THE EFFECT OF ADDITIVE RATE SHOCKS ON DURATION AND IMMUNIZATION: EXAMINING THE THEORY

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ABSTRACT

This paper examines the Macaulay, Hicks, and continuous time formulations of duration by analyzing the term structure dynamics that are necessary to derive these mathematical models. The results presented help to explain the true effects of interest rate risk upon the values of fixed-income securities, as well as provide a rationale for the development of duration models in a general equilibrium framework. In particular, we show that the Hicksian formulation of duration is derived from a uniform infinitesimal additive shock to all yields to maturity on a flat yield curve, while the Macaulay duration is derived from a uniform infinitesimal additive shock to all spot rates. In addition, the Macaulay duration is shown to be inconsistent with a uniform infinitesimal additive shock to any single period spot or forward rate spanning a period less than the maturity of the security. The continuous time representation of duration is examined to show the immunizing condition and how this result is related to the Macaulay and Hicksian durations through the phenomenon of post-shift convexity of investment value. The appendices provide more detailed proofs of the critical concepts discussed in this paper.

INTRODUCTION

By analyzing the effects of additive rate shocks to the term structure it is possible to determine the theoretical limitations of the Macaulay, Hicks, and continuous time formulations of duration, as well as to develop the phenomenon of disequilibrium resulting from the post-shift convexity of investment value. Moreover, these results help to explain both the true effects of interest rate risk upon the values of fixed income securities and the effects of the limiting assumptions on the derived theoretical models. The discussion and analysis presented here begins by examining the measure of duration formulated within the traditional yield to maturity or "internal rate of return" framework. We then allow the term structure to take on unique rates of return for each discounting period and analyze the implications of uniform and single period additive shocks upon duration and immunization strategies.

DURATION

Introduction

In his seminal study on interest rates and bond yields, Macaulay (1938, p. 44) developed several measures of "the time element of a loan". He named these measures "duration". After rejecting a future value weighting model as inappropriate, Macaulay found that weighting each time period by the proportion of each payment, in present value terms, to the present value of the total payment stream resulted in a measure of duration exhibiting the desired properties. In particular, while the maturity of a loan only considers when the final payment occurs, duration assigns weights to the time periods when each cash payment occurs thereby considering the timing and size of the payments. This property of duration is important because it measures the <u>effective</u> life of a payment stream rather than the maturity of the stream. In this way, Macaulay's duration can be considered as expressing the "time element of a loan" more precisely than does maturity.

Given a set of certain payments X(t), t=1,2,...,n, where n is the maturity of the payment stream, and a present value function P(t), where P(t) gives the present value of one dollar to be received at time t, the Macaulay formulation of duration can be expressed as:

$$D_1 = \Sigma_t tX(t)P(t)/\Sigma_t X(t)P(t) \tag{1}$$

The present value function P(t) is inferred from the entire term structure of interest rates, therefore the analysis of D_1 requires an assumption about the term structure behavior. The analysis of duration measures and the associated behavior assumptions will be examined below.

When the discount rate implicit in the present value function P(t) is the constant and continuous "internal rate of return" or yield to maturity "r" then (1) simplifies to:

$$D_2 = \Sigma_t tX(t)\exp[-rt]/\Sigma_t X(t)\exp[-rt] \tag{2}$$

The yield to maturity, r, can be determined by simply knowing the cash flow stream to be received and the present value of this stream. Since the latter are generally easier to obtain than the present value function P(t), D_2 provides a simplified function that is employed in most duration analysis.

<u>Hicksian "Average Period"</u>

In his classic study of value and capital, Hicks (1939, p.186) derives a measure which is equivalent to D_2, referring to this measure as the "average period". One can develop this concept by finding the change in the present value of a certain payment stream with respect to the change in the yield to maturity r. Specifically, let V equal the present value of the certain future payments X(t), t=1,2,...,n, where n is the maturity of the payment stream.[1]

$$V = \Sigma_t X(t)\exp[-rt]$$

(3)

By taking the derivative of V with respect to r one obtains an expression for an infinitesimal change in V resulting from an infinitesimal change in the yield r.

$$dV = \Sigma_t -tX(t)\exp[-rt]dr$$

(4)

Dividing through by V one obtains the desired expression relating the percentage change in V due to a change in the yield r.

$$dV/V = \Sigma_t -tX(t)\exp[-rt]dr/\Sigma_t X(t)\exp[-rt]$$

(5)

