontinuous. This condition is not generally satisfied. By a classical theorem of

Tonelli it can only be satisfied when R is affine in y.[16] Note that additively separable utility is trivially affine since R is independent of y.

An attempt to sidestep the problem by putting the recursive model in an additively separable form also fails due to Tonelli's theorem. The recursive model (Ω, L, R, x) may be represented in a time additive framework by adding state variables. Given an economy, define an additional state variable z by putting

$$\dot{z} = R(t, k, \dot{k}), \qquad z(0) = 0.$$

Set $M(t, k, \dot{k}, z, \dot{z}) = -L(t, k, \dot{k}) \exp z$. The *indirect representation* of the economy is the triple (Ω, M, x). Attainable programs for the indirect representation of the economy are elements of

$$\mathbf{B}(x) = \{(k, z) : k \in \mathbf{A}(x), z(t) = \int_0^t R(s, k, \dot{k})\, ds\}.$$

The *indirect objective functional*, N, is defined by

$$N(k) = \int_0^\infty M(t, k, \dot{k}, z, \dot{z})\, dt.$$

The *indirect programming problem* is to choose an attainable program which achieves $\sup N(k)$. In general, the existence theorems of Balder [3] and Gaines and Peterson [30] do *not* apply to the indirect problem. The set $\mathbf{B}(x)$ need not be compact even though $\mathbf{A}(x)$ is compact. Compactness of $\mathbf{B}(x)$ would follow if the integral of R were continuous in k. However, we have already noted that Tonelli's theorem prevents this unless R is affine in y! If we only know R is convex in y, k_n converges to k weakly in $\mathbf{A}(x)$ implies $z_n(t) \geq -2\epsilon + z(t)$ for n large, which is equation (1) again. This is not enough to yield the compactness of $\mathbf{B}(x)$ needed for the general recursive case.

When R *is* affine in y, a variant of the Upper Semicontinuity Theorem does in fact hold (Corollary 1). For this, we use the Modified Felicity Condition that (F.i), (F.ii') and (F.iii) are satisfied where (F.ii') is given below.

ASSUMPTION F.II′. *There are ρ_{1x} and $\rho_{2x} \in L^1_{loc}$ and a non-negative measurable function λ_x obeying $\int_0^\infty \lambda_x(t) \exp(\int_0^t \rho_{2x}(s)\, ds)\, dt < \infty$ such that $\rho_{1x}(s) \leq R(s, k(s), \dot{k}(s)) \leq \rho_{2x}(s)$ and $-\lambda_x(t) \leq L(t, \dot{k}(t), k(t))$ for all $k \in \mathbf{A}(x)$.*

COROLLARY 1. *Suppose R is affine and the Technology and modified Felicity Conditions are satisfied. Then $\limsup_{n\to\infty} I(k_n) \leq I(k)$.*

Proof. Fix T. Since R is affine in y, both R and $-R$ are convex in y. Combined with the fact that $\rho_{1x}(s) \leq R(s, k(s), \dot{k}(s)) \leq \rho_{2x}(s)$, this shows equation (1) applies to both R and $-R$. Thus $\log \eta_n(t) = \int_0^t R(s, kn(s), \dot{k}_n(s))\, ds$ converges to $\log \eta(t) = \int_0^t R(s, k(s), \dot{k}(s))\, ds$ almost everywhere on $[0, T]$.

[16] Cesari ([17, pg. 107]) gives a version of Tonelli's theorem. It shows continuity of the exponential term in (k, y) would imply R is affine in y.

