Local L^p inequalities for Gegenbauer polynomials

Laura De Carli

Abstract

In this paper we prove new L^p estimates for Gegenbauer polynomials $P_n^{(s)}(x)$. We let $d\mu_s(x) = (1-x^2)^{s-\frac{1}{2}}dx$ be the measure in $(-1,1)$ which makes the polynomials $P_n^{(s)}(x)$ orthogonal, and we compare the $L^p(d\mu_s)$ norm of $P_n^{(s)}(x)$ with that of x^n. We also prove new $L^p(d\mu_s)$ estimates of the restriction of these polynomials to the intervals $[0, z_n]$ and $[z_n, 1]$ where z_n denotes the largest zero of $P_n^{(s)}(x)$.

1. Introduction

In this paper we will prove new L^p estimates for Gegenbauer, (or ultraspherical), polynomials.

The Gegenbauer polynomial of order s and degree n, $P_n^{(s)}(x)$, can be defined, for example, as the coefficients of ω^n in the expansion of the generating function $(1-2x\omega+w^2)^{-s} = \sum_{n=0}^{\infty} \omega^n P_n^{(s)} n(x)$. Gegenbauer polynomials are orthogonal in $L^2(-1,\ 1)$ with the measure $d\mu_s(x) = (1-x^2)^{s-\frac{1}{2}}dx$. Other properties of these polynomials are listed in the next Section.

In this paper we aim to estimate the $L^p(d\mu_s)$ norm of Gegenbauer polynomials and the $L^p(d\mu_s)$ norm of their restrictions to certain intervals of $[-1,1]$ in terms of the $L^p(d\mu_s)$ norm of x^n.

This choice is motivated by the fact that $\lim_{s\to\infty} \widetilde{P}_n^{(s)}(x) = x^n$. This is easy to prove using e.g. the explicit representation (2.2). In [DC] the sharp inequality

$$|P_n^{(s)}(x)| \leq P_n^{(s)}(1)\left(|x|^n + \frac{n-1}{2s+1}(1-|x|^n)\right) \tag{1.1}$$

has been proved for Gegenbauer polynomials of order $s \geq n\dfrac{1+\sqrt{5}}{4}$.

A pointwise comparison between $\widetilde{P}_n^{(s)}(x) = \dfrac{P_n^{(s)}(x)}{P_n^{(s)}(1)}$ and x^n is meaningful only when s is much larger that n.

Gegenbauer polynomials of large degree behave like Bessel functions, in the sense that

$$\lim_{n\to\infty} \frac{P_n^{(s)}\left(\cos\frac{z}{n}\right)}{P_n^{(s)}(1)} = \Gamma\left(s+\frac{1}{2}\right)\left(\frac{z}{2}\right)^{-s+\frac{1}{2}} J_{s-\frac{1}{2}}(z). \tag{1.2}$$

(1.2) easily follows from a well known Mehler-Heine type asymptotic formula for general Jacobi polynomials, (see [Sz], pg. 167).

However, $\widetilde{P}_n^{(s)}(x)$ and x^n have the same L^∞ norm for every $s>0$ and every $n \geq 0$ is. Indeed,

$$\sup_{x\in[-1,1]} \left|\widetilde{P}_n^{(s)}(x)\right| = \sup_{x\in[-1,1]} |x^n| = 1$$

because $|P_n^{(s)}(x)| \le P_n^{(s)}(1)$, (see the next Section).

Also the ratio between the $L^2(d\mu_s)$ norm of $\widetilde{P}_n^{(s)}(x)$ and the $L^2(d\mu_s)$ norm of x^n can be estimated for every n and s.

We prove the following

Proposition 1.1 *The function* $N_2(n,s) = \dfrac{||\widetilde{P}_n^{(s)}||_{L^2(d\mu_s)}}{||x^n||_{L^2(d\mu_s)}}$ *is decreasing with* s*, and*

$$2^{-\frac{n}{2}}\left(\frac{\sqrt{\pi}\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right)^{\frac{1}{2}} = \lim_{s\to\infty} N_2(n,s) < N_2(n,s) \le \lim_{s\to 0} N_2(n,s) = \left(\frac{\sqrt{\pi}\Gamma(n+1)}{2\Gamma\left(n+\frac{1}{2}\right)}\right)^{\frac{1}{2}}. \quad (1.3)$$

Thus,

$$2^{-\frac{n}{2}}\pi^{\frac{1}{4}}n^{\frac{1}{4}} < N_2(n,s) < n^{\frac{1}{4}}. \quad (1.4)$$

It is interesting to observe that $N_2(n,\frac{1}{2}) = 1$. This follows from the explicit formula for $N_2(n,s)$ in Section 2. By Proposition 1.1, $N_2(n,s) = 1$ if and only if $s = \frac{1}{2}$.

Proposition 1.1 shows that while it is true that $\lim\limits_{s\to\infty} \widetilde{P}_n^{(s)}(x) = x^n$, and $\lim\limits_{s\to\infty} ||\widetilde{P}_n^{(s)}(x)||_{L^\infty(d\mu_s)} = ||x^n||_{L^\infty(d\mu_s)}$, it is not true in general that $\lim\limits_{s\to\infty} ||\widetilde{P}_n^{(s)}||_{L^2(d\mu_s)} = ||x^n||_{L^2(d\mu_s)}$.

These consideration suggested us to investigate the ratio of the $L^r(d\mu_s)$ norms of $\widetilde{P}_n^{(s)}(x)$ and x^n for other values of r. We let

$$N_r(n,s) = \frac{||\widetilde{P}_n^{(s)}||_{L^r(d\mu_s)}}{||x^n||_{L^r(d\mu_s)}}, \quad 1 \le r \le \infty.$$

Our next Lemma suggests that $N_r(n,s)$ can be bounded above by a power of $N_2(n,s)$.