By substituting D_2 from equation (2), the above expression simplifies to:

$$dV/V = -D_2 dr. \qquad (6)$$

Thus, for a given infinitesimal change in the yield, dr, the percentage change in the present value of a certain payment stream, dV/V, is proportional to the Hicksian "average period". This relationship underlies the basic notion that duration is a proxy for the riskiness of a fixed income security.[2] As a practical matter, in order for D_2 to be a simplification of D_1, dr should be uniform (equivalent) for all certain payment streams; without the uniform condition one would need to determine the relationship between the yields of different securities in order to specify the <u>relative</u> interest rate risk between the securities. This concept is easily demonstrated by an example. Consider two default-free bonds with different maturities but possessing equivalent durations, as given by (6). If one bond receives a larger shock to its yield than the other bond, then by (6) the first security will exhibit a greater percentage change in value than the latter security. In this example, D_2 can not be an appropriate measure of the relative risk between the two securities unless the relationship between the change in the two yields is known. Thus, when dr is not uniform for all securities it is necessary to know the relationship between the change in the yields of the different securities and, in this case, D_2 will no longer be a practical simplification of D_1.

To demonstrate formally the theoretical limitation of the uniformity of dr for the Hicksian "average period", Ingersoll, Skelton and Weil (ISW) (1978, p. 631) prove the following theorem:

> "Yields to maturity on all assets with known fixed payments can change by the same amount if and only if the yield curve is flat (yields to maturity on pure discount bonds of all maturities are the same) and makes a parallel shift."

The significant result of this theorem is that the yield curve must be flat in order for dr to be uniform for all payment streams. This condition is a severe restriction to place on the term structure and is obviously contradictory to observed behavior.[3]

In order to prove this theorem ISW analyze a change in the value of a portfolio consisting of two pure discount bonds following a uniform infinitesimal additive shock to the term structure. Specifically, let the present value of one dollar to be received in t_i years be expressed as $V = \exp[R(t_i)t_i]$ where $R(t_i)$ is the implicit spot rate for t_i years. When the yield curve is flat then $R(t_i) = r$ and all spot rates will equal the yield to maturity, r, before the term structure shift, and will equal r+dr after the shift, due to the dynamics of the assumed term structure shock. Therefore, the above specifications are sufficient conditions for the theorem to hold. The necessary condition that the yield curve is flat and makes a parallel shift, following ISW, is proven by contradiction. Assume the yield curve is not flat and all yields and spot rates undergo a uniform infinitesimal additive shock. Consider a portfolio of two pure discount bonds paying one dollar with certainty at times t_1 and t_2, where $t_1 \neq t_2$ and $R(t_1) \neq R(t_2)$. (Note that assuming $R(t_1) \neq R(t_2)$ causes $r \neq R(t_i)$, i=1,2.) The present value of the portfolio can be expressed in two equivalent formulations since the interest rate can be measured by either the spot rates or the yield.

$$V = \exp[-R(t_1)t_1] + \exp[-R(t_2)t_2] \quad (7a)$$

$$= \exp[-rt_1] + \exp[-rt_2] \quad (7b)$$

The assumed shift in the yield curve (all yields to maturity and all spot rates receive a uniform infinitesimal additive shock) implies that $dr=dR(t_1)=dR(t_2)$. Given the assumed yield curve shift, the change in the value of the portfolio can be found by taking the derivative of V with respect to the rate of interest.

$$dV = -\{t_1\exp[-R(t_1)t_1]dR(t_1) + t_2\exp[-R(t_2)t_2]dR(t_2)\} \quad (8a)$$

$$= -\{t_1\exp[-rt_1]dr + t_2\exp[-rt_2]dr\} \quad (8b)$$

Multiplying V in (7a,b) by t_1 and adding dV/dr from (8a,b), remembering that $dr=dR(t_1)=dR(t_2)$ from the assumed dynamics of the term structure shock, results in the expression:

$$t_1V + dV/dr = (t_1-t_2)\exp[-R(t_2)t_2] = (t_1-t_2)\exp[-rt_2] \quad (9)$$

since the remaining terms with the factor t_1 cancel. The above equality will hold if and only if $R(t_2) = r$ which, of course, is a contradiction. Therefore, yields to maturity can make an identical infinitesimal shift, $dr=dR(t_1)=dR(t_2)$, only when all yields to maturity are equal, $r=R(t_1)=R(t_2)$, which can only occur when the yield curve is flat and makes a parallel shift.

Consequently, it is apparent that the Hicksian "average period" or D_2 is not a general measure of a bond's risk, but rather is severely restricted by the assumption of a uniform infinitesimal shift of all yields on a flat (parallel) yield curve. The restrictive assumptions exist because the yield to maturity is adopted as the conventional interest rate measure.