Apply the Lower Semicontinuity Theorem to $F_0(t,\chi,\eta,\xi,\theta)$ defined by

$$F_0(t,\chi,\eta,\xi,\theta) = \begin{cases} \eta L(t,\chi,\xi) + \lambda_x(t)\exp(\int_0^t \rho_{2x}(s)\,ds) & \text{for } \xi \in G(t,\chi) \\ +\infty & \text{otherwise} \end{cases}.$$

Note $F_0 \geq 0$. Thus

$$\begin{aligned} &\liminf \left\{ \int_0^T \lambda_x(t)\exp\left(\int_0^t \rho_x(s)\,ds\right)\,dt + \int_0^T \eta_n(t)L(t,k_n,\dot{k}_n)\,dt \right\} \\ &\quad \geq \int_0^T \lambda_x(t)\exp\left(\int_0^t \rho_{2x}(s)\,ds\right)\,dt + \int_0^T \eta(t)L(t,k,\dot{k})\,dt. \end{aligned}$$

Since T is arbitrary and $F_0 \geq 0$, we let $T \to \infty$ to obtain

$$\begin{aligned} &\liminf \left\{ \int_0^\infty \lambda_x(t)\exp(\int_0^t \rho_{2x}(s)\,ds)\,dt - I(k_n) \right\} \\ &\quad \geq \int_0^\infty \lambda_x(t)\exp(\int_0^t \rho_{2x}(s)ds)dt - I(k). \end{aligned}$$

As the additional integrals are finite, $\limsup I(k_n) \leq I(k)$. ∎

Proof of Existence Theorem. Let $k_n \in \mathbf{A}(x)$ be a maximizing sequence for I. Since $\mathbf{A}(x)$ is compact, we can find a $k^* \in \mathbf{A}(x)$ and a subsequence, also denoted k_n with $k_n \to k^*$. By the Upper Semicontinuity Theorem for Objectives, $\limsup I(k_n) \leq I(k^*)$. Thus $J(x) = \limsup I(k_n) \leq I(k^*)$. As $k^* \in \mathbf{A}(x)$, $I(k^*) \leq J(x)$. Therefore $I(k^*) = J(x)$ and k^* is the desired optimal path. ∎

Corollary 2. *Suppose* (Ω, L, R, x) *satisfies the Technology and modified Felicity Conditions. Then* $P(x)$ *has a solution.*

The proof of the corollary is the same as the theorem, except for two points. First, Corollary 1 must be used instead of the Upper Semicontinuity Theorem for Objectives. Second, the bounds on L and R in (F.ii') ensure the supremum is finite. After some modification, this approach also applies to undiscounted additively separable models. These models have the additional difficulty that the supremum may not be finite. The way around this problem is to use the overtaking criterion investigated by Brock and Haurie [13] and Yano [72]. In this case the value loss is used in place of the felicity function. The value loss objective functional has a supremum, zero, which is attained by the golden rule. Good programs are then those with finite value loss. As the correct form of the overtaking criterion for recursive utility remains unknown, the remainder of this paper will focus on discounted models.

4.2. *One-Sector Models*

One type of model that satisfies the technology conditions is the one-sector growth model. Denote consumption by $c(t)$, capital by $k(t)$ and net investment by $\dot{k}(t)$. Assume the gross production function $f\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ is continuous in (k,t) and increasing in k. Further, there is a continuous function $\tau(t)$ with $0 \leq f(k,t) \leq \tau(t)(1+k)$ for $k \geq 0$. This growth

condition is automatically satisfied if f is concave in k. Capital depreciates at rate $\beta \in [0,1]$, thus the net production function is $g(k,t) = f(k,t) - \beta k$. Consumption c is given by $c(t) = g(k,t) - \dot{k}(t) = f(k,t) - \beta k(t) - \dot{k}(t)$. Define $G(t,k) = [-\beta k, g(k,t)]$ and $\Omega = \{(t,k,y) \in \mathbb{R} + \times \mathbb{R} + m \times \mathbb{R}m : y \in G(t,k)\}$. This defines the *standard technology*.

