A Faber–Krahn inequality for solutions of Schrödinger’s equation

L. De Carli, S.M. Hudson*

Florida International University, Math, 11200 S.W. 8th Street, Miami, FL 33199, United States

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Abstract

We consider nontrivial solutions of $-\Delta u(x) = V(x)u(x)$, where $u \equiv 0$ on the boundary of a bounded open region $D \subset \mathbb{R}^n$, and $V(x) \in L^\infty(D)$. We prove a sharp relationship between $\|V\|_\infty$ and the measure of $D$, which generalizes the well-known Faber–Krahn theorem. We also prove some geometric properties of the zero sets of the solution of the Schrödinger equation $-\Delta u(x) = V(x)u(x)$.

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1. Introduction

We study nontrivial solutions of the Schrödinger equation

$-\Delta u(x) = V(x)u(x), \quad (1.1)$

where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, which vanish on the boundary of a bounded open region $D \subset \mathbb{R}^n$, $n \geq 1$. We say that $u$ is nontrivial if it does not vanish identically in $D$. 

* Corresponding author.

E-mail addresses: decarli@fiu.edu (L. De Carli), hudsons@fiu.edu (S.M. Hudson).
We establish a sharp relationship between the potential $V$ and the measure of $D$. Let $B(0, 1)$ denote the unit ball in $\mathbb{R}^n$, and set $\omega_n = |B(0, 1)| = \frac{\pi^n}{n!}$. Let $j = j_{\frac{n}{2} - 1}$ be the first zero of the Bessel function $J_{\frac{n}{2} - 1}(x)$. Our main result is:

**Theorem 1.1.** Suppose that $u \in C(\overline{D})$ is a nontrivial solution of (1.1) in the distribution sense. Suppose that $u \equiv 0$ on $\partial D$, and $V \in L^\infty(D)$. Then

$$|D| \cdot \|V\|_\infty^{\frac{n}{2}} \geq j^n \omega_n. \tag{1.2}$$

We show below that dilations and constant multiples of

$$u_*(x) = |x|^{1-\frac{n}{2}} J_{\frac{n}{2} - 1}(|x|), \tag{1.3}$$

where $J_n(r)$ is the Bessel function of the first kind, give equality in (1.2), so the constant $C = j^n \omega_n$ in the theorem is sharp. In the proof of this theorem, and of related ones in Section 2, we can assume without loss of generality that $u > 0$ on $D$; if $u$ changes sign on $D$, we can apply the theorem on the subset where $u > 0$ instead. Note that the formula $|D| \cdot \|V\|_\infty^{\frac{n}{2}}$ is dilation-invariant, so we may also assume that $\|V\|_\infty = 1$. In [4], the authors proved (1.2), but with a smaller constant $c$. When $n = 2$, for example, we obtained $c = 4\pi$; the constant in (1.2) is $C = \pi j^2$ with $j \sim 2.4048$.

When $V(x) \equiv \lambda$, a constant, $u$ is an eigenfunction for the Dirichlet problem:

$$\begin{cases} -\Delta u(x) = \lambda u(x) & x \in D \\ u \equiv 0 & x \in \partial D. \end{cases} \tag{1.4}$$

The well-known Faber–Krahn inequality (see e.g. [2]) states that, for any bounded domain $D$ of fixed volume $|D|$, the smallest possible eigenvalue of the Dirichlet problem (1.4) occurs when $D$ is a ball. That is, if $D^*$ is the ball centered at the origin with $|D| = |D^*|$, and $\lambda_1(D)$ is the first eigenvalue of the Dirichlet problem (1.4), then $\lambda_1(D) \geq \lambda_1(D^*)$. When $D = B(0, 1)$, the smallest eigenvalue of (1.4) is $\lambda_1(D) = j^2$, and the eigenfunctions are constant multiples of $u_*(j x)$, where $u_*$ and $j$ are defined as in Theorem 1.1. Thus, our result generalizes the Faber–Krahn result, with the same extremals. Another interesting generalization of the Faber–Krahn inequality appears in [10]; assuming that $\partial D$ is smooth, $|D|$ is fixed, and $v : D \to \mathbb{R}^n$ is bounded, the smallest possible eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta u(x) + v \cdot \nabla u = \lambda u(x) & x \in D \\ u \equiv 0 & x \in \partial D, \end{cases}$$

occurs when $D$ is a ball and $v$ is constant.

Neither our Theorem 1.1 nor the Faber–Krahn inequality applies on unbounded domains in $\mathbb{R}^2$; a counterexample is given in Section 2. Our proof fails in this case mainly because it depends on Green’s identity. However, with mild assumptions on $u$ at infinity, both theorems hold when $D$ is unbounded. See Theorem 2.7. One can slightly relax the assumption that $u$ vanishes on the boundary, and then prove (1.2) with a slightly smaller constant; see Proposition 2.4 in Section 2. This is a key idea in the proofs of Theorems 1.1 and 2.7.

One can easily apply Theorem 1.1 to eigenfunctions of the operator $-\Delta - V$. Such a function satisfies $-\Delta u(x) - V(x)u(x) = \lambda u(x)$. By setting $V_2 = V + \lambda$, we are back to (1.1), with a different potential, and Theorem 1.1 implies $|D| \cdot \|V + \lambda\|_\infty^{\frac{n}{2}} \geq j^n \omega_n$.
We now show that the constant in Theorem 1.1 is best possible. Let \( u = u_\ast \), as defined in (1.3). It is radial, so we may abuse notation slightly and write \( u(x) = u(|x|) = u(r) \). Also, \( \Delta u = u_{rr} + (n - 1)r^{-1}u_r \).