Macaulay's Duration

The knowledge of distinct spot discount rates provides additional information enabling the financial economist to value fixed income securities without relying on the yield to maturity as a measure of the rate of interest. The Macaulay formulation of duration, D_1, incorporates information of the distinct spot or forward rates within the model by defining individual discount functions for each future time period. Therefore, the Macaulay D_1 formulation may prove to be more useful than the Hicksian D_2 formulation, since the latter must be derived under restrictive term structure assumptions. However, here we show that Macaulay's duration is not a panacea for dealing with interest rate risk, as it is also subject to limiting assumptions in its derivation. This section begins by deriving an expression for the Macaulay duration. The analysis continues by examining a non-uniform shock to the term structure which is inconsistent with the derivation of D_1, and then considers a specific uniform term structure shock that is consistent with D_1.

Consider the present value of a pure discount bond paying one dollar with certainty at time T:

$$V = \exp[-R(t)t]$$

where R(t) is the spot rate implied by the term structure continuum and is therefore defined for each discounting period t, t=[0,T]. The expression exp[-R(t)t] is a distinct discount function defined for each discounting period implied by the initial term structure and is thus analogous to the present value function, P(t), expressed in (1). Taking the derivative of V with respect to R(t) results in an expression relating the change in V due to a change in the spot rate R(t):

$$dV = -t \exp[-R(t)t]dR(t) \tag{10}$$

Dividing through by V one obtains an expression relating the percentage change in V resulting from a change in the spot rate R(t):

$$dV/V = -t \exp[-R(t)t]dR(t)/\exp[-R(t)t] \tag{11}$$

Substituting for D_1 the above expression simplifies to:

$$dV/V = -D_1 dR(t) \tag{12}$$

Thus Macaulay's duration, D_1, is proportional to the percentage change in V. As previously noted, the R(t) values are defined by the term structure for each time period t, $t=(0,\infty]$. To understand the theoretical limitations of D_1 it is necessary to examine the effect of specific term structure shocks upon its derivation.

The condition where only a single spot rate change occurs will be analyzed to show explicitly a simple non-uniform term structure shock that is inconsistent with the derivation of D_1. In their paper on duration and the measurement of basis risk, Cox, Ingersoll and Ross (CIR) (1979, p. 52) prove the following theorem:

"a bond with a long duration will not necessarily be affected proportionally more than a bond of a short duration by a given change in the spot rate of interest...".

To understand the development of this theorem, consider a pure discount bond expressed as a function of the single period spot rate plus a geometric series of the implied forward rates. Let the present value of a pure discount bond V paying one dollar at time T be expressed as

$$V = (1+r_1)^{-1} \prod_i (1+\hat{r}_i)^{-1} \tag{13}$$

where the initial one period spot rate is r_1 and the $\hat{r}_i$ values are the implied forward rates for the time periods i-1 to i, i=2,3,...,T. Taking the derivative of V with respect to r_1, thereby treating all forward rates as if unchanged, results in the expression:

$$dV = -(1+r_1)^{-2} dr_1 \prod_i (1+\hat{r}_i)^{-1} \tag{14}$$

$$= -V dr_1/(1+r_1) \tag{15}$$

Thus, the percentage price change in the pure discount bond, assuming that only a single additive shock to the <u>initial</u> spot rate occurs, is:

$$dV/V = -dr_1/(1+r_1) \tag{16}$$

This shows that the percentage change of the bond is independent of the duration and maturity, and in fact is equivalent for all pure discount bonds <u>regardless</u> of duration, rather than being proportional to the duration. Note that this result is not contradictory to the previous derivation of D_1. That is, the percentage change in the value of a pure discount bond will be proportional to duration <u>given</u> that the spot rate defined for the entire period of the bond is the rate that undergoes the infinitesimal change.

The above clearly shows that D_1 can not be derived from a single change in the one period spot rate or, for that matter, the change in any one period forward rate. The question then arises as to what, if any, term structure behavior is consistent with the derivation of D_1.

ISW (1978, p.631) examine the above problem by showing that D_1 is proportional to the percentage change of a certain payment stream when:

"the yield curve (the continuously compounded yield-to-maturity R(t) on pure discount bonds) undergoes a uniform additive displacement dR(t) = dR for all t."[4]

ISW begin their analysis by considering the ratio of the durations of two pure discount bonds, which is mathematically proportional to the ratio of the percentage changes in the values of the two bonds. Consider the relationship between the ratio of the percentage changes in the price of the two pure discount bonds with maturities t_1 and t_2, $t_1 \neq t_2$, and the ratio of the respective derivatives shown in the following expression:

$$\frac{dV_1/V_1}{dV_2/V_2} = \frac{-t_1\ dR(t_1)}{-t_2\ dR(t_2)} \tag{17}$$

A necessary condition for Macaulay's duration, D_1, to be proportional to the percentage change in the value of a pure discount bond is that $dR(t_1) = dR(t_2)$. The above expression must equal t_1/t_2 in order for Macaulay's duration to be proportional to the percentage change in the value of a pure discount <u>bond</u> since the ratio of the percentage changes in the values of the two bonds must be proportional to the ratio of the durations of the two bonds. Thus, the above expression will equal t_1/t_2 if and only if $dR(t_1) = dR(t_2)$, which can occur only when the term structure undergoes a uniform infinitesimal additive displacement. It is important to realize that this result is obtained without making any assumptions concerning the level of any spot rate implied by the initial or post-shift term structures. Consequently, D_1 is derived under more general conditions than is D_2, since to derive D_2 one must assume that the initial and post-shift term structures are both flat.

