APPENDIX TO CHAPTER

4 Measuring Interest-Rate Risk: Duration

In our discussion of interest-rate risk, we saw that when interest rates change, a bond with a longer term to maturity has a larger change in its price and hence more interest-rate risk than a bond with a shorter term to maturity. Although this is a useful general fact, in order to measure interest-rate risk, the manager of a financial institution needs more precise information on the actual capital gain or loss that occurs when the interest rate changes by a certain amount. To do this, the manager needs to make use of the concept of duration, the average lifetime of a debt security's stream of payments.

The fact that two bonds have the same term to maturity does not mean that they have the same interest-rate risk. A long-term discount bond with ten years to maturity, a so-called *zero-coupon bond*, makes all of its payments at the end of the ten years, whereas a 10% coupon bond with ten years to maturity makes substantial cash payments before the maturity date. Since the coupon bond makes payments earlier than the zero-coupon bond, we might intuitively guess that the coupon bond's *effective maturity*, the term to maturity that accurately measures interest-rate risk, is shorter than it is for the zero-coupon discount bond.

Indeed, this is exactly what we find in example 1.

APPLICATION Rate of Capital Gain

Calculate the rate of capital gain or loss on a ten-year zero-coupon bond for which the interest rate has increased from 10% to 20%. The bond has a face value of $1,000.

Solution

The rate of capital gain or loss is −49.7%.

$$g = \frac{P_{t+1} - P_t}{P_t}$$

where

$$P_{t+1} = \text{price of the bond one year from now} = \frac{\$1{,}000}{(1 + 0.20)^9} = \$193.81$$

$$P_t = \text{price of the bond today} = \frac{\$1{,}000}{(1 + 0.10)^{10}} = \$385.54$$

Thus:

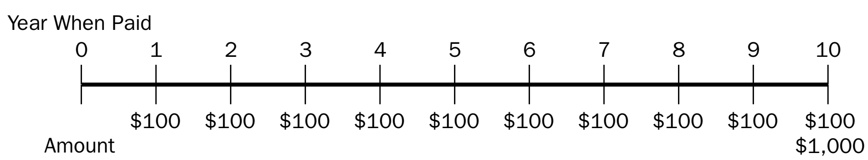
$$g = \frac{\$193.81 - \$385.54}{\$385.54}$$

$$g = -0.497 = -49.7\%$$

But as we have already calculated in Table 2 in Chapter 4, the capital gain on the 10% ten-year coupon bond is -40.3%. We see that interest-rate risk for the ten-year coupon bond is less than for the ten-year zero-coupon bond, so the effective maturity on the coupon bond (which measures interest-rate risk) is, as expected, shorter than the effective maturity on the zero-coupon bond.

Calculating Duration

To calculate the duration or effective maturity on any debt security, Frederick Macaulay, a researcher at the National Bureau of Economic Research, invented the concept of duration more than half a century ago. Because a zero-coupon bond makes no cash payments before the bond matures, it makes sense to define its effective maturity as equal to its actual term to maturity. Macaulay then realized that he could measure the effective maturity of a coupon bond by recognizing that a coupon bond is equivalent to a set of zero-coupon discount bonds. A ten-year 10% coupon bond with a face value of $1,000 has cash payments identical to the following set of zero-coupon bonds: a $100 one-year zero-coupon bond (which pays the equivalent of the $100 coupon payment made by the $1,000 ten-year 10% coupon bond at the end of one year), a $100 two-year zero-coupon bond (which pays the equivalent of the $100 coupon payment at the end of two years), … , a $100 ten-year zero-coupon bond (which pays the equivalent of the $100 coupon payment at the end of ten years), and a $1,000 ten-year zero-coupon bond (which pays back the equivalent of the coupon bond's $1,000 face value). This set of coupon bonds is shown in the following time line:



This same set of coupon bonds is listed in column (2) of Table 1, which calculates the duration on the ten-year coupon bond when its interest rate is 10%.

To get the effective maturity of this set of zero-coupon bonds, we would want to sum up the effective maturity of each zero-coupon bond, weighting it by the percentage of the total value of all the bonds that it represents. In other words, the duration of this set of zero-coupon bonds is the weighted average of the effective maturities of the individual zero-coupon bonds, with the weights equaling the proportion of the total value represented by each zero-coupon bond. We do this in several steps in Table 1. First we calculate the present value of each of the zero-coupon bonds when the interest rate is 10% in column (3). Then in column (4) we divide each of these present values

TABLE 1 Calculating Duration on a $1,000 Ten-Year 10% Coupon Bond When Its Interest Rate Is 10%