Since \( J_{\frac{n}{2} - 1}(r) \) satisfies the Bessel equation \( y'' + r^{-1}y' + \left(1 - \left(\frac{n}{2} - 1\right)^2 r^{-2}\right)y = 0, u(r) \) satisfies the equation \( r^{\frac{n}{2} - 2}(ru_{rr} + (n - 1)ru_r + ru) = 0, \) which is equivalent to \( -\Delta u = u \).

Notice that \( u \) satisfies (1.1) with potential \( V = 1 \), and that \( u(j) = 0, \) so \( D = B(0, j) \). So, \( u \) is an extremal for Theorem 1.1 and (after a dilation, if necessary) for the Faber–Krahn Theorem as well.

Theorem 1.1 also holds when \( n = 1 \); the best constant is \( C = \pi \) and the basic extremal function is \( u_\ast(x) = \cos x \). The proof in this case is similar, but requires a few easy modifications, which are left to the reader.

Many authors have investigated the unique continuation properties of the solutions of (1.1). Usually, one assumes that \( V(x) \in L^p(D), u \) is in some Sobolev space, and \( D \) is connected. For example, when \( V \in L^\infty(D) \) and \( u = 0 \) on some open set \( Z \subset D \), then \( u \equiv 0 \) on \( D \). This also holds when \( Z \) is a single point, if \( u \) vanishes to infinite order at \( Z \). See for example the survey paper of Wolff [16] and the references cited there.

Theorem 1.1 can be viewed as a unique continuation property of the solutions of (1.1). If \( u \) vanishes on the boundary of \( D \), and if \( |D| \) is too small for (1.2), then \( u \equiv 0 \) in \( D \). This work originated as an attempt to bridge the gap in unique continuation results, between \( \dim(Z) = 0 \) and \( \dim(Z) = n \). A well-known special case occurs when \( V = 0 \) on \( D \), so that \( u \) is harmonic. Then, the maximum principle implies that \( u = 0 \) on \( D \). Except for this old result, not much is known about geometric properties of level curves of harmonic functions; we feel this deserves further study. See [4] for references, and also the recent preprint [7].

We prove Theorems 1.1 and 2.7 in Section 2, along with a similar result, Theorem 2.6, which does not refer to \( |D| \), but involves \( \|u\|_{L^p(D)} \) instead.

The geometric properties of the zero sets of solutions of the Schrödinger equations (1.1) are fascinating and largely unexplored, even when \( V \) is constant. When \( V \) is a nonzero eigenvalue of the Laplacian on a smooth manifold, Donnelly and Fefferman (see [5,6]) and Savo [14] have estimated the total arc length (nodal length) of \( Z \) in terms of the eigenvalue. See also [12,13], and the references cited there.

In Section 3, we examine extension properties of the zero sets of solutions of (1.1), and how they relate to known results for harmonic functions. For example, a harmonic function that vanishes on a half-line must vanish on the whole line (see [4,7]). But in Theorem 3.1, we prove that there exist solutions of (1.1) with \( V = -\frac{\Delta u}{u} \neq 0 \), that vanishes on a half line in \( \mathbb{R}^2 \), and not on the whole line.

2. Proof of Theorem 1.1 and related results

We shall study the sets \( A_\alpha = \{x \in D : u(x) > \alpha\} \) for \( \alpha \geq 0 \), and how the Lebesgue measure \( |A_\alpha| \), depends on \( \alpha \), which leads to some interesting geometric differential equations. One may approach Theorem 1.1 with an eigenfunction expansion of \( u \) (we are indebted to Leckband for this remark), or with methods explored in [3], but those methods do not give the stronger related results proved in Propositions 2.4, 2.6 and 2.7. Also, though our proposed extremal \( u_\ast \) is radial, it seems difficult or impossible to prove directly a symmetrization result. Instead, we will compare the differential equations for \( u \) with those for \( u_\ast \).

In this section, we let \( V_u = -\frac{\Delta u}{u} \); when there is no ambiguity, we may drop the subscript \( u \).

Let \( V_\ast = -\frac{\Delta u_\ast}{u_\ast} \). Below, we will define new variables for \( u \) and \( u_\ast \) simultaneously using notation
similar to this, except when this notation may lead to confusion. For example, \( \{ x : u_*(x) > \beta \} \) will be denoted by \( B_\beta \).

Multiplying by a constant, and translating if needed, we can normalize both functions so that \( \|u\|_{L^\infty} = u(0) = u_*(0) = 1 \). The theorem is also dilation-invariant, so we may also assume \( \|V\|_{L^\infty} = \|V_*\|_{L^\infty} = j^2 \).

Our goal is to prove \( |D| \geq \omega_n = |B(0, 1)| \). Since the domain of \( u_* \) is \( B(0, 1) = B_0 \), and \( |D| = |A_0| \), our goal can also be written as \( |A_0| \geq |B_0| \).

First, we consider a special case of Theorem 1.1, in which \( u \) is a Morse function, so that its critical points are isolated and non-degenerate, (see e.g. [11]). This allows integration on most of the level sets of \( u \), the ones containing no critical point. Also, we will assume for now that \( u \) has a maximum value of 1 at a unique point \( x_0 \in D \), which we may assume is \( x_0 = 0 \). We will assume that \( u \) agrees with a second-degree polynomial in some neighborhood of 0; then we say \( u \) has a polynomial cap. After proving this case in Proposition 2.1 below, and making some further modifications (see Proposition 2.4), we will provide a density argument to prove Theorem 1.1.

We will assume in the rest of the proof that \( n \geq 2 \) and leave the minor modifications required when \( n = 1 \) to the reader.

**Proposition 2.1.** Suppose that \( u \) is a Morse function with a polynomial cap, which satisfies the assumptions of Theorem 1.1. Suppose that \( \|V\|_{L^\infty} = j^2 \). Then \( |D| \geq \omega_n \).