The first and third technology conditions are clearly satisfied. The second technology condition is also satisfied. To see this, we study the path of pure accumulation. As a preliminary step, define undepreciated capital by $K(t) = e^{\beta t}k(t)$. Undepreciated capital obeys $0 \leq \dot{K} \leq F(K,t)$ where $F(K,t) = e^{\beta t}f(e^{-\beta t}K,t)$ is undepreciated gross output. Since F obeys the growth condition whenever f does, and since $K \to e^{-\beta t}K$ is continuous in $\mathcal{C}$, it is enough to consider the no-depreciation case ($\beta = 0$).

Let $\tau_T = \sup\{\tau(t) : 0 \leq t \leq T+1\}$ and $M = \sup\{|f(k,t)| : 0 \leq t \leq T+1, |k-k0| \leq b\} \leq \tau_T(1 + |k_0| + b)$, where b is an arbitrary positive number. Then $b/M \geq 1/2\tau_T$ for b sufficiently large. The Peano Existence Theorem (Hartman [33, pg. 10]) now shows that any solution to $\dot{k} = f(k,t)$ with $k(0) = x$ on $[0,T)$ can be extended to $[0, T + 1/2\tau_T)$. Thus a solution $k(t|x)$ exists on $[0,\infty)$.[17] As f is increasing, this solution is unique [33, pg. 34]. It follows that $v(t) \leq k(t|x)$ whenever $\dot{v} \leq f(v,t)$ with $v(0) \leq x$ [33, pg. 26]. Hence $0 \leq \dot{v} \leq \tau(t)(1 + v(t)) \leq \tau(t)(1 + k(t|x)) = \mu_x(t)$ whenever $v \in \mathbf{A}(x)$. Since μ_x is continuous, it is locally integrable, and the third condition is met. If the production function is not increasing, maximal solutions to the equation $\dot{k} = f(k,t)$ [33, Chapter 3] can be used to establish the same result.

The advantage of the above approach is that it yields the tightest possible bound on μ_x. Our strategy is to use μ_x to obtain (F.ii'). For this it is important that μ_x be as small as possible. The examples following Corollary 3 will further illustrate this point. A cruder bound can easily be obtained. Solving the differential equation $\dot{k} = \tau(t)(1+k)$ shows that feasible paths must obey $k(t) \leq (1+x)\exp(\int_0^t \tau(s)\,ds) - 1$.

One model that gives rise to a standard technology is the affine technology $f(k) = rk + w$, where $r > \beta$ and $w \geq 0$ are given parameters. A variant of this model has been investigated by Uzawa [67] and Nairay [51].[18] It can be regarded as a deterministic version of the income fluctuation problem studied by Schechtman [58] and Schechtman and Escudero [59]. Following the above technique yields the upper bounds $[x + (w/r)]e^{rt} - (w/r)$ for the capital stock and $(rx + w)e^{rt}$ for investment. Note that this example does not satisfy the Inada Conditions at 0 or "infinity." These conditions are $f'(0+) = \infty$ and $f'(k) - \beta < 0$ for k sufficiently large. Although the Inada conditions are important for demonstrating the existence of a steady state, they are not relevant for the existence of optimal paths. In addition, the affine model's technology is *superproductive*: $f'(k) - \beta > 0$ for *all* $k > 0$. We could also apply this to the more general case where r and w are allowed to vary with time. This would represent the budget constraint faced by a consumer when given the time path

[17] The fact that $f(k,t) \leq \tau(t)(1+k)$ plays a crucial role. When g exhibits increasing returns to scale, the path of pure accumulation may blow up in finite time. Consider $f(k,t) = k^\alpha$ for $\alpha > 1$. The path of pure accumulation from $x = 1$ is $k^{\alpha-1} = 1/[(1-\alpha)t + 1]$. Thus $k(1/(\alpha-1)) = +\infty$.