Lemma 1.2 *For every* $s > 0$*,* $n \ge 1$*, and* $r \ge 2$*,*

$$N_r(n,s) \le N_2(n,s)^{\frac{2}{r}} \left(\frac{r}{2}\right)^{\frac{1}{r}\left(s+\frac{1}{2}\right)}, \quad (1.5)$$

and

$$N_r(n,s) \le n^{\frac{1}{2r}} \left(\frac{r}{2}\right)^{\frac{1}{r}\left(s+\frac{1}{2}\right)}. \quad (1.6)$$

When $s \to 0$ *this upper bound is sharp, in the sense that the power of* n *in (1.6) cannot be replaced by a smaller power.*

The proof of the Lemma is in Section 3.

Numerical evidence suggests that $N_r(n,s) \le N_2(n,s)^{\frac{2}{r}}$ when $s \ge \frac{1}{2}$ and $1 \le r \le \infty$. When $0 \le s < \frac{1}{2}$ we conjecture instead that $N_r(n,s) \ge N_2(n,s)^{\frac{2}{r}}$.

The upper bound in Lemma 1.2 can be improved if we restrict $\widetilde{P}_n^{(s)}(x)$ to the intervals $\{1 \le |x| \le z_n\}$ and $(-z_n, z_n)$, where z_n denotes the largest positive zero of $P_n^{(s)}(x)$.

Our main result is the following.

Theorem 1.3 *For every $n > 2,\ s > 0,$ and $r \geq 1$,*

$$\sin^{\frac{2}{r}}\left(\frac{\pi}{n+1}\right)(1-z_n^2)^{\frac{1}{r}(s+\frac{1}{2})} \leq \frac{||\widetilde{P}_n^{(s)}||_{L^r(\{1\leq|x|\leq z_n\},\ d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} \leq p(n,s)^{(s+\frac{1}{2})\frac{1}{[\frac{n+1}{2}]r}}, \tag{1.7}$$

where $p(n,s) = \prod_{j=1}^{n}(1-z_j) = \dfrac{\Gamma(s)\,\Gamma(n+2\,s)}{2^n\,\Gamma(2\,s)\,\Gamma(n+s)}$ *is as in (2.15).*

Using Stirling's formula, it is possible to prove that $\lim_{s\to\infty} p(n,s)^{s+\frac{1}{2}} = e^{-\frac{n(n-1)}{4}}$, and thus $\lim_{s\to\infty}\dfrac{||\widetilde{P}_n^{(s)}||_{L^r(\{1\leq|x|\leq z_n\},\ d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} \leq \lim_{s\to\infty} p(n,s)^{(s+\frac{1}{2})\frac{2}{nr}} = e^{-\frac{n-1}{2r}}$.

We have recalled in the next Section that $z_n < \cos\left(\frac{\pi}{n+1}\right)\sqrt{\frac{(n-1)(n+2s-2)}{(n+s-2)(n+s-1)}}$, (see (2.12)), and so

$$\begin{aligned}
&\lim_{s\to\infty}\sin^{\frac{2}{r}}\left(\frac{\pi}{n+1}\right)(1-z_n^2)^{s+\frac{1}{2}}\\
&> \sin^{\frac{2}{r}}\left(\frac{\pi}{n+1}\right)\lim_{s\to\infty}\left(1-\frac{(n-1)(n+2s-2)\cos^2\left(\frac{\pi}{n+1}\right)}{(n+s-2)(n+s-1)}\right)^{s+\frac{1}{2}}\\
&= \sin^{\frac{2}{r}}\left(\frac{\pi}{n+1}\right)e^{-\frac{2}{r}(n-1)\cos^2\left(\frac{\pi}{n+1}\right)}.
\end{aligned}$$

From the inequalities above and (1.7) follows that

$$\sin^{\frac{2}{r}}\left(\frac{\pi}{n+1}\right)e^{-\frac{2}{r}(n-1)\cos^2\left(\frac{\pi}{n+1}\right)} < \lim_{s\to\infty}\frac{||\widetilde{P}_n^{(s)}||_{L^r(\{1\leq|x|\leq z_n\},\ d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} < e^{-\frac{n-1}{2r}}. \tag{1.8}$$

This upper bound is not sharp; in fact we have proved in Proposition 1.1 that $\lim_{s\to\infty} N_2(n,s) = (\pi n)^{\frac{1}{4}}2^{-\frac{n}{2}}$, while Lemma 1.3 yields $\lim_{s\to\infty}\dfrac{||\widetilde{P}_n^{(s)}||_{L^r(\{1\leq|x|\leq z_n\},\ d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} \leq e^{-\frac{n-1}{4}}$, and $e^{-\frac{n-1}{4}} > (\pi n)^{\frac{1}{4}}2^{-\frac{n}{2}}$ for every $n \geq 2$. However, Theorem (1.3) is interesting because it provides an upper and lower bound for the $L^r(\{1\leq|x|\leq z_n\},\ d\mu_s)$ norm of $\widetilde{P}_n^{(s)}(x)$ and is valid for every $r \geq 1$.

Since $\lim_{s\to\infty} z_n = 0$, (see the next Section), it is natural to conjecture that $N_r(n,s)$ is bounded above by a constant independent of s. In order to prove this conjecture we should prove that also the ratio of the $L^r(d\mu_s)$ norm of $\widetilde{P}_n^{(s)}(x)$ in $(-z_n, z_n)$ and $||x^n||_{L^r(d\mu_s)}$ is a bounded function of s.

In the next Theorem we estimate the $L^r(d\mu_s)$ norm of $\widetilde{P}_n^{(s)}(x)$ in $(-z_n, z_n)$ through interpolation.