**Proof.** Since \( u \) is continuous, \( |A_\alpha| \) is strictly decreasing with \( \alpha \). We claim it is also continuous. Clearly, \( \lim_{\alpha \to \alpha^+} |A_\alpha| = |A_*| \). Let \( L_\alpha \) be the level set where \( u = \alpha \). Since \( u \) is a Morse function, its Lebesgue measure in \( \mathbb{R}^n \) is \( |L_\alpha| = 0 \). We get

\[
\lim_{\alpha \to \alpha^-} |A_\alpha| = |\{ z : u(z) \geq \alpha \}| = |L_\alpha| + |A_\alpha| = |A_*|
\]

which proves continuity. Likewise, \( |B_\beta| \) is continuous and decreasing in \( \beta \). Since \( |B_\beta| \) is an injective function of \( \beta \), it has an inverse defined on \([0, \omega_n]\). So, we can define a continuous function \( \beta(\alpha) \) by setting

\[
|B_\beta| = |A_\alpha|,
\]

where \( 0 \leq \alpha \leq 1 \). Note that we can assume \( |A_0| \leq \omega_n \), otherwise we are done. Next, we will prove that

\[
\beta(\alpha) \leq \alpha
\]

(2.1)

for all \( 0 \leq \alpha \leq 1 \). Then, setting \( \alpha = 0 \), we get \( |A_0| = |B_\beta(0)| \geq |B_0| \), which proves Proposition 2.1. \( \square \)

To prove (2.1) we need some notation for average values. Set \( \text{avg}(\alpha) = \frac{1}{|A_\alpha|} \int_{A_\alpha} u(x)dx \).

Likewise, let \( \text{avg}_*(\beta) = \frac{1}{|B_\beta|} \int_{B_\beta} u_*(x)dx \). Recall \( L_\alpha \) is the set where \( u = \alpha \). We also let \( L_{*,\beta} \) be the sphere where \( u_* = \beta \). Let \( Q \) be the set of critical values of \( u \) (and let \( 0 \in Q \)). The critical points of \( u \) are isolated, and form a closed subset of \( D \), so \( Q \) is closed in \([0, 1]\).

**Lemma 2.2.** If \( \alpha \notin Q \) then

\[
(a) \quad \frac{d|A_\alpha|}{d\alpha} = \int_{L_\alpha} \frac{1}{|\nabla u(x)|} \, ds(x), \text{ where } ds \text{ is } (n - 1)\text{-dimensional Hausdorff measure on } L_\alpha.
\]

A similar formula holds for the level sets of \( u_* \).
By the chain rule,

\[ \frac{d\beta}{d\alpha} \geq \frac{\text{avg}_u(\beta)}{\text{avg}(\alpha)}. \]  

(2.2)

**Proof.** (a) Since \( \alpha \notin Q \), \( L_\alpha \) contains no critical points of \( u \), so \( \int_{L_\alpha} \frac{1}{|\nabla u(x)|} \, ds(x) \) is well defined, and is continuous on some interval \( \alpha_1 < y < \alpha_2 \) containing \( \alpha \). We apply the co-area formula (see e.g. [8])

\[
\int_{\Omega} g(x)|\nabla u(x)| \, dx = \int_{-\infty}^{\infty} \left( \int_{[\alpha(y)]} g(x) ds(x) \right) \, dy
\]

with \( g = \frac{1}{|\nabla u|} \) and \( \Omega = A_\alpha \setminus A_{\alpha_2} \). Since \( \{u^{-1}(y)\} = L_y \) (for \( \alpha \leq y \leq \alpha_2 \)) we get

\[
|A_\alpha| - |A_{\alpha_2}| = \int_{\alpha_2}^{\alpha_2} \int_{L_y} \frac{1}{|\nabla u(x)|} \, ds(x) \, dy.
\]

(2.3)

Part (a) follows by applying \( \frac{d\beta}{d\alpha} \) to both sides. To prove (2.2), we observe that

\[
\mathcal{H}_{n-1}(L_\alpha)^2 \leq \int_{L_\alpha} |\nabla u(x)| \, ds(x) \int_{L_\alpha} |\nabla u(x)|^{-1} \, ds(x) = IJ
\]

by Hölder’s inequality. Here \( \mathcal{H}_{n-1}(L_\alpha) \) is the \((n-1)\)-dimensional Hausdorff measure of \( L_\alpha \), and \( J = -\frac{d\alpha}{d\alpha \beta} \), by part (a). The outward normal derivative on \( L_\alpha \) is \( u_\eta = -|\nabla u| \), so by Green’s theorem,

\[
I = \int_{L_\alpha} -u_\eta(x) \, ds(x) = \int_{A_\alpha} -\Delta u \, dx \leq \|V\|_\infty \int_{A_\alpha} u \, dx = \|V\|_\infty |A_\alpha| \, \text{avg}(\alpha).
\]

A similar formula holds for \( u_\gamma \), but with equality, since \( |\nabla u_\gamma| \) is constant on each \( L_{\eta,\beta} \), and \( V_{\gamma} = -\|V\|_\infty \) is also constant. Since \( |A_\alpha| = |B_{\beta}| \), we can divide \( I \) and \( I_\gamma \) by \( \|V\|_\infty |A_\alpha| \) and get \( I \geq \frac{\text{avg}_u(\beta)}{\text{avg}(\alpha)} \). Note that \( \partial A_\alpha \subseteq L_\alpha \) and \( \partial B_{\beta} = L_{\eta,\beta} \). Since \( |A_\alpha| = |B_{\beta}| \), and \( B_{\beta} \) is a ball, the isoperimetric principle implies

\[
IJ \geq (\mathcal{H}_{n-1}(L_\alpha))^2 \geq (\mathcal{H}_{n-1}(\partial A_\alpha))^2 \geq (\mathcal{H}_{n-1}(L_{\eta,\beta}))^2 = I_\gamma J_\gamma.
\]

By the chain rule,

\[
\frac{d\beta}{d\alpha} = \frac{d\beta}{d|B_\beta|} \frac{d|B_\beta|}{d\alpha} = \frac{d\beta}{d|B_\beta|} \frac{d|A_\alpha|}{d|B_\beta|} = \frac{J}{I} \geq \frac{\text{avg}_u(\beta)}{\text{avg}(\alpha)},
\]

proving Lemma 2.2. \( \square \)

The rest of the proof of (2.1) is a propagation argument, starting at the polynomial cap, where \( \alpha = 1 \).