[18] There are two important differences in Nairay's work. First, he only requires $k \geq -w/r$ rather than $k \geq 0$. This can be handled through an appropriate modification of Ω. Second, he does not impose a lower bound on $\dot{k}$. However, he uses convexity properties to show $\dot{k}$ is bounded on optimal paths. Without loss of generality we can assume k is bounded and the technology conditions are satisfied.

of interest rates and wages, as in the discrete-time Ramsey equilibrium (Becker and Foias [8], Becker, Boyd and Foias [7]).

Another example failing the Inada condition at 0 is given by $f(k) = 2k/(1+k)$. Clearly $f'(0) = 2 > 1 \geq \beta$. As $f(k) = (1/2 + 1/2k)^{-1}$, this production function is a member of the C.E.S. class with elasticity of substitution equal to $1/2$ and labor input 1. The Cobb-Douglas model has $f(k) = Ak^{\alpha}$ with parameters $A > 0$ and $0 < \alpha < 1$. When $\beta = 0$, this model is superproductive. When $\beta > 0$, the Inada condition at "infinity" implies the existence of a positive maximum sustainable stock b defined by $f(b) = \beta b$.

Non-convex technologies are also permitted. One such example would be $f(k) = \pi/4 + \arctan(k-1)$. Such convex-concave technologies were first studied by Clark [20] and Skiba [60]. This will also cover the stock non-convexities in investment models (Davidson and Harris [21]), but not flow non-convexities.

Epstein [25, 26] has introduced a generalization of Uzawa's [67] recursive utility function.[19] A felicity function, u, and a discounting function, v, are defined in terms of consumption c. We assume the felicity function is negative, continuous, concave and increasing with $u(0) > -\infty$ and the discounting function is continuous concave and increasing with $v(c) \geq v(0) = \rho > 0$ for all $c \geq 0$. Now take $L(t,k,y) = -u(f(k,t) - y - \beta k)$ and $R(t,k,y) = -v(f(k,t) - y - \beta k)$. These objectives are a continuous-time version of Koopmans' [39] recursive preferences. Note that $I(0) = u(0)/(1-\rho) > -\infty$, so good programs exist.

Examples of admissible discounting functions include: $v(c) = 1 + \arctan c^{\alpha}$ for $0 < \alpha \leq 1$; $v(c) = 1 + \operatorname{arcsec}(1+c)$, and $v(c) = 2 - e^{-c}$. The first two are bounded from above by $1 + \pi/2$, the last by 2. In all of these cases $\rho = 1$. Other examples of discounting functions may be unbounded above, for example, $v(c) = 1 + \log(1+c)$ and $v(c) = 1 + c^{\alpha}$, $0 < \alpha \leq 1$. Under a standard technology, the affine form $v(c) = a + bc$ with $a > 0, b \geq 0$ is permitted by the modified Felicity Conditions. Of course, $b = 0$ yields the additively separable utility function. The affine case can be generalized by removing the upper bound on u, although care is then needed to ensure that the modified Felicity conditions hold. At the other extreme, when u is a constant, we obtain the Epstein-Hynes [28] utility function. We will restrict our attention to felicity and discounting functions that satisfy either the modified or unmodified Felicity Conditions. However, Nairay [51] introduces a transformation that can recast certain other types of felicity functions, in particular, those considered by Uzawa [67], into our framework.

Production functions and objectives meeting the above conditions define the *standard recursive one sector model* (f, β, u, v, x).

COROLLARY 3. *Every standard recursive one sector model (f, β, u, v, x) has an optimal solution.*

This corollary generalizes previous existence theory for both the one sector Epstein model and the additively separable model to include both time varying and certain types of non-convex technologies.

With Epstein-Hynes utility ($u = -1$), any standard technology yields a standard model.

[19] Epstein's [27] generating function is analogous to Koopmans' aggregator. This permits a more general formulation of recursive utility.

To see this, note $c = f(k) - \beta k - k \leq f(k) \leq \tau(t)(1 + k(t|x))$ when the technology is standard. With v increasing, this means that $v(c) \leq v(\tau(t)[1 + k(t|x)]) = \rho_x(t)$. As $\rho_x(t))$ is continuous, $\rho_x \in L^1_{loc}$, and the Felicity Conditions are satisfied.