Theorem 1.4 *For every $r \geq 2,\ s > 0$ and $n \geq 2$,*

$$\frac{||\widetilde{P}_n^{(s)}||_{L^r((-z_n,\ z_n),\ d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} \leq N_2(n,s)^{\frac{2}{r}}\left(\frac{n(n+2s)}{2s+1}\right)^{1-\frac{2}{r}}\left(z_n^2\,\frac{\frac{nr}{2}+s+1}{n-\frac{1}{2}}\right)^{n(\frac{1}{2}-\frac{1}{r})}$$

where z_n denotes the largest zero of $P_n^{(s)}$. Furthermore,

$$\lim_{s\to\infty} \frac{||\widetilde{P}_n^{(s)}||_{L^r((-z_n,\ z_n),\ d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} \leq n^{1-\frac{2}{r}} 2^{n(\frac{1}{2}-\frac{1}{r})} N_2(n,s)^{\frac{2}{r}}.$$

From Theorems 1.3 and 1.4 and Proposition 1.1 we can easily prove the following

Corollary 1.5 *For every $n \geq 2$ and every $r \geq 2$, $\lim_{s\to\infty} N_r(n,s)$ is finite. If $r < 4$ this limit is bounded above by a constant that does not depend on n.*

2. Preliminaries

Let us briefly review the main properties of the ultraspherical polynomials. For more details we refer to [Sz]. The ultraspherical polynomials can be defined through Rodriguez' formula

$$(1-x^2)^{s-\frac{1}{2}} P_n^{(s)}(x) = \frac{(-1)^n \Gamma(s+\frac{1}{2})\Gamma(n+2s)}{\Gamma(2s)\Gamma(n+s+\frac{1}{2})\Gamma(n+1)2^n} \left(\frac{d}{dx}\right)^n (1-x^2)^{n+s-\frac{1}{2}}. \tag{2.1}$$

When $s > 0$ we have the following explicit expression

$$P_n^{(s)}(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \frac{\Gamma(n-m+s)}{\Gamma(s)\Gamma(m+1)\Gamma(n-2m+1)} (2x)^{n-2m}. \tag{2.2}$$

Note that $P_n^{(s)}(x)$ is either even or odd.

The ultraspherical polynomials or order $s = 0$ are related to the Tchebicheff polynomials $T_n(x) = \cos(n\cos^{-1}(x))$ by the following limit relation.

$$\lim_{s\to 0} s^{-1} P_n^{(s)}(x) = \frac{2}{n} T_n(x). \tag{2.3}$$

The L^2 norm of $P_n^{(s)}(x)$ with respect to the measure $d\mu_s(x) = (1-x^2)^{s-\frac{1}{2}}dx$ in $(-1,\ 1)$ can be explicitly computed. It is

$$||P_n^{(s)}||^2_{L^2(d\mu_s)} = \int_{-1}^{1} |P_n^{(s)}(x)|^2 (1-x^2)^{s-\frac{1}{2}} dx = \frac{\pi 2^{1-2s}\Gamma(n+2s)}{(n+s)(\Gamma(s))^2\Gamma(n+1)}. \tag{2.4}$$

When $s > 0$ the maximum of of $P_n^{(s)}(x)$ in $[-1,\ 1]$ can be explicitly computed. We have:

$$\sup_{-1\leq x\leq 1} |P_n^{(s)}(x)| = P_n^{(s)}(1) = \frac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)}, \quad s > 0. \tag{2.5}$$

$N_2(n,s)$ can be explicitly computed as well. Indeed, the $L^r(d\mu_s)$ norm of x^n is

$$||x^n||_{L^r(d\mu_s)} = \left(\int_{-1}^{1} x^{nr}(1-x^2)^{s-\frac{1}{2}} dx\right)^{\frac{1}{r}}$$

$$= \beta^{\frac{1}{r}}\left(\frac{1}{2}(nr+1),\ s+\frac{1}{2}\right) = \left(\frac{\Gamma\left(\frac{1}{2}(nr+1)\right)\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(\frac{nr}{2}+s+1\right)}\right)^{\frac{1}{r}}, \tag{2.6}$$

where $\beta(a,\, b)$ is the standard Beta function. Thus,

$$N_2(n,s) = \left(\frac{2^{-n}\sqrt{\pi}\Gamma(n+1)\Gamma\left(s+\frac{1}{2}\right)\Gamma(n+s)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+s\right)\Gamma\left(\frac{n}{2}+s+\frac{1}{2}\right)} \right)^{\frac{1}{2}}. \tag{2.7}$$

This expression has been simplified with the aid of the well known duplication formula for the Gamma function $\dfrac{\Gamma(2x)}{\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)} = \dfrac{2^{2x-1}}{\sqrt{\pi}}$. It is interesting to note that $N_2(n,\, \frac{1}{2}) \equiv 1$.

The derivatives of ultraspherical polynomials are constant multiples of ultraspherical polynomials. From (2.2) easily follows that

$$\frac{d}{dx} P_n^{(s)}(x) = 2s P_{n-1}^{(s+1)}(x), \tag{2.8}$$

and if we let $\widetilde{P}_n^{(s)}(x) = \frac{P_n^{(s)}(x)}{P_n^{(s)}(1)}$, from (2.8) and 2.5 follows that

$$\frac{d}{dx}\widetilde{P}_n^{(s)}(x) = \frac{n(n+2s)}{1+2s}\widetilde{P}_{n-1}^{(s+1)}(x) \tag{2.9}$$

$$\frac{d^2}{d^2x}\widetilde{P}_n^{(s)}(x) = \frac{n(n-1)(n+2s)(n+2s+1)}{(1+2s)(3+2s)}\widetilde{P}_{n-2}^{(s+2)}(x). \tag{2.10}$$

$P_n^{(s)}(x)$ satisfies the following differential equation:

$$(1-x^2)y'' - (2s+1)xy' + n(n+2s)y = 0. \tag{2.11}$$

The zeros of ultraspherical polynomials have important and well studied properties. The literature on the subject is extensive and we will not attempt to survey it. We refer to [E] and the references cited there.

The following properties are well known, and are shared also by other systems of orthogonal polynomials.