**Lemma 2.3.** There is an interval \( a_0 < \alpha < 1 \) where \( \frac{d\beta}{d\alpha} \) is strictly increasing with \( \alpha \).

**Proof.** With a slight abuse of notation,

\[
u_\alpha(x) = u_\alpha(|x|) = u_\alpha(r) = cr^{1 - \frac{n}{2}} J_{\frac{n}{2} - 1}(jr),
\]

where \( c \) is chosen so that \( u_\alpha(0) = 1 \). Also, recall from the introduction that \( -\Delta u_\alpha = j^2 u_\alpha \). Let the radius of the disk \( B_\beta \) be \( r_\beta \), so \( u_\alpha(r_\beta) = \beta \), and \( |B_\beta| = \omega_n r_\beta^n \). Since \( (u_\alpha)_r(0) = 0 \),
(u_*)_{rr}(0) = \lim_{s \to 0} \frac{(u_*)_{rr}(s)}{s}. So
\[ j^2 = j^2 u_*(0) = -\Delta u_*(0) = -n(u_*)_{rr}(0). \]
From the Taylor series of u_ at r = 0 (at \( \beta = 1 \)) we get
\[ \frac{du_*}{d(r^2_\beta)} = \frac{(u_*)_{rr}(0)}{2} = -\frac{j^2}{2n}, \]
and since \( \beta = u_*(r_\beta) \), we get
\[ \frac{d|B_\beta|^2}{d\beta} = \frac{d|B_\beta|^2}{d(r^2_\beta)} \frac{d(r^2_\beta)}{d\beta} = -\omega_n^2 \frac{2n}{j^2}. \] (2.4)
It is easy to check that this derivative from the left is continuous at \( \beta = 1 \).
We need a similar estimate for the derivative of \( |A_\alpha| \). Recall that \( u(x) \) attains its maximum at the origin, \( u(0) = 1 \), that \( \left| \Delta u \right| = |V| \leq j^2 \), and that in a neighborhood of the origin it has the form \( u(x) = 1 + \sum_{1 \leq i \leq k \leq n} a_{ik} x_i x_k \). Since \( u \) is Morse, the Hessian matrix of \( u \) at the origin, which we denote by \( H \), is nonsingular, and since the origin is a maximum, its eigenvalues are negative. Applying an orthogonal transformation of coordinates, we can replace \( H \) by a diagonal matrix \( H_d \). In the new coordinates, \( u(y) = 1 - \sum_{i=1}^n \lambda_i y_i^2 \), where \( -2\lambda_1 - \cdots - 2\lambda_n \) are the eigenvalues of \( H \). The ellipsoid where \( 1 + \sum_{1 \leq i \leq k \leq n} a_{ik} x_i x_k > 0 \) has the same volume as the one \( 1 - \sum_{i=1}^n \lambda_i y_i^2 > 0 \), which is \( \omega_n (\lambda_1 \cdots \lambda_n)^{-1/2} \). So,
\[ \Delta u(0) = \text{trace } H = \text{trace } H_d = -2(\lambda_1 + \cdots + \lambda_n). \]
Since \( -\Delta u(0) = V_u(0)u(0) = V_u(0) \leq j^2 \), we have
\[ \lambda_1 + \cdots + \lambda_n \leq \frac{j^2}{2}. \]
Given this constraint, the minimum possible value of \( K = (\lambda_1 \cdots \lambda_n)^{-1/2} \) occurs when every \( \lambda_i = \frac{j^2}{2n} \), so \( K \geq \left( \frac{2n}{j^2} \right)^{n/2} \). For large enough \( \alpha < 1 \), \( A_\alpha \) is a dilate of the ellipsoid mentioned above, so
\[ |A_\alpha| = \omega_n (1 - \alpha)^{n/2} K \]
and
\[ \frac{d|A_\alpha|^2}{d\alpha} = -(K \omega_n)^{n/2} \leq -\omega_n^{n/2} \frac{2n}{j^2} \] (2.5)
which also applies at \( \alpha = 1 \). We claim that equality does not occur in (2.5). This is clear, unless \( K = \left( \frac{2n}{j^2} \right)^{n/2} \), but in that case \( u \) is radial; \( u(r) = 1 - \lambda r^2 \). So, \( -\Delta u = 2n\lambda \) is constant, and \( V_u \) has a local maximum at 0. So, \( V_u(0) > -\|V_u\|_{\infty} = -j^2 \), and again equality is impossible in (2.5).
Comparing this improvement of (2.4) and (2.5), and recalling that \( |A_\alpha| = |B_\beta| \), we get
\[ \frac{d\beta}{d\alpha} = \frac{d|A_\alpha|^2}{d\alpha} \div \frac{d|B_\beta|^2}{d\beta} > 1 = \frac{\beta}{\alpha} \] (2.6)
at \( \alpha = 1 \). By continuity, \( \frac{d \beta}{d \alpha} > \frac{\beta}{\alpha} \) on some interval \((a_0, 1)\). By the quotient rule, \( \frac{d}{d\alpha} \left( \frac{\beta}{\alpha} \right) > 0 \) there, so \( \frac{\beta}{\alpha} \) increases there, proving Lemma 2.3.