This corollary also applies to Uzawa models that do not satisfy the conditions of Nairay's existence theorem. He assumed u log-concave in order to quote Magill's [45] existence theorem. Consider $u(c) = -\exp\{-[c + (1/\sqrt{2})]^2\}$. This is a concave, bounded, increasing felicity function that satisfies our hypotheses, yet it is not log-concave. This example also shows our conditions are weaker than Magill's.

With additively separable utility, things become more complicated. Consider the case where $f(k,t) \leq \alpha(1+k)$, so $k \leq (1+x)e^{\alpha t}$ as above. Suppose $u \geq 0$, and define the *upper asymptotic exponent* by $\eta = \limsup_{c\to\infty}[\log u(c)/\log c]$. Hence $[|u(c)|\exp(-rt)]$ is bounded at infinity by $\exp(\eta \log c - \rho)t$ and thus by $\exp(\eta\alpha - \rho)t$. The modified Felicity Conditions will be met whenever $\eta\alpha < \rho$. It is clear that the smaller the bound on k (smaller α), the easier it is to satisfy $\eta\alpha < \rho$, thus the importance of the smallest possible choice of μ_x. When $u(c) = \log c$, $\eta = 0$ and any positive discount factor ρ will yield the existence of optimal paths, provided there is a good program. Similar considerations apply when $u(c) = -c^\eta$ with $\eta < 0$. Variations on this will also work if f is time-dependent. Note the similarity to the results for discrete-time models obtained by Brock and Gale [12], McFadden [47] and Boyd [11].[20]

4.3. *Multi-Sector Models*

Models with heterogeneous goods also fall into our framework. One such model is the problem of harvesting two species in a predator-prey relationship (Haurie and Hung [35], Haurie [34]). Benhabib and Nishimura [9] give examples of n-sector models without joint production. Yet another example is the continuous-time von Neumann model used by Magill [45]. Given a production set Π, a *reduced process* is a pair $(z(t), q(t)) \in \Pi \subset \mathbb{R}^{2m}_+$, $q(t) = c(t) + \dot{z}(t)$ with $c, \dot{z} \geq 0$ where z gives the input stock and q is the flow of output. The production set Π satisfies:

Production Set.

(1) *The set* Π *is a closed convex cone.*
(2) *If* $(k,y) \in \Pi$ *and* $k = 0$*, then* $y = 0$*.*
(3) *There is an* $(k,y) \in \Pi$ *with* $y > 0$*.*
(4) *If* $(k,y) \in \Pi$*,* $k' \geq k$ *and* $0 \leq y' \leq y$*, then* $(k',y') \in \Pi$*.*

By setting $\Omega = \mathbb{R}_+ \times \Pi$, so $G(t,k) = \{y \in \mathbb{R}^m_+ : (k,y) \in \Pi\}$, we can recast the model in our framework. Suppose $G(t,k)$ is not bounded. Take $y_n \in G(t,k)$ with $\|y_n\| \to \infty$. Then $(k/\|y_n\|, y_n/\|y_n\|) \in \Pi$ since Π is a cone. Let y be a cluster point of $y_n/\|y_n\|$. Then $(0,y) \in \Pi$ since Π is closed. Thus $y = 0$ by condition (2). But $\|y\| = 1$, and this contradiction shows $G(t,k)$ is bounded. As $G(t,k)$ is clearly closed, the investment correspondence is compact-valued.

An example of such a production set is the von Neumann [69] technology given by $\Pi =$

[20] Other types of one-sector models, such as the joint production model of Liviatan and Samuelson [43], also fit neatly into our framework.