All zeros of $P_n^{(s)}(x)$ are real and simple and lie in $[-1,\, 1]$. Since $\dfrac{d}{dx}P_n^{(s)}(x) = 2sP_{n-1}^{(s+1)}(x)$, Rolle's theorem implies that between any two zeros of $P_n^{(s)}(x)$ there is a zero of $P_{n-1}^{(s+1)}(x)$.

We will denote by $z_{n,\,k}(s)$, $k = 1,\ \dots,\ n$, the zeros of $P_n^{(s)}(x)$ enumerated in increasing order. That is, $-1 < z_{n,1}(s) < \dots < z_{n,n}(s) < 1$. When there is no risk of confusion, we will just let $z_{n,j}(s) = z_j$.

To the best of our knowledge, the best available upper bound for z_n is in [ADGR].

$$z_n < \sqrt{\frac{(n-1)(n+2s-2)}{(n+s-2)(n+s-1)}}\cos\left(\frac{\pi}{n+1}\right), \quad n \geq 1. \tag{2.12}$$

The inequality (2.12) improves the following inequality due to Elbert, (see [E]).

$$z_n < \frac{\sqrt{(n-1)(n+2s+1)}}{n+s} = \sqrt{1-\left(\frac{s+1}{n+s}\right)^2}. \tag{2.13}$$

2.1 Four useful Lemmas

Lemma 2.1 *Let z_1 ... z_n be the zeros of $P_n^{(s)}(x)$ arranged in increasing order. Then*

$$P_n^{(s)}(x) = \prod_{k=\frac{[n+1]}{2}}^{n} \frac{x^2 - z_k^2}{1 - z_k^2}, \tag{2.14}$$

and

$$\prod_{k=\frac{[n+1]}{2}}^{n} (1 - z_k^2) = \prod_{k=1}^{n} (1 - z_k) = \frac{\Gamma(s)\,\Gamma(n + 2\,s)}{2^n\,\Gamma(2\,s)\,\Gamma(n + s)}. \tag{2.15}$$

Furthermore

$$P_n^{(s)}(x) \le x^n \textit{for} \ \ x \ge z_n \tag{2.16}$$

Proof. We have already observed that the zeros of $P_n^{(s)}(x)$ are symmetric with respect to $x = 0$. When n is odd, $P_n^{(s)}(x)$ vanishes also at $x = 0$. Therefore, if we let $M(n, s) = \dfrac{2^n\,\Gamma(n + s)}{\Gamma(1 + n)\,\Gamma(s)}$ be the coefficient of x^n in the explicit expression (2.2) we can factorize $P_n^{(s)}(x)$ as follows:

$$P_n^{(s)}(x) = M(n, s) \prod_{k=1}^{n} (x - z_k) = M(n, s) \begin{cases} \displaystyle\prod_{k=\frac{n}{2}}^{n} (x^2 - z_k^2) & \text{if } n \text{ is even,} \\ \displaystyle x \prod_{k=\frac{n-1}{2}}^{n} (x^2 - z_k^2) & \text{if } n \text{ is odd.} \end{cases} \tag{2.17}$$

Thus,

$$\widetilde{P}_n^{(s)}(x) = \frac{P_n^{(s)}(x)}{P_n^{(s)}(1)} = \begin{cases} \displaystyle\prod_{k=\frac{n}{2}}^{n} \frac{x^2 - z_k^2}{1 - z_k^2} & \text{if } n \text{ is even,} \\ \displaystyle x \prod_{k=\frac{n-1}{2}}^{n} \frac{x^2 - z_k^2}{1 - z_k^2} & \text{if } n \text{ is odd} \end{cases}$$

which is (2.14).

Since $\frac{x^2 - z_k^2}{1 - z_k^2} \le x^2$, (2.16) follows.

Let $p(n, s) = \displaystyle\prod_{k=1}^{n} (1 - z_k)$. Note that $p(n, s) = \dfrac{P_n^{(s)}(1)}{M(n, s)}$, and since $P_n^{(s)}(1)$ is as in (2.5),

$$p(n, s) = \frac{\Gamma(s)\,\Gamma(n + 2\,s)}{2^n\,\Gamma(2\,s)\,\Gamma(n + s)}$$

as required.

The inequality (2.16) can also be proved from the following Lemma.

Lemma 2.2 *for every $n > 1$ and $s > 0$, $\widetilde{P}_n^{(s)}(x) \le x\widetilde{P}_{n-1}^{(s+1)}(x)$ in $[z_n,\ 1]$.*

Proof. Our key tool is the differential equation (2.11). That is,

$$n(n+2s)y = (2s+1)xy' - (1-x^2)y'', \tag{2.18}$$

where $y = P_n^{(s)}(x)$.

We divide both sides of (2.18) by $P_n^{(s)}(1)$ and recall that, by (2.9) and (2.10), $\frac{d}{dx}\widetilde{P}_n^{(s)}(x) = \frac{n(n+2s)}{1+2s}\widetilde{P}_{n-1}^{(s+1)}(x)$ and $\frac{d^2}{d^2x}\widetilde{P}_n^{(s)}(x) = \frac{n(n-1)(n+2s)(n+2s+1)}{(1+2s)(3+2s)}\widetilde{P}_{n-2}^{(s+2)}(x)$. We obtain the following three term relation.

$$\frac{(n-1)(n+2s+1)}{(1+2s)(3+2s)}(1-x^2)\widetilde{P}_{n-2}^{(s+2)}(x) - x\widetilde{P}_{n-1}^{(s+1)}(x) + \widetilde{P}_n^{(s)}(x) = 0,$$

and since $\widetilde{P}_{n-2}^{(s+2)}(x)$ is positive in $[z_n,\ 1]$, we gather

$$\widetilde{P}_n^{(s)}(x) < x\widetilde{P}_{n-1}^{(s+1)}(x),$$

as required.