Clearly, there is a smallest value of \( a_0 \in [0, 1] \) for which Lemma 2.3 holds. We now assume that \( a_0 \) is minimal, and will prove that \( a_0 = 0 \). Assume \( a_0 > 0 \) (to get a contradiction).

If \( \alpha \geq a_0 \), then by substitution, (2.6), the previous remarks, and Lemma 2.3,

\[
\text{avg}_* (\beta) = \int_{\beta}^{1} |B_b| db = \int_{\alpha}^{1} |B_{\beta(a)}| \frac{db}{da} da \geq \int_{\alpha}^{1} |A_a| \frac{b}{a} da > \frac{\beta \text{avg}(\alpha)}{\alpha}
\]

so that

\[
\frac{\text{avg}_* (\beta)}{\text{avg}(\alpha)} > \frac{\beta}{\alpha}.
\]  

(2.7)

But the terms in (2.7) are continuous, so there is some \( 0 < a_1 < a_0 \) such that (2.7) also holds for \( \alpha > a_1 \). By Lemma 2.2

\[
\frac{d \beta}{d \alpha} \geq \frac{\text{avg}_* (\beta)}{\text{avg}(\alpha)} > \frac{\beta}{\alpha}
\]

whenever \( \alpha > a_1 \) is not in \( Q \). As in the previous proof, this with the quotient rule implies that \( \frac{\beta}{\alpha} \) has a positive derivative there. There are finitely many \( \alpha > a_1 \) in \( Q \), and \( \frac{\beta}{\alpha} \) is continuous, even at those points, so it increases on \( a_1 < \alpha < 1 \). So, \( a_0 \) is not minimal for Lemma 2.3, unless \( a_0 = 0 \).

Since \( \beta(1) = 1 \), monotonicity of \( \frac{\beta}{\alpha} \) implies (2.1) and Proposition 2.1.

Now we want to use the density of Morse functions to prove Theorem 1.1. But we lose control of some potentials where \( u \approx 0 \), and we need to modify Proposition 2.1 first. In effect, we must replace \( D \) by some \( A_\epsilon \).

**Proposition 2.4.** Suppose that \( v \) is Morse on \( D \) with a polynomial cap, that \( v > \epsilon \) on \( D \) and that \( v = \epsilon \) on the boundary. Then

\[
|D| \cdot \|V_v\|_{L^\infty} \geq C_\epsilon,
\]

where \( C_\epsilon \to C \), the constant in Theorem 1.1, as \( \epsilon \to 0 \).

**Proof.** The proof of Proposition 2.4 is almost the same as the proof of Proposition 2.1, but the lower bound for \( \alpha \) is \( \epsilon \), instead of 0. Again, we can assume \( \|V_v\|_{L^\infty} = j^2 \). The propagation proof in Proposition 2.1 gives \( a_0 = \epsilon \) instead of 0. Thus, \( \frac{\beta}{\alpha} \) increases to 1 on the interval \( \epsilon \leq \alpha \leq 1 \), so that \( \beta(\epsilon) \leq \epsilon \), and \( |D| = |A_\epsilon| = |B_{\beta(\epsilon)}| \geq |B_\epsilon| \to \omega_n \), as \( \epsilon \to 0 \). Setting \( C_\epsilon = j^n |B_\epsilon| \) proves Proposition 2.4. \( \square \)

To prove Theorem 1.1, we will approximate \( u \) with a Morse function \( v \) that satisfies the assumptions of Proposition 2.4 on a set \( E \approx D \) with a potential \( V_v \approx V \) there. We will define \( v \) and \( E \) after some preliminary approximations.

**Proof of Theorem 1.1.** Let \( u \in C(D) \) with \( V \in L^\infty(D) \), \( u = 0 \) on \( \partial D \), and \( u > 0 \) on \( D \). Let \( 0 < \gamma < 0.01 \) be arbitrary, but sufficiently small. We can assume \( \|u\|_{\infty} = 1 \). By compactness, there is a \( \overline{\gamma} > 0 \), so that no ball of diameter \( \overline{\gamma} \) intersects both \( \partial D \) and \( A_{2\gamma} \). So, if \( \Omega = \{x \in D : \text{dist}(x, \partial D) > \overline{\gamma}\} \), then \( A_{2\gamma} \subseteq \Omega \). Let \( \phi(x) \in C_0^\infty(B(0, 1)) \) be a smooth positive
bump function with \( \| \phi \|_1 = 1 \). Let \( \phi_\gamma(x) = \frac{1}{\gamma^p} \phi(x/\gamma) \). For every \( \gamma \leq \gamma \), and \( y \in \Omega \), define
\[
w(y) = w_\gamma(y) = u \ast \phi_\gamma(y) = \int_{B(0, \gamma)} u(y - x) \phi_\gamma(x) \, dx \geq 0
\]
in \( C^\infty(\Omega) \). Since \( u \) satisfies (1.1) in the distribution sense, we see that
\[
|\Delta w(y)| = |(\Delta u) \ast \phi_\gamma(y)| \leq \| V \|_{L^\infty(\Omega)} (u \ast \phi_\gamma)(y) \leq \| V \|_{\infty} w(y).
\]
So, \( \| V_w \|_{\infty} \leq \| V \|_{\infty} \).