Post-Shift Convexity Implies Arbitrage

The above discussion shows that the duration measures D_1 and D_2 are both derived by assuming a uniform infinitesimal additive displacement of all spot and forward rates implied by the initial term structure.[5] This section analyzes the consequences of relaxing this assumption by allowing the term structure to receive a _non_-infinitesimal uniform additive shock. By analyzing these term structure dynamics it can be shown that the functional form of the post-shift investment value is convex. The functional form referred to here is a ratio of the post-shift values of two different payment streams, where both streams have the same initial durations and the same initial values. The ratio, first derived by Fisher and Weil (1971), is formed by dividing the post-shift value of a multiple payment stream by the post-shift value of a single pure discount bond. The value of the Fisher-Weil ratio can show how well a portfolio of pure discount bonds replicates the price behavior of the pure discount bond following a shock to the term structure.

In their analysis of post-shift convexity, ISW contend that convexity is inconsistent with a general equilibrium behavior of the term structure; by definition a general equilibrium condition would preclude any arbitrage opportunity. ISW show that a non-infinitesimal shock to the term structure implies just such an arbitrage opportunity, thus the term structure can not behave solely in this manner and at the same time remain in equilibrium. The importance of this concept is that D_1 and D_2 are not theoretically consistent with a uniform <u>non</u>-infinitesimal additive shock to the term structure, even though uniform <u>infinitesimal</u> additive shocks are consistent with the derivation of D_1 and D_2.

ISW (1978) develop their arbitrage argument by analyzing the first and second order conditions of the Fisher-Weil ratio. First and second order conditions were first derived under equivalent specifications using forward rates of interest by Fisher and Weil (1971). Fisher and Weil also showed the post-shift value of investment to be convex with respect to the term structure shock. By applying the arbitrage argument that follows, ISW contend that uniform non-infinitesimal additive shifts can not be the only manner in which the term structure behaves.

Let $V'=\exp[-R'(t_i)t_i]$, be the post-shift value of a pure discount bond paying one dollar with certainty at time t_i following a uniform non-infinitesimal additive shock, δ, where $R'(t_i)=R(t_i)+\delta$ is the post-shift spot rate. The Fisher-Weil ratio, Q, is derived in Appendix A by incorporating the post-shift spot rate $R'(t_i)$. The ratio is constructed by dividing the post-shift value of the single pure discount bond, V, maturing at time T into the post-shift value of a portfolio consisting of two pure discount bonds, V', $Q=V'/\exp[-R'(T)T]$.

$$Q = \{(t_2-T)\exp[\delta(T-t_1)] + (T-t_1)\exp[\delta(T-t_2)]\} \; 1/(t_2-t_1) \tag{18}$$

By evaluating Q we can determine the relative effects of a uniform additive displacement of the term structure, δ, upon the values of the investment alternatives. When $\delta = 0$, Q will be equal to one or the pre-shift value. The first and second derivatives of Q with respect to δ are, respectively,

$$dQ/d\delta = K\ \{\exp[\delta(T-t_1)] - \exp[\delta(T-t_2)]\}$$

(19)

$$d^2Q/d\delta^2 = K\ \{(T-t_1)\exp[\delta(T-t_1)] + (t_2-T)\exp[\delta(T-t_2)]\}$$

(20)

where $K = (t_2-T)(T-t_1)/(t_2-t_1)$. Examining (19) and (20) we see that the post-shift value Q=1 is a unique minimum value for Q, since $dQ/d\delta \gtrless 0$ when $\delta \gtrless 0$ and $d^2Q/d\delta^2 > 0$ for all values of δ. The relationship between Q and δ is shown in Figure I.

Figure I about here

It is apparent that the post-shift value of the bond portfolio will never be less than the value of the pure discount bond maturing at time T. In fact, the greater the absolute value of δ the greater will be the profit realized by buying the bond portfolio and selling the pure discount bond V. Therefore, when δ is non-infinitesimal, the post-shift value of the bond portfolio will be greater than that of the pure discount bond maturing at time T. ISW contend that the term structure can not behave only in this manner (all spot or forward rates change by the amount δ, a constant) since, if it did, investors would hold portfolios of perpetual and instantaneous short-term pure discount bonds and sell bonds of intermediate maturities, thereby realizing a sure profit after the term structure shock. This obviously does not occur, therefore the term structure must behave in some manner other than only uniform additive non-infinitesimal displacements.