The following Lemma improves a Lemma in [DC]

Lemma 2.3 *For every* $0 \le |x| \le z_k$*,* $s > 0$ *and* $n \ge 2$*,*

$$|\widetilde{P}_n^{(s)}(x)| \le \frac{n(n+2s)}{2s+1}\xi_{n-1}^{n-1} z_n. \tag{2.19}$$

Proof. It is well known, (see e.g. [Sz]), that the local maxima of $|P_n^{(s)}(x)|$ are increasing. The critical points of $P_n^{(s)}(x)$ are the zeros of $P_{n-1}^{(s+1)}(x)$, and hence $|P_n^{(s)}(x)|$, restricted to the interval $[0,\ z_n]$, attains its maximum at the largest zero of $P_{n-1}^{(s+1)}(x)$, which we can denote by ξ_{n-1}. Thus, for every $x \in [-z_n, z_n]$, $\widetilde{P}_n^{(s)}(x) \le \widetilde{P}_n^{(s)}(\xi_{n-1})$.

By the mean value theorem,

$$\widetilde{P}_n^{(s)}(z_n) - \widetilde{P}_n^{(s)}(\xi_{n-1}) = (z_n - \xi_{n-1})\frac{\partial}{\partial x}\widetilde{P}_n^{(s)}(\xi)$$

where $\xi_{n-1} < \xi < z_n$. By (2.9),

$$-\widetilde{P}_n^{(s)}(\xi_{n-1}) = (z_n - \xi_{n-1})\frac{n(n+2s)}{2s+1}\widetilde{P}_{n-1}^{(s+1)}(\xi)$$

and since $\widetilde{P}_{n-1}^{(s+1)}(x) \le x^{n-1}$ in $[\xi_{n-1}, 1]$ and $-\widetilde{P}_n^{(s)}(\xi_{n-1}) = |\widetilde{P}_n^{(s)}(\xi_{n-1})|$, we can infer that

$$\widetilde{P}_n^{(s)}(x) \le \xi^{n-1}(z_n - \xi_{n-1})\frac{n(n+2s)}{2s+1} < \xi_{n-1}^{n-1} z_n \frac{n(n+2s)}{2s+1},$$

as required.

The following Lemma concerns the monotonicity of ratios of Gamma functions.

Lemma 2.4 *a) The function*

$$x \to \frac{\Gamma(x)\, x^y}{\Gamma(x+y)}, \quad x > 0,$$

is decreasing when $0 < y \le 1$ *and is increasing when* $y > 1$. *Therefore,*

$$\frac{\Gamma(x)\, x^y}{\Gamma(x+y)} \le \lim_{x\to\infty} \frac{\Gamma(x)\, x^y}{\Gamma(x+y)} = 1 \tag{2.20}$$

when $y > 1$, *and the inequality reverses when* $y < 1$.

b) The function

$$x \to \frac{\Gamma(x)\,(x+y)^y}{\Gamma(x+y)}, \quad x > 0,$$

is decreasing for every $y > 0$, *and*

$$\frac{\Gamma(x)\, x^y}{\Gamma(x+y)} \ge \lim_{x\to\infty} \frac{\Gamma(x)\, x^y}{\Gamma(x+y)} = 1. \tag{2.21}$$

Proof. We prove only a) since the proof of b) is almost the same. Let $g_y(x) = \dfrac{\Gamma(x)\, x^y}{\Gamma(x+y)}$. When $y = 0$ and $y = 1$, then $g_y(x) \equiv 1$, so we assume either $y > 1$ or $0 < y < 1$.

To investigate the monotonicity of $g_y(x)$ we study the sign of the derivative of

$$\ln g_y(x) = y \ln x + \ln(\Gamma(x)) - \ln(\Gamma(x+y)).$$

The logarithmic derivative of $\Gamma(z)$ is

$$\frac{\Gamma'(z)}{\Gamma(z)} = \gamma - \frac{1}{z} - \sum_{m=1}^{\infty} \left(\frac{1}{z+m} - \frac{1}{m} \right)$$

where γ is Euler's constant. Therefore,

$$\begin{aligned}(\ln g_y(x))' &= \frac{y}{x} - \sum_{m=0}^{\infty} \frac{1}{x+m} - \frac{1}{x+y+m} \\ &= y \left(\frac{1}{x} - \sum_{m=0}^{\infty} \frac{1}{(x+m)(x+m+y)} \right).\end{aligned}$$

Note that

$$\frac{1}{x} = \sum_{m=0}^{\infty} \frac{1}{x+m} - \frac{1}{x+m+1} = \sum_{m=0}^{\infty} \frac{1}{(x+m)(x+m+1)}.$$

Thus,

$$\begin{aligned}(\ln g_y(x))' &= y \left(\sum_{m=0}^{\infty} \frac{1}{(x+m)(x+m+1)} - \sum_{m=0}^{\infty} \frac{1}{(x+m)(x+m+y)} \right) \\ &= y \sum_{m=0}^{\infty} \frac{y-1}{(x+m+1)(x+m+y)}.\end{aligned} \tag{2.22}$$

When $y > 1$ the function in (2.22) is positive and when $y < 1$ it is negative. Therefore, $\ln g_y(x)$ is increasing whenever $y > 1$ and is decreasing whenever $0 \le y < 1$, as required.

(2.20) follows by Stirling's formula.

3 Most of the proofs

Proof of Theorem 1.3. We use the factorization in (2.14). Suppose that n is even, since the proof is similar in the other case. By Hölder inequality,

$$||\widetilde{P}_n^{(s)}||_{L^r((z_n,\ 1),\ d\mu_s)} = \left(\int_{z_n}^1 \prod_{j=\frac{n}{2}}^{n} \left(\frac{t^2 - z_j^2}{1 - z_j^2}\right)^r (1 - t^2)^{s-\frac{1}{2}}\right)^{\frac{1}{r}}$$

$$\leq \prod_{j=\frac{n}{2}}^{n} \left(\int_{z_n}^1 \left(\frac{t^2 - z_j^2}{1 - z_j^2}\right)^{\frac{nr}{2}} (1 - t^2)^{s-\frac{1}{2}}\right)^{\frac{2}{nr}} = \prod_{j=\frac{n}{2}}^{n} J(z_j),$$

where we have let $J(z_j) = \left(\int_{z_n}^1 \left(\frac{t^2 - z_j^2}{1 - z_j^2}\right)^{\frac{nr}{2}} (1 - t^2)^{s-\frac{1}{2}} dt\right)^{\frac{2}{nr}}$.