As \( \gamma \to 0 \), \( w \to u \) uniformly on \( \Omega \). For some \( 0 < \gamma < \gamma \), sufficiently small, \( \| w - u \|_{L^\infty(\Omega)} < \gamma \). The Morse functions are dense in \( C^2(\Omega) \), so we can choose one, \( m \), such that \( \| m - w \|_{C^2(\Omega)} < \frac{\gamma^2}{2} \). But we need a Morse function on \( \Omega \) with a polynomial cap. We can make \( \| u - m \|_{\infty} \) as small as we like on \( \Omega \), and know that \( u \) is small on \( \overline{\Omega} - A_2 \gamma, \) so we can assume \( m \) has a maximum value somewhere on \( \Omega \).

**Lemma 2.5.** Let \( m \) be a Morse function on \( \Omega \), with a maximum value at some point \( x_0 \in \Omega \). Then, for every \( \epsilon > 0 \), there is a Morse function \( v \) with a polynomial cap at \( x_0 \), such that \( \| m - v \|_{C^2(\Omega)} < \epsilon \).

**Remark.** Later, we will set this \( \epsilon = \frac{\gamma^2}{2} \). We use \( \epsilon \) also in Proposition 2.4, and will set that \( \epsilon = 4 \gamma \).

**Proof.** We may assume \( m(0) = 1 \) is the maximum value of \( m \). Let \( p(x) \) be the 2nd order Taylor polynomial for \( m \) centered at \( 0 \). Since \( m \) is Morse, this critical point is non-degenerate. So, \( p(x) < 1 \) on a deleted neighborhood of \( x = 0 \) and
\[
p(x) = 1 + \sum_{i,j=1}^n x_i x_j a_{i j}.
\]
We will define \( v \) below such that \( v = m \) except on a ball \( B(0, \delta) \), where \( \delta > 0 \) can be chosen arbitrarily small. Let \( \Psi(x) \) be a radially decreasing \( C^\infty \) function, supported in \( B(0, \delta) \), with \( \Psi \equiv 1 \) on \( B(0, \delta/2) \) and \( \| \nabla \Psi \| \leq \frac{10}{\delta} \). Set \( v = p \Psi + m(1 - \Psi) \). To show that \( v \) is Morse, we will check that it has no critical point in \( x \in B(0, \delta) \) (we assume \( x \neq 0 \) here, and in similar remarks below). Note:
\[
\nabla v = \nabla p \Psi + \nabla m(1 - \Psi) + (p - m) \nabla \Psi.
\]
By Euler’s homogeneous function theorem
\[
x \cdot \nabla p(x) = 2(p(x) - 1),
\]
and
\[
x \cdot \nabla m(x) = 2(p(x) - 1) + O(\|x\|^3).
\]
Also, \( |x \cdot \nabla \Psi| < 10 \), and \( p - m = O(\|x\|^3) \). Combining these we get
\[
x \cdot \nabla v = 2(p(x) - 1) + O(\|x\|^3) < 0
\]
on any small enough \( B(0, \delta) \). By compactness, \( p(x) - 1 \) has a maximum value \( \rho < 0 \) on the sphere \( |x| = 1 \). By homogeneity, \( p(x) - 1 < \rho |x|^2 \) for all \( x \). So, \( x \cdot \nabla v < 0 \) on any small enough ball, \( B(0, \delta) \), and \( \nabla v \neq 0 \) there. So, the only critical points of \( v \) are the ones it shares with \( m \), and \( v \) is also Morse.
For small enough $\delta$, $\|m-v\|_{C^2(\Omega)} < \epsilon$, because $m-v = \Psi(m-p)$; if $D^k$ is some derivative of order $0 \leq k \leq 2$, then $|D^k \Psi| \leq K\delta^{-k}$ and so $|D^k(m-p)| \leq K\delta^{-k}$ and so $|D^k(m-v)| \leq K\delta < \epsilon$.

Finally, if the maximum value of $v$ occurs at 0 and at some other point(s) in $D$, we can perturb $v$ slightly by adding a very small constant multiple of $\Psi$ to it. The method above shows that the new $v$ has no new critical points, and is still a Morse function.  

Setting $\epsilon = \frac{\nu^2}{2}$ above, and combining the previous two approximations, we get $\|v - w\|_{C^2(\Omega)} < \gamma^2$. Note that $\|\Delta(v - w)\|_{L^\infty(\Omega)} < n\gamma^2$. If $y \in \Omega$ with $u(y) \leq 2\gamma$, then

$$v(y) \leq u(y) + |w(y) - u(y)| + |v(y) - w(y)| < 4\gamma,$$

so $E_{4\gamma} = \{y \in \Omega : v(y) > 4\gamma\} \subset A_{2\gamma}$, and $v = 4\gamma$ on the boundary. By Proposition 2.4,

$$|E_{4\gamma}| \cdot \|V_v\|_{L^\infty(E_{4\gamma})} \geq C_\gamma. \text{ Since } |D| \geq |E_{4\gamma}|, \text{ it is enough to show that}$$

$$\limsup_{\gamma \to 0} \|V_v\|_{L^\infty(E_{4\gamma})} \leq \|V\|_{L^\infty(D)}. \quad (2.9)$$

On $E_{4\gamma} \subset A_{2\gamma}$, we have $|w - u| < \gamma$ and $u(y) > 2\gamma$, so $w(y) \geq \gamma$ and

$$|V_v(y)| = \left| \frac{\Delta v(y)}{v(y)} \right| \leq \frac{|\Delta w(y)| + n\gamma^2}{w(y) - \gamma^2} \leq \frac{\|V_w\|_{L^\infty} w(y) + n\gamma^2}{w(y) - \gamma^2} \leq \frac{\|V_w\|_{L^\infty} + n\gamma}{1 - \gamma}.$$ 

The last step uses the fact that the function $x \to \frac{\|V_w\|_{L^\infty} x + n\gamma^2}{x - \gamma^2}$ is decreasing on $[\gamma, \infty)$. Since $\|V_w\|_{L^\infty} \leq \|V\|_{L^\infty}$, this proves (2.9) and Theorem 1.1.  