In summary, we first derived expressions for the Macaulay and Hicksian formulations of duration, and then examined the theoretical implications of additive term structure shocks upon the derivation of these models. The Hicksian formulation of D_2 is shown to be consistent with a uniform infinitesimal additive shock (parallel shift) to a flat yield curve. The Macaulay formulation of D_1 is shown to derive from a uniform infinitesimal additive shock to all spot or forward rates, and D_1 is shown to be inconsistent with a noninfinitesimal additive shock to any spot or forward rate defined for a period less than the maturity of a bond. The Fisher/Weil ratio is then examined to show the post-shift convexity of investment value contradicts a no-arbitrage argument; thus, ISW show that the term structure can not behave with only uniform non-infinitesimal additive shifts. These results prove that the Macaulay and Hicksian formulations of duration can not be a complete measure of interest rate risk inherent in fixed income securities, and that these formulations of duration can only be considered as static measures of risk resulting from unique term structure dynamics.

IMMUNIZATION

Early Development

A basic underlying principle of the immunization strategy is discussed by Samuelson (1945) in his analysis of interest rate risk and its effect upon the value of a fund with two distinct cash flow streams, one stream consisting of cash in-flows while the other stream only has cash out-flows. Taking the derivative of the fund's value with respect to the "interest rate per annum averaged over time", Samuelson (1945, p. 19) develops, except for an inconsequential error, a measure equivalent to the Hicksian formulation of duration and concludes:

> "Increased interest rates will help any organization whose (weighted) average time period of disbursements is greater than the average time period of its receipts."

This result might mislead one to conclude that interest rate risk can be eliminated by equating the duration of the receipts with the duration of the disbursements. However, if either the receipts or the disbursements consist of a multiple payment stream then one must consider the effect that the behavior of the term structure may have upon the results of applying any duration measure to achieve a balance between cash in-flows and cash out-flows.

Redington (1952 p. 289) was the first to define and use the term immunization in the context of investment valuation. He defines immunization as:

> "the investment of the assets in such a way that the existing business is immune to a general change in the rate of interest."

Redington's goal in writing for an actuarial audience was to equate the future value of an investment to future liability. In general, the objective of an immunization strategy is to invest current funds in such a manner that a future liability can be paid regardless of the subsequent behavior of interest rates. Without an immunizing technique, the payment stream earned prior to the dispensing of the outstanding liability will be subject to a stochastic reinvestment rate. Therefore, when only multiple payment streams are available it is uncertain how to initially invest ones funds to meet a future liability. Thus, an immunization strategy is intended to overcome the interest rate and price risk inherent when one makes a current investment, such that this investment will grow over time to equal a future liability.

As shown in the duration analysis above, an assumed term structure behavior implies a specific duration measure. Thus, the ability of any measure of duration to describe the price behavior of fixed-income securities will depend upon the appropriateness of the term structure shock assumed in the derivation of the duration measure. In particular, Redington obtained a conclusion similar to Fisher and Weil and ISW regarding the convexity of the post-shift value of investment, since Redington choose to simplify his analysis by adopting a measure of duration equivalent to the Hicksian formulation.[6] Moreover, several of the discussants of Redington's paper commented on his result of post-shift convexity and it is clear that the discussants were aware that Redington's result was "too good" to be accepted without question. Never-the-less, it was not until financial theorists developed a general theory of equilibrium and arbitrage arguments that the post-shift convexity of investment value was understood to be a direct result of an assumed shock to the term structure inconsistent with general equilibrium dynamics of the term structure.

Bierwag's Analysis

Bierwag (1978) helped to develop a deeper understanding of the concept of immunization by examining duration within a continuous time framework. Integrating over all spot rates on the term structure continuum permits the identification of the value of the security at any instant of time prior to maturity, given that the term structure is known. His research analyzes the post-shift value of a fixed income security with a certain payment stream following a uniform additive shock to the term structure. Following Bierwag's notation we adopt a continuous discount model with the cash flow stream being paid continuously throughout the life of the security. Let the value of such a security be expressed as:

$$V = \int_0^n C(t)\exp[-h(0,t)t]dt \tag{21}$$

where n is maturity, $h(0,t)$ is the rate of growth of an investment over the time period 0 to t, and $C(t)$ is the continuously paid cash flow stream.