(1) Year	(2) Cash Payments (Zero-Coupon Bonds) ($)	(3) Present Value (*PV*) of Cash Payments (*i* = 10%) ($)	(4) Weights (% of total *PV* = *PV*/$1,000) (%)	(5) Weighted Maturity (1 × 4)/100 (years)
1	100	90.91	9.091	0.09091
2	100	82.64	8.264	0.16528
3	100	75.13	7.513	0.22539
4	100	68.30	6.830	0.27320
5	100	62.09	6.209	0.31045
6	100	56.44	5.644	0.33864
7	100	51.32	5.132	0.35924
8	100	46.65	4.665	0.37320
9	100	42.41	4.241	0.38169
10	100	38.55	3.855	0.38550
10	1,000	385.54	38.554	3.85500
Total		1,000.00	100.000	6.75850

by $1,000, the total present value of the set of zero-coupon bonds, to get the percentage of the total value of all the bonds that each bond represents. Note that the sum of the weights in column (4) must total 100%, as shown at the bottom of the column.

To get the effective maturity of the set of zero-coupon bonds, we add up the weighted maturities in column (5) and obtain the figure of 6.76 years. This figure for the effective maturity of the set of zero-coupon bonds is the duration of the 10% ten-year coupon bond because the bond is equivalent to this set of zero-coupon bonds. In short, we see that ***duration is a weighted average of the maturities of the cash payments***.

The duration calculation done in Table 1 can be written as follows:

$$DUR = \sum_{t=1}^{n} t \frac{CP_t}{(1+i)^t} \Big/ \sum_{t=1}^{n} \frac{CP_t}{(1+i)^t} \tag{1}$$

where DUR = duration
t = years until cash payment is made
CP_t = cash payment (interest plus principal) at time t
i = interest rate
n = years to maturity of the security

This formula is not as intuitive as the calculation done in Table 1, but it does have the advantage that it can easily be programmed into a calculator or computer, making duration calculations very easy.

If we calculate the duration for an 11-year 10% coupon bond when the interest rate is again 10%, we find that it equals 7.14 years, which is greater than the 6.76 years for the ten-year bond. Thus we have reached the expected conclusion: ***All else being equal, the longer the term to maturity of a bond, the longer its duration.***

You might think that knowing the maturity of a coupon bond is enough to tell you what its duration is. However, that is not the case. To see this and to give you more practice in calculating duration, in Table 2 we again calculate the duration for the ten-year 10% coupon bond, but when the current interest rate is 20%, rather than 10% as in Table 1. The calculation in Table 2 reveals that the duration of the coupon bond at this higher interest rate has fallen from 6.76 years to 5.72 years. The explanation is fairly straightforward. When the interest rate is higher, the cash payments in the future are discounted more heavily and become less important in present-value terms relative to the total present value of all the payments. The relative weight for these cash payments drops as we see in Table 2, and so the effective maturity of the bond falls. We have come to an important conclusion: ***All else being equal, when interest rates rise, the duration of a coupon bond falls.***

The duration of a coupon bond is also affected by its coupon rate. For example, consider a ten-year 20% coupon bond when the interest rate is 10%. Using the same procedure, we find that its duration at the higher 20% coupon rate is 5.98 years versus 6.76 years when the coupon rate is 10%. The explanation is that a higher coupon rate means that a relatively greater amount of the cash payments are made earlier in the life of the bond, and so the effective maturity of the bond must fall. We have thus established a third fact about duration: ***All else being equal, the higher the coupon rate on the bond, the shorter the bond's duration.***

TABLE 2 Calculating Duration on a $1,000 Ten-Year 10% Coupon Bond When Its Interest Rate Is 20%

(1) Year	(2) Cash Payments (Zero-Coupon Bonds) ($)	(3) Present Value (*PV*) of Cash Payments (i = 20%) ($)	(4) Weights (% of total *PV* = *PV*/$580.76) (%)	(5) Weighted Maturity (1 × 4)/100 (years)
1	100	83.33	14.348	0.14348
2	100	69.44	11.957	0.23914
3	100	57.87	9.965	0.29895
4	100	48.23	8.305	0.33220
5	100	40.19	6.920	0.34600
6	100	33.49	5.767	0.34602
7	100	27.91	4.806	0.33642
8	100	23.26	4.005	0.32040
9	100	19.38	3.337	0.30033
10	100	16.15	2.781	0.27810
10	$1,000	161.51	27.808	2.78100
Total		580.76	100.000	5.72204

STUDY GUIDE

To make certain that you understand how to calculate duration, practice doing the calculations in Tables 1 and 2. Try to produce the tables for calculating duration in the case of an 11-year 10% coupon bond and also for the 10-year 20% coupon bond mentioned in the text when the current interest rate is 10%. Make sure your calculations produce the same results found in this appendix.