In order to compare $J(z_j)$ with $||x^n||_{L^r(d\mu_s)}$ we let $\dfrac{t^2 - z_j^2}{1 - z_j^2} = x^2$, so that $t = \sqrt{x^2(1 - z_j^2) + z_j^2}$ and $dt = \dfrac{x(1 - z_j^2)}{\sqrt{x^2(1 - z_j^2) + z_j^2}}\, dx$. Note that $x \leq \dfrac{x}{\sqrt{x^2(1 - z_j^2) + z_j^2}} \leq 1$.

With this substitution, $(1 - t^2)^{s-\frac{1}{2}} = \left((1 - z_j^2)(1 - x^2)\right)^{s-\frac{1}{2}}$, and

$$J(z_j)^{\frac{nr}{2}} = (1 - z_j^2)^{s+\frac{1}{2}} \int_{\frac{z_n^2 - z_j^2}{1 - z_j^2}}^{1} x^{nr}(1 - x^2)^{s-\frac{1}{2}} \frac{x dx}{\sqrt{x^2(1 - z_j^2) + z_j^2}}$$

$$\leq (1 - z_j^2)^{s+\frac{1}{2}} \int_0^1 x^{nr}(1 - x^2)^{s-\frac{1}{2}} dx = (1 - z_j^2)^{s+\frac{1}{2}} ||x^n||^r_{L^r(d\mu_s)}$$

and

$$||\widetilde{P}_n^{(s)}||_{L^r((z_n,\ 1),\ d\mu_s)} \leq \prod_{j=1}^{\frac{n}{2}} J(z_j)$$

$$\leq ||x^n||_{L^r(d\mu_s)} \prod_{j=\frac{n}{2}}^{n} \left((1 - z_j^2)^{s+\frac{1}{2}}\right)^{\frac{2}{nr}} = ||x^n||_{L^r(d\mu_s)} p(n, s)^{(s+\frac{1}{2})\frac{2}{nr}},$$

as required.

To prove the other inequality we observe that

$$\frac{x^2 - z_j^2}{1 - z_j^2} \geq \frac{x^2 - z_n^2}{1 - z_n^2}$$

whenever $j \leq \frac{n}{2}$. Therefore,

$$||\widetilde{P}_n^{(s)}||_{L^r((z_n,\ 1),\ d\mu_s)} \geq \left(\int_{z_n}^1 \left(\frac{t^2 - z_n^2}{1 - z_n^2}\right)^{\frac{nr}{2}} d\mu_s(t)\right)^{\frac{1}{r}}.$$

We use again the substitution $\dfrac{t^2 - z_n^2}{1 - z_n^2} = x^2$, so that and

$$||\widetilde{P}_n^{(s)}||^r_{L^r((z_n,\ 1),\ d\mu_s)} \geq (1 - z_n^2)^{s+\frac{1}{2}} \int_0^1 \frac{x^{nr+1}}{\sqrt{x^2(1-z_n^2)+z_n^2}}(1-x^2)^{s-\frac{1}{2}}dx$$

$$= (1 - z_n^2)^{s+\frac{1}{2}} \int_0^1 x^{nr}\psi(x, z_n^2)(1-x^2)^{s-\frac{1}{2}}dx, \tag{3.23}$$

where we have let $\psi(x,t) = \dfrac{x}{\sqrt{x^2(1-t)+t}}$.

The easy inequality $\psi(x,t) \geq x$ is not enough to prove (1.7). We use the elementary inequality $a^2 + b^2 - 2ab \geq 0$, with $a = \psi(x,t)^{\frac{1}{2}}$ and $b \in \mathbf{R}$, to infer that
$\psi(x,t) \geq 2b(\psi(x,t))^{\frac{1}{2}} - b^2 \geq 0$ for every $b \in \mathbf{R}$. From (3.23) follows that

$$(1-z_n^2)^{-(s+\frac{1}{2})}||\widetilde{P}_n^{(s)}||^r_{L^r((z_n,\ 1),\ d\mu_s)}$$

$$\geq \left(\int_0^1 2b\,(\psi(x,\ {z_n}^2))^{\frac{1}{2}}x^{nr}(1-x^2)^{s-\frac{1}{2}}dx - b^2||x^n||_{L^r(d\mu_s)}\right). \tag{3.24}$$

Our next task is to choose b so to maximize the function in (3.24).

It is easy to verify that $(\psi(x,t))^{\frac{1}{2}}$ is a convex whenever $-1 < x < 1$, and thus, by Taylor formula,

$$\psi^{\frac{1}{2}}(x,t) \geq (\psi(x,0))^{\frac{1}{2}} + t\frac{\partial}{\partial t}(\psi(x,0))^{\frac{1}{2}} = 1 - t\frac{1-x^2}{2x^2},$$

and

$$||\widetilde{P}_n^{(s)}||^r_{L^r((z_n,\ 1),\ d\mu_s)}(1-z_n^2)^{-(s+\frac{1}{2})}$$

$$\geq 2b\int_0^1 \left(1 - {z_n}^2\frac{1-x^2}{2x^2}\right)x^{nr}(1-x^2)^{s-\frac{1}{2}}dx - b^2||x^n||^r_{L^r(d\mu_s)}$$

$$= ||x^n||^r_{L^r(d\mu_s)}\left(2b - b^2 - b\,{z_n}^2\,\frac{\int_0^1 x^{nr-2}(1-x^2)^{s-\frac{1}{2}}dx}{||x^n||^r_{L^r(d\mu_s)}}\right)$$

$$= b\left(2 - b - {z_n}^2\left(\frac{2s+1}{2(nr-1)}\right)\right)||x^n||^r_{L^r(d\mu_s)}.$$