The term structure is assumed to begin at an equilibrium state and there is assumed to be no interest rate uncertainty:

$$h(0,t_1)t_1 = h(0,t_0)t_0 + h(t_0,t_1)(t_1-t_0) \tag{22}$$

where $0<t_0<t_1$ and $h(t_0,t_1)(t_1-t_0)$ is the implied forward rate over the time period t_0 to t_1. If (22) does not hold then it would be possible to borrow and lend over different time periods and earn a return greater than $h(0,t_1)t_1$ over the time period t_1. This is easily shown by example. Assume that the forward rate $h(t_0,t_1)$ is such that:

$$h(0,t_1)t_1 - h(0,t_0)t_0 < h(t_0,t_1)(t_1-t_0) \tag{23}$$

In this case, one could borrow at the rate $h(0,t_1)$ for the period t_1 and invest those funds for a period of t_0 at the rate $h(0,t_1)$ and then at time t_0 reinvest the proceeds at $h(t_0,t_1)$ until time t_1. This strategy would ensure a profit since the interest earned on the borrowed funds will exceed the borrowing cost, $h(0,t_1)t_1$. In a complete and competitive market this opportunity can not exist, therefore (22) must hold if the term structure is in equilibrium.

In order to introduce interest rate uncertainty into the model, Bierwag explicitly assumes a uniform infinitesimal additive shock occurring instantly after the purchase of the security. Equilibrium once again prevails after the initial shock and the new equilibrium condition remains until the security matures. Thus, the equilibrium condition given by (22) is assumed to hold for both the pre-shift and post-shift state of the term structure. The analysis continues by considering separately the post-shift values of the reinvestment returns of the cash flow stream and the present value of the remaining cash flow payments at some instant of time after the term structure shock. Allowing the time following the term structure shock to vary permits one to identify the "average period" where the security is immunized with respect to a pure discount bond.

The next step is to isolate a time period m after the term structure shock where m<n and n is the time remaining until maturity when the term structure shock occurs. In Appendix B expressions representing the present value of the remaining cash flow stream at time m, T, and the post-shift value of the reinvested cash flow stream at time m, Q, are derived. These expressions are, respectively:

$$T(\theta) = \exp[h(o,m)m] \int_{m}^{n} C(t)\exp[-h(o,t)t]\exp[-(t-m)\theta]dt \qquad (24)$$

$$Q(\theta) = \exp[h(o,m)m] \int_{0}^{m} C(t)\exp[-h(o,t)t]\exp[(m-t)\theta]dt \qquad (25)$$

where θ is the term structure shock. The derivative of $T(\theta)$, $m<t<n$, and $Q(\theta)$, $t<m$, with respect to the additive shock θ are, respectively:

$$T'(\theta) = \exp[h(0,m)m] \int_{m}^{n} C(t)\exp[-h(0,t)t]\{-(t-m)\exp[-(t-m)\theta]\}dt \qquad (26)$$

with $T'(\theta) <0$ since $t-m>0$, and

$$Q'(\theta) = \exp[h(0,m)m] \int_{0}^{m} C(t)\exp[-h(0,t)t]\{(m-t)\exp[(m-t)\theta]\}dt \qquad (27)$$

with $Q'(\theta) >0$ since $m-t>0$.

It follows that $Q''(\theta) >0$ and $T''(\theta) >0$. Figure II shows the relationship between $T(\theta)$, $Q(\theta)$ and θ. The reinvested cash flows, $Q(\theta)$, which occur until time m, increase in value as the additive shock to the term structure increases. The opposite occurs with the value of all remaining cash flows, T(θ), since T(θ) increases as the additive shock decreases. The result is similar to Redington, Fisher and Weil, and ISW, since the above condition results in a minimum value for T(θ)+Q(θ) when θ=0, while when θ is non-infinitesimal the curve is convex. The difference in Bierwag's analysis from that of Redington, Fisher and Weil, and ISW is: (1) the consideration of duration within a continuous time framework, and (2) the distinction between reinvested cash flows and the present value of the payment stream remaining. The second distinction allows the analysis to be extended to show the immunized condition.

Figure II about here

In appendix B the post-shift value of the continuous payment stream at time m, T(θ)+Q(θ), following a uniform additive shock to the term structure, is derived. In Appendix C, T'(θ)+Q'(θ) is evaluated at θ=0, resulting in:

$$T'(0) + Q'(0) = V(m-D_3)\exp[h(0,m)m] \quad (28)$$

where D_3 is the continuous time representation of Macaulay's duration defined as:

$$D_3 = 1/V \; {}_0^{n} \; tC(t)\exp[-h(0,t)t]dt \quad (29)$$

It is apparent from (28) that an immunizing condition exists when $m=D_3$ since Q'(0) + T'(0) will then equal zero; that is the change in the values of the reinvested payment stream and the remaining unpaid future receipts, at time m, will be equal but opposite in sign, therefore canceling with one another. The net effect of equating the investment horizon to D_3 is the value at time m of the continuous payment stream is known to be at least as much as a pure discount bond maturing at time m.