$\{(k,y) : (-k,y) \leq (-A,B)z, z \geq 0, y \geq 0\}$ where (A,B) is a pair of $n \times m$ matrices with non-negative entries such that for any j there is an i with $a_{ij} > 0$ and for any i there is a j with $b_{ij} > 0$.

Define $\alpha(k,y) = \sup\{\alpha \in \mathbb{R} : y \geq \alpha k\}$ whenever $(k,y) \in \Pi$. The following theorem concerning maximal balanced growth is due to von Neumann [69] and Gale [31].

Von Neumann Equilibrium Theorem. *There exists a vector of prices p^*, an interest rate r^*, a capital stock k^*, and an expansion rate α^*, such that*

(1) $(r^*p^*, p^*)(-k,y) \leq 0$ *for all* $(k,y) \in \Pi$.
(2) $\alpha^* = \alpha(k^*,y^*) = \sup\{\alpha(k,y) : (k,y) \in \Pi, (k,y) \neq 0\}$ *where* $y^* = \alpha^* k^*$.
(3) $0 < \alpha^* = r^* < \infty$, $p^* \geq 0$, $k^* \geq 0$.

Magill [45] considers models that satisfy the KMT condition $p^*y^* > 0$ (see Kemeny, Morgenstern and Thompson [38] or Takayama [64]). Either regularity ($k > 0$ whenever $(k,y) \in \Pi$ and $\alpha(k,y) = \alpha^*$) or Gale's [31] irreducibility criterion implies the KMT condition. For k in the attainable set $\mathbf{A}(x) = \{k \in \mathcal{A} : (k,\dot{k}) \in \Pi, \dot{k} \geq 0 \text{ and } 0 \leq k(0) \leq x\}$, Magill [45] shows $\max\{|\dot{k}(t)|, |c(t)|\} \leq A\exp[(\alpha * + \epsilon)t]$ a.e. for any $\epsilon > 0$. An examination of his proof reveals that ϵ can be taken as zero when p^* is strictly positive. Define $R(k,y) = \sup\{v(c) : (k, y+c) \in \Pi, c \in \mathbb{R}^n\}$, where v is any Epstein-Hynes discounting function defined over n consumption goods. Taking $\mu_x(t) = A\exp[(\alpha * + \epsilon)t]$ and $\rho_x(t) = v(\mu_x(t), \ldots, \mu_x(t))$ satisfies both the Technology and Felicity Conditions. Thus optimal paths exist.

In the additively separable case, curvature conditions (asymptotic elasticity, asymptotic exponent) are required for existence of optimal paths. Under the appropriate curvature conditions, existence follows in a straightforward manner without using the complex weighting schemes employed by Magill [45].

All these results, in both one- and multi-sector models, also apply to time-varying felicity and discounting functions. One way these might arise is through an uncertain lifetime (Yaari [71]). When applied to recursive utility it results in one of the generalized recursive utility functionals studied by Streufert [61, 62]. The same technique, when applied to additively separable utility, yields a time-varying discount factor à la McKenzie [48] or Mitra [49]. If, in addition, the probability of death has a Poisson distribution, the uncertain lifetime simply alters the discount factor (Blanchard [10], Chang [18]).

5. Conclusion

We conclude the paper by mentioning several problems for further investigation. One problem is to consider recursive preferences defined by a generating function (Epstein [27]). Epstein restricted his attention to Mackey-continuous, concave utility functions. These are automatically weakly upper-semicontinuous. Of course, generating functions could be considered in our reduced-form framework. The question of what conditions on the generating function ensure weak upper-semicontinuity remains open. The combination of Cesari's theorem with monotonicity is not available there. An alternative method would have to be found. The more general lower closure theorems for orientor fields based on weak convergence (see Cesari [17]) may prove of use in this future project.