One additional fact about duration makes this concept useful when applied to a portfolio of securities. Our examples have shown that duration is equal to the weighted average of the durations of the cash payments (the effective maturities of the corresponding zero-coupon bonds). So if we calculate the duration for two different securities, it should be easy to see that the duration of a portfolio of the two securities is just the weighted average of the durations of the two securities, with the weights reflecting the proportion of the portfolio invested in each.

APPLICATION Duration

A manager of a financial institution is holding 25% of a portfolio in a bond with a five-year duration and 75% in a bond with a ten-year duration. What is the duration of the portfolio?

Solution

The duration of the portfolio is 8.75 years.

$$(0.25 \times 5) + (0.75 \times 10) = 1.25 + 7.5 = 8.75 \text{ years}$$

We now see that ***the duration of a portfolio of securities is the weighted average of the durations of the individual securities, with the weights reflecting the proportion of the portfolio invested in each***. This fact about duration is often referred to as the *additive property of duration*, and it is extremely useful, because it means that the duration of a portfolio of securities is easy to calculate from the durations of the individual securities.

To summarize, our calculations of duration for coupon bonds have revealed four facts:

1. The longer the term to maturity of a bond, everything else being equal, the greater its duration.
2. When interest rates rise, everything else being equal, the duration of a coupon bond falls.
3. The higher the coupon rate on the bond, everything else being equal, the shorter the bond's duration.
4. Duration is additive: The duration of a portfolio of securities is the weighted average of the durations of the individual securities, with the weights reflecting the proportion of the portfolio invested in each.

Duration and Interest-Rate Risk

Now that we understand how duration is calculated, we want to see how it can be used by managers of financial institutions to measure interest-rate risk. Duration is a particularly useful concept, because it provides a good approximation, particularly when interest-rate changes are small, for how much the security price changes for a given change in interest rates, as the following formula indicates:

$$\%\Delta P \approx -DUR \times \frac{\Delta i}{1+i} \tag{2}$$

where $\%\Delta P = (P_{t+1} - P_t)/P_t$ = percent change in the price of the security from t to $t+1$ = rate of capital gain
DUR = duration
i = interest rate

APPLICATION Duration and Interest-Rate Risk

A pension fund manager is holding a ten-year 10% coupon bond in the fund's portfolio and the interest rate is currently 10%. What loss would the fund be exposed to if the interest rate rises to 11% tomorrow?

Solution

The approximate percentage change in the price of the bond is −6.15%.

As the calculation in Table 1 shows, the duration of a ten-year 10% coupon bond is 6.76 years.

$$\%\Delta P \approx -DUR \times \frac{\Delta i}{1+i}$$

where DUR = duration = 6.76
Δi = change in interest rate = 0.11 − 0.10 = 0.01
i = current interest rate = 0.10

Thus:

$$\%\Delta P \approx -6.76 \times \frac{0.01}{1+0.10}$$

$$\%\Delta P \approx -0.0615 = -6.15\%$$

APPLICATION Duration and Interest-Rate Risk

Now the pension manager has the option to hold a ten-year coupon bond with a coupon rate of 20% instead of 10%. As mentioned earlier, the duration for this 20% coupon bond is 5.98 years when the interest rate is 10%. Find the approximate change in the bond price when the interest rate increases from 10% to 11%.

Solution

This time the approximate change in bond price is -5.4%. This change in bond price is much smaller than for the higher-duration coupon bond:

$$\%\Delta P \approx -DUR \times \frac{\Delta i}{1 + i}$$

where DUR = duration = 5.98
Δi = change in interest rate = $0.11 - 0.10 = 0.01$
i = current interest rate = 0.10

Thus:

$$\%\Delta P \approx -5.98 \times \frac{0.01}{1 + 0.10}$$

$$\%\Delta P \approx -0.054 = -5.4\%$$

The pension fund manager realizes that the interest-rate risk on the 20% coupon bond is less than on the 10% coupon, so he switches the fund out of the 10% coupon bond and into the 20% coupon bond.

Applications 3 and 4 have led the pension fund manager to an important conclusion about the relationship of duration and interest-rate risk: ***The greater the duration of a security, the greater the percentage change in the market value of the security for a given change in interest rates. Therefore, the greater the duration of a security, the greater its interest-rate risk.***

This reasoning applies equally to a portfolio of securities. So by calculating the duration of the fund's portfolio of securities using the methods outlined here, a pension fund manager can easily ascertain the amount of interest-rate risk the entire fund is exposed to. As we will see in Chapter 9, duration is a highly useful concept for the management of interest-rate risk that is widely used by managers of banks and other financial institutions.