The function $b \to b\left(2 - {z_n}^2\left(\dfrac{2s+1}{2(nr-1)}\right) - b\right)$ attains its maximum when $b = \dfrac{1}{2}\left(2 - {z_n}^2\,\dfrac{2s+1}{2(nr-1)}\right)$, and so

$$||\widetilde{P}_n^{(s)}||^r_{L^r((z_n,\ 1),\ d\mu_s)} \geq (1-z_n^2)^{s+\frac{1}{2}}\left(1 - {z_n}^2\,\frac{2s+1}{4(nr-1)}\right)^2||x^n||^r_{L^r(d\mu_s)}.$$

We are left to prove that

$$c(n,s,r) = 1 - {z_n}^2\,\frac{2s+1}{4(nr-1)}$$

is always positive. We use the upper bound for z_n in (2.13), so to obtain

$$1 - {z_n}^2\,\frac{2s+1}{4(nr-1)} \geq 1 - \frac{(n-1)(2s+1)(n+2s-2)\cos^2\left(\frac{\pi}{n+1}\right)}{4(nr-1)(n+s-2)(n+s-1)}$$

$$\geq 1-\frac{(n-1)(2s+1)(n+2s-2)\cos^2\left(\frac{\pi}{n+1}\right)}{4(n-1)(n+s-2)(n+s-1)}.$$

It is easy to verify that the function above decreases with s and hence,

$$1-{z_n}^2\,\frac{2s+1}{4(nr-1)}\geq \lim_{s\to\infty} 1-\frac{(2s+1)(n+2s-2)\cos^2\left(\frac{\pi}{n+1}\right)}{4(n+s-2)(n+s-1)}$$

$$=1-\cos^2\left(\frac{\pi}{n+1}\right)=\sin^2\left(\frac{\pi}{n+1}\right)$$

and from that (1.7) follows.

Proof of Proposition 1.1 To prove that $N_2(n,s)$ decreases with s we study the function $s\to \log(N_2(n,s))$; $N_2(n,s)$ is decreasing in s if and only $\frac{\partial}{\partial s}\log(N_2(n,s))$ is negative.

We recall that $N_2(n,s)=\left(\dfrac{2^{-n}\sqrt{\pi}\Gamma(n+1)\Gamma\left(s+\frac{1}{2}\right)\Gamma(n+s)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+s\right)\Gamma\left(\frac{n}{2}+s+\frac{1}{2}\right)}\right)^{\frac{1}{2}}$.

The partial derivative of $\log(N_2(n,s))$ with respect to s is

$$\frac{\partial}{\partial s}\log(N_2(n,s)$$

$$=\frac{1}{2}\sum_{m=0}^{\infty}\left(\frac{1}{\frac{n}{2}+s+m}+\frac{1}{\frac{n}{2}+s+\frac{1}{2}+m}-\frac{1}{\frac{1}{2}+s+m}-\frac{1}{n+s+m}\right)$$

$$=-\frac{n(n+1)}{2}\sum_{m=0}^{\infty}\frac{4m+2n+4s+1}{(m+n+s)(2m+2s+1)(2m+n+2s)(2m+n+2s+1)}$$

which is negative, as required.

(1.4) follows by Stirling formula.

Proof of Lemma 1.2. We use Riesz interpolation theorem. When $r\geq 2$,

$$||P_n^{(s)}||_{L^r(d\mu_s)}\leq ||P_n^{(s)}||_{L^2(d\mu_s)}^{\frac{2}{r}}||P_n^{(s)}||_{L^\infty(d\mu_s)}^{1-\frac{2}{r}},$$

or equivalently

$$||\widetilde{P}_n^{(s)}||_{L^r(d\mu_s)}\leq ||\widetilde{P}_n^{(s)}||_{L^2(d\mu_s)}^{\frac{2}{r}}$$

since $||\widetilde{P}_n^{(s)}||_{L^\infty(d\mu_s)}=1$. From the inequality above follows that

$$\frac{||\widetilde{P}_n^{(s)}||_{L^r(d\mu_s)}}{||x^n||_{L^r(d\mu_s)}}\leq\left(\frac{||\widetilde{P}_n^{(s)}||_{L^2(d\mu_s)}}{||x^n||_{L^2(d\mu_s)}}\right)^{\frac{2}{r}}\frac{||x^n||_{L^2(d\mu_s)}^{\frac{2}{r}}}{||x^n||_{L^r(d\mu_s)}}$$

$$=N_2(n,s)^{\frac{2}{r}}\left(\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{nr}{2}+s+1\right)}{\Gamma\left(\frac{1}{2}(nr+1)\right)\Gamma(n+s+1)}\right)^{\frac{1}{r}}. \tag{3.25}$$

We can argue as in Lemma 1.1 to show that the function

$$n \to \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{nr}{2}+s+1\right)}{\Gamma\left(\frac{1}{2}(nr+1)\right)\Gamma(n+s+1)}$$

is increasing, and is then bounded above by its limit at $n \to \infty$, which is $\left(\frac{r}{2}\right)^{s+\frac{1}{2}}$.

(2.20) follows from Lemma 1.1.

We are left to prove that the upper bound in (2.21) is actually sharp when $s = 0$.