A second problem is to extend the existence theorem to allow jumps in the state variables. Time-additive utility models with state constraints, such as irreversible investment, are known to exhibit state jumps when the dynamic process has a binding constraint. This problem is particularly acute in the realm of perfect foresight equilibrium analysis.[21]

A third problem is to permit the horizon length to be determined endogenously as part of the solution. The optimal horizon may be finite or infinite. The natural resource literature provides a number of interesting models where the horizon is unbounded. Significant implications of the model may depend on whether or not the terminal time is finite.[22]

The existence theory in our paper places the analysis of a class of recursive utility models on a firm foundation. A fourth problem would be to start the analysis of the optima themselves. Sensitivity analysis is an obvious first step. What is the effect of the initial stock on the feasible set, the value function and optimal paths? How do continuous changes in the felicity function affect the value function and optimal paths?

A fifth problem would be to demonstrate the existence of shadow prices or dual variables which support an optimum.[23] Previous work in the area of recursive, but not time-additive, utility has focused on applications of the Volterra variational derivative in aggregate models (see Wan [70], Ryder and Heal [57], Epstein and Hynes [28] and Sung [63]). Duality methods for the recursive case have been explored under the hypothesis of a twice continuously differentiable value function and single capital good. Epstein and Hynes [28] combined this with standard Pontryagin conditions. Alternatively, Chang [19] followed a dynamic programming approach. Many questions about duality theory for recursive utility remain open. What is the correct form of the dual for the general recursive utility programming problem? Do supporting prices exist which obey a form of the no-arbitrage and transversality conditions? Can support prices be used to completely characterize an optimum? Support prices are linear functionals and therefore are elements of a dual space. Are those support prices representable in an economically interesting form? For example, are they represented as an integral functional? These are possible problems for future work on the theoretical foundations of optimal capital accumulation with a recursive objective.

Appendix

The procedure we use to show that the $\mathcal{C}$ and weak topologies are not equivalent is adapted from standard arguments showing the weak and norm topologies are not equivalent on infinite-dimensional Banach spaces. In fact, weak topologies are usually not characterized by sequences, and thus not metrizable. We will show that, contrary to Balder, the weak topology is not stronger than the $\mathcal{C}$ topology.

[21] Araujo and Scheinkman [1] and Romer [56] may be consulted for a detailed discussion of the economic importance of the jump case. Existence theory for this problem in time-additive models was the subject of Murray [50].

[22] Toman [66] provides examples and existence results for a class of time-additive models with an unbounded horizon.

[23] Duality theory for concave time-additive models has been the subject of many papers. The paper by Araujo and Scheinkman [1] represents the current state of that research.

Suppose the contrary and let $V = \{f \in \mathcal{A} : \|f\|_{\infty,1} < 1\}$. This is open in $\mathcal{C}$, thus there is a basic weakly open set N with $N \subset V$. By the definition of the weak topology there are $\epsilon, m > 0$ and $g_n \in L^\infty$ with compact support, $n = 1, \ldots, m$ with $N = \{f \in \mathcal{A} : |f(0)| < \epsilon$ and $|\int_0^\infty \dot{f} g_n| < \epsilon$ for $n = 1, \ldots, m\}$.

We can regard g_n as a linear functional on $\mathcal{A}$. As such, its null space has codimension at most one. Thus the intersection of the null spaces of the g_n has codimension at most m. As $\mathcal{A}$ is infinite-dimensional, there is a non-zero $h \in \mathcal{A}$ with $\int_0^\infty \dot{h} g_n = 0$ for all $n = 1, \ldots, m$. Now let $H = 2h/\|h\|_{\infty,1}$. Clearly $H \in N$ but $H \notin V$. This contradiction shows that the weak topology cannot be stronger than the $\mathcal{C}$ topology on $\mathcal{A}$.

In fact, we have shown that any basic, weakly open neighborhood of zero contains functions that are not in V. Thus we can construct a net that converges weakly to zero, but does not converge in $\mathcal{C}$. Simply use the weakly open neighborhoods of zero as the directed set and, for each neighborhood N, let x_N be an element of N that is not in V.

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