Remark. The same density argument shows that Proposition 2.4 extends to functions $v \in C(\overline{D})$ which are not necessarily Morse with polynomial caps.

The following theorem arose from conversations with Julian Edward.

**Theorem 2.6.** Let $1 \leq p < \infty$ and $D \subset \mathbb{R}^n$ bounded, with $n \geq 1$. Suppose that $u \in C(D)$ is a nonzero solution of (1.1), with $\|V\|_{L^\infty} = 1$, and $u = 0$ on $\partial D$. Then

$$\frac{\|u\|_{L^p(D)}}{\|u\|_{L^\infty(D)}} \geq c = \frac{\|u_u\|_{L^p(B(0,1))}}{\|u_u\|_{L^\infty(B(0,1))}},$$

where $u_u$ is given by (1.3). The constant $c$ is sharp.

**Proof.** Again, by density, we can assume that $u$ is a Morse function. We can normalize $u$ and $u_u$, as before, so that $\|u\|_{L^\infty(D)} = \|u_u\|_{L^\infty(D)} = 1$ and then must prove $\|u\|_p \geq \|u_u\|_p$. Also, $\|V\|_{L^\infty} = \|V_u\|_{L^\infty} = 1$.

The proof of Proposition 2.1, and inequality (2.1) in particular, show that $\beta(\alpha) \leq \alpha$ for all $0 \leq \alpha \leq 1$. Since $|B_\beta|$ decreases with $\beta$, we have $|A_\alpha| = |B_{\beta(\alpha)}| \geq |B_\alpha|$. But $\|u\|_p$ can be written in terms of $|A_\alpha|$:

$$\|u\|_p = p \int_0^1 \alpha^{p-1} |A_\alpha| d\alpha \geq p \int_0^1 \alpha^{p-1} |B_\alpha| d\alpha = \|u_u\|_p^p,$$

which proves Theorem 2.6.  

Remark. If \( \|V\|_\infty = k \neq 1 \), then we can apply the theorem to \( v(x) = u(k^{-1/2}x) \) since the \( L^\infty \) norm of \( V_v = -\frac{\Delta u}{v} \) is 1. In this case, we get
\[
\frac{\|u\|_{L^p(D)}}{\|u\|_{L^\infty(D)}} \geq c \cdot \|V\|^{-\frac{1}{sp}}.
\]

We now consider Theorem 1.1, including the Faber–Krahn inequality as a special case, on unbounded domains. It is easy to extend these to unbounded domains in \( \mathbb{R}^1 \). Unfortunately, if we do not impose conditions on \( u \) at infinity, there are fairly simple counterexamples which show that neither result is valid in \( \mathbb{R}^2 \).

For example, Let \( v(x, y) = e^x \sin(y) \), and \( D = \{(x, y) \in \mathbb{R}^2 : 0 < y < \pi\} \). Then \( v \) satisfies the hypotheses of Theorem 1.1 on \( D \), with \( V \equiv 0 \). Since \( |D| = \infty \), (1.2) is meaningless. Let \( D_2 = \{(x, y) \in \mathbb{R}^2 : |y| < (1 + x^2)^{-1}\} \), so that \( |D_2| < \infty \). There is a conformal map \( f : D_2 \to D \), which preserves boundaries, and \( u = v \circ f \) is harmonic on \( D_2 \). So, Theorem 1.1 and the Faber–Krahn inequality fail on \( D_2 \) for \( u \). However, with very mild assumptions on \( u \) at infinity, the theorem holds.

**Theorem 2.7.** Suppose that \( u \) and \( D \) satisfy the hypotheses of Theorem 1.1 except that \( D \) is not bounded in \( \mathbb{R}^n \). If \( \lim_{r \to \infty} \inf |x| = r \lim_{x \to \infty} |u(x)| = 0 \), then (1.2) still holds.

**Proof.** It is enough to prove an inequality of the form (2.8). We may assume \( u > 0 \) on \( D \), and that \( u(x_0) = 1 \) at some point \( x_0 \in D \). Let \( 0 < r < .01 \), as in the proof of Theorem 1.1. Choose \( r \) large enough such that \( \overline{A_{2r}} \cap B(0, r) \neq \emptyset \), \( x_0 \in B(0, r) \), and \( |u(x)| < 2r \) when \( |x| = r \). Now restrict \( u \) to the domain \( D_r = D \cap B(0, r) \) and replace \( u \) by \( u/\|u\|_\infty \) and \( \gamma \) by \( \gamma/\|u\|_\infty \). So, we can assume that \( \|u\|_\infty = 1 \), and after a dilation, we can also assume \( \|V\|_\infty = \|V_x\|_\infty \). The rest of the proof is the same as that of Theorem 1.1, but on the bounded domain \( D_r \). Here, \( u \) may not be \( \equiv 0 \) on the boundary of \( D_r \), as in the proof of Theorem 1.1, but we used this assumption only to ensure a positive distance between \( A_{2r} \) and \( \partial D \). That is ensured now by our choice of \( r \), and by the compactness of \( \overline{A_{2r}} \subset D_r \). We conclude that \( |D_r| \|V\|_\infty \geq C_{\gamma} \) where \( C_{\gamma} \to C \), the constant in Theorem 1.1, as \( \gamma \to 0 \), which proves (1.2). \( \square \)

3. **Continuation of \( Z \)**

Geometric properties of the zero set \( Z \) of a solution of the Schrödinger equation (1.1) in \( \mathbb{R}^n \) are not completely understood yet. They are quite subtle, even for a harmonic function on the plane. For example, Flatto proved in [9] that \( Z \subset \mathbb{R}^2 \) may contain curves of the form of \( y = x^2 \) but not on curves of the form of \( y = x^3 \).\(^1\) In [4] (see also [15,7]), we proved that a harmonic function \( u \) on \( \mathbb{R}^2 \) which vanishes on a nontrivial segment of an analytic curve \( \Gamma \) must vanish on the whole curve. Also, if \( u \) does not vanish elsewhere, it must be linear on \( \mathbb{R}^2 \).