The above discussion on immunization began by relating the early developments of post-shift convexity by Redington to the later contributions made by Fisher and Weil and ISW. Bierwag's continuous time representation of duration is then examined to show the immunization of a continuous payment stream against a uniform infinitesimal additive shock to the term structure. Moreover, the continuous time representation of the post-shift value of a continuous payment stream is shown to be a convex function similar to the Fisher-Weil ratio.

CONCLUSION

This paper examines a body of literature published in the late 1970's which has become an important foundation of modern duration and immunization theory. This literature was the forerunner to a major thrust of modern duration and immunization theory that examines duration measures derived under general equilibrium conditions of the term structure. In fact, the same individuals who show us the fundamentals of modern duration and immunization theory examined in this paper are the same theorists who have initiated the analysis of duration measures derived under general equilibrium conditions.[7] With this foundation, financial economists may some day develop a complete integration of investment valuation and term structure theory.

APPENDICES

These appendices are provided as an extension of the proofs appearing in the papers cited. The original proofs omit a number of steps, causing readers to find these proofs to be abstruse. The mathematical analysis that follows has been "simplified" from the original form to include the steps which had been originally omitted for brevity. A "*" indicates an equation that does not appear in the original paper.

APPENDIX A

This appendix develops the result of post-shift convexity by deriving the Fisher-Weil ratio, Q, of the post-shift value of a portfolio of two pure discount bonds to the post-shift value of a single pure discount bond. The notation adopted is that of ISW (1979) and explicitly shows the result of a uniform non-infinitesimal additive shock to the term structure.

The initial values of these alternatives are assumed to be equal:

$$V = n_1V_1 + n_2V_2 \tag{A-1}$$

where n_k is the proportion of the portfolio invested in bond k. Also, one assumes that the durations of the alternative investments are equal:

$$t_1n_1V_1/V + t_2n_2V_2/V = T \tag{A-2}$$

Solving (A-1) for n_2V_2 and substituting into (A-2) results in:

* $$n_1V_1t_1 + (V-n_1V_1)t_2 = VT \tag{A-3}$$

Rearranging terms one obtains:

$$n_1V_1 = V(t_2-T)/(t_2-t_1) \tag{A-4}$$

The post-shift value of V_i is given by the expression:

* $$V'_i = \exp[-R'(t_i)t_i] \tag{A-5}$$

where $R'(t_i)=R(t_i)+\delta$. Expanding $R'(t_i)$, V'_i becomes

$$* \quad V'_i = \exp-[R(t_i)+\delta]t_i \quad \text{(A-6)}$$

$$* \quad = \exp[-R(t_i)t_i] \exp[-\delta t_i] \quad \text{(A-7)}$$

$$= V_i \exp[-\delta t_i] \quad \text{(A-8)}$$

The post-shift value of the portfolio is then:

$$V' = n_1V'_1 + n_2V'_2 \quad \text{(A-9)}$$

$$= n_1V_1\exp[-\delta t_1] + n_2V_2\exp[-\delta t_2] \quad \text{(A-10)}$$

Substituting (A-4) and a similarly derived expression for n_2V_2 into (A-10) results in the expression:

$$* \quad V' = V(t_2-T) \exp[-\delta t_1]/(t_2-t_1) + V(t_1-T) \exp[-\delta t_2]/(t_1-t_2) \quad \text{(A-11)}$$

$$V' = \{ (t_2-T)\exp[-\delta t_1] + (T-t_1)\exp[-\delta t_2] \} V/(t_2-t_1) \quad \text{(A-12)}$$

We can compare the post-shift values of the alternative investments by analyzing the ratio, Q, of the post-shift values of the portfolio to the pure discount bond maturing at time T.

$$Q = V'/\exp[-R'(T)T] \quad \text{(A-13)}$$

$$* \quad = V'/\exp[-R(T)T]\exp[-\delta T] \quad \text{(A-14)}$$

$$= V'\exp[\delta T]/V. \quad \text{(A-15)}$$

Substituting (A-12) into the above expression for Q results in equation (18) in the text:

$$Q = \{ (t_2-T)\exp[\delta(T-t_1)] + (T-t_1)\exp[\delta(T-t_2)] \} 1/(t_2-t_1) \quad \text{(A-}$$

16)

Appendix B

This appendix, following Bierwag (1978), develops the post-shift values at time m of both the reinvested payment stream and the payment stream to be received after time m.