Recalling that $\lim_{s\to 0} s^{-1} P_n^{(s)}(x) = \frac{2}{n}\cos(nx)$, (see Section), we can see that

$$\lim_{s\to 0} ||\widetilde{P}_n^{(s)}||_{L^r(d\mu_s)} = \lim_{s\to 0} \int_{-1}^{1} |\widetilde{P}_n^{(s)}(x)|^r (1-x^2)^{s-\frac{1}{2}} dx \tag{3.26}$$

$$= \int_0^\pi |\cos(nt)|^r dt = \frac{\sqrt{\pi}\,\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(1+\frac{r}{2}\right)}.$$

We have used the change of variable $x = \cos t$ in the integral in (3.26). Therefore,

$$N_r(n,0) = \lim_{s\to 0} \frac{||\widetilde{P}_n^{(s)}||_{L^r(d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} = \left(\frac{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{nr}{2}+1\right)}{\Gamma\left(\frac{r}{2}+1\right)\Gamma\left(\frac{1}{2}(nr+1)\right)}\right)^{\frac{1}{r}}.$$

By Lemma 2.4, the function $n^{-\frac{1}{2r}} N_r(n,0)$ is increasing, and its limit is $\left(\frac{\left(\frac{r}{2}\right)^{\frac{1}{2}}\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r}{2}+1\right)}\right)^{\frac{1}{r}}$.

Proof of Theorem 1.4. We use Lemma 2.3 and interpolation. When $r \geq 2$,

$$||P_n^{(s)}||_{L^r((-z_n,\ z_n)\ d\mu_s)} \leq ||P_n^{(s)}||_{L^2(d\mu_s)}^{\frac{2}{r}} ||P_n^{(s)}||_{L^\infty(-z_n,\ z_n)}^{1-\frac{2}{r}},$$

or equivalently

$$||\widetilde{P}_n^{(s)}||_{L^r((-z_n,\ z_n),\ d\mu_s)} \leq \left(\frac{n(n+2s)}{2s+1}\xi_{n-1}^{n-1} z_n\right)^{1-\frac{2}{r}} ||\widetilde{P}_n^{(s)}||_{L^2(d\mu_s)}^{\frac{2}{r}}.$$

From the inequality above follows that

$$\frac{||\widetilde{P}_n^{(s)}||_{L^r((-z_n,\ z_n),\ d\mu_s)}}{||x^n||_{L^r(d\mu_s)}}$$

$$\leq \left(\frac{n(n+2s)}{2s+1}\xi_{n-1}^{n-1} z_n\right)^{1-\frac{2}{r}} \left(\frac{||\widetilde{P}_n^{(s)}||_{L^2(d\mu_s)}}{||x^n||_{L^2(d\mu_s)}}\right)^{\frac{2}{r}} \frac{||x^n||_{L^2(d\mu_s)}^{\frac{2}{p}}}{||x^n||_{L^r(d\mu_s)}} \tag{3.27}$$

$$\leq \left(\frac{n(n+2s)}{2s+1}{z_n}^n\right)^{1-\frac{2}{r}} N_2(n,s)^{\frac{2}{r}} \left(\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{nr}{2}+s+1\right)}{\Gamma\left(\frac{1}{2}(nr+1)\right)\Gamma(n+s+1)}\right)^{\frac{1}{r}}.$$

We use Lemma 2.4 to estimate the ratio of the Gamma functions in the inequality above.

First we apply the Lemma to the ratio

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}(nr+1)\right)} = (n-\frac{1}{2})^{-\frac{nr}{2}+n-1}\frac{\Gamma\left(n-\frac{1}{2}\right)(n-\frac{1}{2})^{\frac{nr}{2}-n+1}}{\Gamma\left(\frac{1}{2}(nr+1)\right)} = (n-\frac{1}{2})^{-\frac{nr}{2}+n}\frac{\Gamma(x)x^y}{\Gamma(x+y)}$$

with $x = n - \frac{1}{2}$ and $y = \frac{nr}{2} - n + 1$. Since $y > 1$, $\dfrac{\Gamma(x)x^y}{\Gamma(x+y)} < 1$.

Then we apply the Lemma to the ratio

$$\frac{\Gamma\left(\frac{nr}{2}+s+1\right)}{\Gamma(n+s+1)} = \left(\frac{nr}{2}+s+1\right)^{\frac{nr}{2}-n}\frac{\Gamma(x+y)}{(x+y)^y\,\Gamma(x)}$$

$x = n+s+1$ and $y = \frac{nr}{2} - n$. The ratio $\dfrac{\Gamma(x+y)}{(x+y)^y\,\Gamma(x)}$ is always increasing, and so it is < 1. Therefore,

$$\left(\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{nr}{2}+s+1\right)}{\Gamma\left(\frac{1}{2}(nr+1)\right)\Gamma(n+s+1)}\right)^{\frac{1}{r}} \leq \left(\frac{\frac{nr}{2}+s+1}{n-\frac{1}{2}}\right)^{n(\frac{1}{2}-\frac{1}{r})},$$

and

$$\frac{||\widetilde{P}_n^{(s)}||_{L^r((-z_n,\ z_n),\ d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} \leq \left(\frac{n(n+2s)}{2s+1}\right)^{1-\frac{2}{r}}\left(z_n^2\,\frac{\frac{nr}{2}+s+1}{n-\frac{1}{2}}\right)^{n(\frac{1}{2}-\frac{1}{r})}$$

as required.

By (2.12), $z_n^2 < \dfrac{(n-1)(n+2s-2)\cos^2\left(\frac{\pi}{n+1}\right)}{(n+s-2)(n+s-1)}$, and so

$$\frac{||\widetilde{P}_n^{(s)}||_{L^r((-z_n,\ z_n),\ d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} \leq \left(\frac{n(n+2s)}{2s+1}\right)^{1-\frac{2}{r}}\left(\frac{(n-1)(n+2s-2)\cos^2\left(\frac{\pi}{n+1}\right)}{(n+s-2)(n+s-1)} \times \frac{\frac{nr}{2}+s+1}{n-\frac{1}{2}}\right)^{n(\frac{1}{2}-\frac{1}{r})}$$

When $s \to \infty$ the right hand side tends to $n^{1-\frac{2}{r}}\left(\dfrac{4(n-1)\cos^2\left(\frac{\pi}{n+1}\right)}{2n-1}\right)^{n(\frac{1}{2}-\frac{1}{r})}$. It is easy to prove that the function in parenthesis is an increasing function of n, and its limit is 2. This concludes the proof of the Theorem.

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