A theorem of Bers describes the local structure of \( Z \subset \mathbb{R}^2 \) for nontrivial solutions of \( -\Delta u = \lambda u \). In a neighborhood of each point, \( Z \) is either a smooth curve or the intersection of \( n \) smooth curves at equal angles. In the \( n \)-dimensional case, one can give only a metric description of \( Z \). See [1,12].

This property is not valid when \( V \) is a bounded potential without any regularity assumption, as the following theorem shows.

\(^1\) Actually, the difference between these cases seems to be more algebraic than geometric.
Theorem 3.1. There is a function $u: \mathbb{R}^2 \to \mathbb{R}$ with $V = -\frac{\Delta u}{u} \in L^\infty$, that vanishes on a half line in $\mathbb{R}^2$, but not on the whole line. Also, $\|V\|_\infty > 0$ can be made arbitrarily small.

Proof. Let $w = \text{Im} (z^3) = r^3 \sin(3\theta)$. Let $0 \leq \phi(\theta) \leq 1$ be a smooth cut-off function such that $\phi = 0$ on $\left[\frac{7\pi}{6}, 2\pi\right]$ and $\phi = 1$ on $\left[\frac{\pi}{6}, \pi\right]$. Let

$$u = \lambda x + \phi(\theta) \cdot w(x, y)$$

and let $u(0, 0) = 0$. So, $u$ is harmonic on the regions where $\frac{\pi}{6} < \theta < \pi$ and where $\frac{7\pi}{6} < \theta < 2\pi$. Also, $u$ vanishes on the negative $y$-axis, but $u < 0$ on the positive $y$-axis.

We can check that $V$ is bounded on the region $0 \leq \theta \leq \frac{\pi}{6}$. Now, $\Delta u = [\Delta \phi] \cdot w(x, y) + 2\nabla \phi \cdot \nabla w$. Also, $\Delta \phi = \frac{\phi''}{r^2} \leq \frac{C}{r^2}$, while $|w| = |\text{Im} (z^3)| \leq |z^3| = r^3$, and $|\nabla \phi| = \frac{\phi'}{r} \leq \frac{C}{r}$, while $|\nabla w| = |3z^2| = 3r^2$. So, $|\Delta u| \leq Cr + 6cr$ on this region. But $w = r^3 \sin(3\theta) \geq 0$ and $\phi \geq 0$ there, so $u = \lambda x + \phi w \geq x = \lambda r \cos(\theta) > \frac{\lambda r}{2}$, and

$$|V| = \left|\frac{\Delta u}{u}\right| \leq \frac{Cr + 6cr}{\lambda r/2} = 2\lambda^{-1}(C + 6c).$$

On the opposite region, $\pi \leq \theta \leq \frac{7\pi}{6}$, we have $u < 0$. But $|u| = |\lambda x + \phi w| > \frac{\lambda r}{2}$, and the rest is identical. This argument shows that $V$ can be arbitrarily small, because $\|V\|_\infty = O(\lambda^{-1})$. □

Remarks. There are analogous functions in higher dimensions. For example, if $u$ is as in Theorem 3.1, set $w(x, y, z) = u(x, y)$. Then $w$ vanishes on a half-plane, but not on the whole plane. Also, $w$ satisfies the equation $-\Delta w(x, y, xz) = V(x, y)w(x, y, z)$, where $V$ is as in Theorem 3.1 and is bounded.

It seems possible to adjust the example to make $u$ vanish on an arc of a circle, for example. But by Theorem 1.1, it cannot vanish on the entire circle, unless the circle or $\|V\|_\infty$ is large. Also, by well-known unique continuation results, $u$ cannot vanish on an open set. The function $u$ defined in the proof above vanishes on a curve $Z$, which is the union of a half-line and a curve, both extending to infinity. It is interesting to note that no solution of (1.1) can vanish only on a half-line. The proof below also shows that no solution of (1.1) can vanish on a line segment, and have the same sign on both sides of that segment.

Theorem 3.2. There is no $C^2$ function $u: \mathbb{R}^2 \to \mathbb{R}$ with $V = -\frac{\Delta u}{u} \in L^\infty$, that vanishes on a half line $Z$ in $\mathbb{R}^2$, and at no other points.

Proof. Assume such a $u$ exists, to get a contradiction. Then $u$ cannot change sign, and we may assume $u > 0$ on $\mathbb{R}^2 - Z$. We can assume $Z$ is the half-line $\{(x, 0) : x \geq -1\}$. So, $u > 0$ on $B(0, 1)$, except that $u = 0$ on the diameter, $B(0, 1) \cap Z$. Also, $u_y, u_x, \Delta u$ and therefore $u_yy$, all vanish on that diameter. Define $w(x, y)$ on $B(0, 1)$ by $w(x, y) = u(x, y)$ when $y \geq 0$ and $w = 0$ when $y < 0$. Since $|\Delta u| \leq Cu$, we also get $|\Delta w| \leq Cw$. This is clear, except perhaps on $Z$, but it is easy to check that on $Z$, $\Delta w = w_{yy} = u_{yy} = \Delta u = 0$. Since $V_w = -\frac{\Delta w}{w}$ is bounded, $w$ must vanish on $B(0, 1)$, a standard unique continuation result. But this contradicts our assumption that $u > 0$ off $Z$, proving the Theorem. □

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References