An additive uniform displacement, θ, of all continuous discount rates can be written as $h'(0,t) = h(0,t)+\theta$. After the term structure shock the new equilibrium condition becomes:

$$h'(0,m)m = h'(0,t)t + h'(t,m)(m-t) \qquad \text{(B-1)}$$

for $0<t<m$. Rearranging terms:

* $$h'(t,m)(m-t) = h(0,m)m - h(0,t)t + (m-t)\theta \qquad \text{(B-2)}$$

for $0<t<m$. For $m<t<n$ the post-shift condition becomes:

$$h'(0,t)t = h'(0,m)m + h'(m,t)(t-m) \qquad \text{(B-3)}$$

and by rearranging terms:

* $$h'(m,t)(t-m) = h(0,t)t - h(0,m)m + (t-m)\theta \qquad \text{(B-4)}$$

for $m<t<n$ and the period of time remaining until maturity is n-t. The post-shift value, at time m, of the remaining certain cash flows to be received after time m, where $m<t<n$, will be equal to:

$$T(\theta) = \int_m^n C(t)\exp[-h'(m,t)(t-m)]dt \qquad \text{(B-5)}$$

Substituting (B-4) into (B-5) obtains:

* $$T(\theta) = \int_m^n C(t)\exp[h(0,m)m]\exp[-h(0,t)t]\exp[-(t-m)\theta]dt \qquad \text{(B-6)}$$

The factor $\exp[h(0,m)m]$ is known at time m and therefore is a constant which may be factored outside the integral to obtain (24) in the text:

$$T(\theta) = \exp[h(0,m)m] \int_m^n C(t)\exp[-h(0,t)t]\exp[-(t-m)\theta]dt \qquad \text{(B-7)}$$

The post-shift value of all reinvested cash flows at time m, where

$0<t<m$, will be equal to:

$$Q(\theta) = \int_0^m C(t)\exp[h'(t,m)(m-t)]dt \qquad \text{(B-8)}$$

Substituting (B-1) into (B-8) obtains:

* $$Q(\theta) = \int_0^m C(t)\exp[h(0,m)m]\exp[-h(0,t)t]\exp[(m-t)\theta]dt \qquad \text{(B-9)}$$

The constant factor $\exp[h(0,m)m]$ may be factored outside the integral sign to obtain equation (25) in the text:

$$Q(\theta) = \exp[h(0,m)m] \int_0^m C(t)\exp[-h(0,t)t]\exp[(m-t)\theta]dt \qquad \text{(B-10)}$$

Appendix C

This appendix follows Bierwag's development of the continuous time representation of duration showing the immunized condition. The change in the post-shift value of $T(\theta) + Q(\theta)$ due to a change in θ is given by (26) and (27) in the text:

* $$Q'(\theta)+T'(\theta) = \exp[h(0,m)m] \left\{ \int_0^m C(t)\exp[-h(0,t)t](m-t)\exp[(m-t)\theta]dt + \int_0^m C(t)(m-t)\exp[-h(0,t)t]\exp[(m-t)\theta]dt \right\} \qquad \text{(C-1)}$$

When (C-1) is evaluated at $\theta=0$ we can obtain the immunizing condition:

* $$Q'(0) + T'(0) = \exp[h(0,m)m] \int_0^n C(t) \{ m\exp[-h(0,t)t] - t\exp[-h(0,t)t] \}dt \qquad \text{(C-2)}$$

Define the continuous time representation of duration, D_3, as:

$$D_3 = 1/V \int_0^n t\, C(t) \exp[-h(0,t)t]dt \qquad \text{(C-3)}$$

Now, since

* $$mV = \int_0^n C(t)\, m \exp[-h(0,t)t]dt \qquad \text{(C-4)}$$

expression (C-2) becomes, after substituting for D_3V and mV:

$$Q'(0) + T'(0) = (mV-D_3V) \exp[h(0,m)m] \qquad \text{(C-5)}$$

which is equivalent to (28) in the text.

FOOTNOTES

[1] In his original exposition, Hicks derives an equivalent result using discrete compounding. The continuous compounding convention used here was first derived by L. Fisher (1966).

[2] This paper does not analyze other elements of risk inherent in fixed income securities, such as default and callability risk, by assuming that the payment stream is known with certainty. The models developed in this paper are therefore better suited to analyze the behavior of default free securities.

[3] The restriction may not be a severe problem for long maturity bonds. For example, Livingston and Jain (1982) show that par bond yield curves will become flat for long maturities and Schaefer (1977) shows that yield curves become asymptotically horizontal and will approach the perpetuity yield curve at long maturities.

[4] ISW develop a corollary to the above theorem that states D_1 is proportional to the percentage change in the present value of a certain payment stream when the percentage change in the discount function, $dP(t)/P(t)$, is equivalent for all t.

[5] ISW (1979) show that uniform infinitesimal additive shock to the forward rate structure is equivalent to an identical shock to the spot rate structure.

[6] Both Fisher and Weil (1971) and ISW (1978) derive the post-shift convexity result employing Macaulay's duration D_1. Their analysis is more general but can be applied to the Hicksian measure, D_2, to obtain the same post-shift convexity result given in Redington (1952).

[7] For example, see Cox, Ingersoll and Ross (1979) and Bierwag, Kaufman and Toevs (1982).

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