Local $L^p$ inequalities for Gegenbauer polynomials

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Abstract

In this paper we prove new $L^p$ estimates for Gegenbauer polynomials $P_n^{(s)}(x)$. We let $d\mu_s(x) = (1 - x^2)^{s-\frac{1}{2}} dx$ be the measure in $(-1,1)$ which makes the polynomials $P_n^{(s)}(x)$ orthogonal, and we compare the $L^p(d\mu_s)$ norm of $P_n^{(s)}(x)$ with that of $x^n$. We also prove new $L^p(d\mu_s)$ estimates of the restriction of these polynomials to the intervals $[0, z_n]$ and $[z_n, 1]$ where $z_n$ denotes the largest zero of $P_n^{(s)}(x)$.

1. Introduction

In this paper we will prove new $L^p$ estimates for Gegenbauer, (or ultraspherical), polynomials.

The Gegenbauer polynomial of order $s$ and degree $n$, $P_n^{(s)}(x)$, can be defined, for example, as the coefficients of $\omega^n$ in the expansion of the generating function

$$ (1 - 2x \omega + \omega^2)^{-s} = \sum_{n=0}^{\infty} \omega^n P_n^{(s)} n(x). $$

Gegenbauer polynomials are orthogonal in $L^2(-1, 1)$ with the measure $d\mu_s(x) = (1 - x^2)^{s-\frac{1}{2}} dx$. Other properties of these polynomials are listed in the next Section.

In this paper we aim to estimate the $L^p(d\mu_s)$ norm of Gegenbauer polynomials and the $L^p(d\mu_s)$ norm of their restrictions to certain intervals of $[-1, 1]$ in terms of the $L^p(d\mu_s)$ norm of $x^n$.

This choice is motivated by the fact that $\lim_{s \to \infty} \tilde{P}_n^{(s)}(x) = x^n$. This is easy to prove using e.g. the explicit representation (2.2). In [DC] the sharp inequality

$$ |P_n^{(s)}(x)| \leq P_n^{(s)}(1) \left( |x|^n + \frac{n - 1}{2s + 1} (1 - |x|^n) \right) $$

(1.1)

has been proved for Gegenbauer polynomials of order $s \geq n + \frac{1 + \sqrt{5}}{4}$.

A pointwise comparison between $\tilde{P}_n^{(s)}(x) = \frac{P_n^{(s)}(x)}{P_n^{(s)}(1)}$ and $x^n$ is meaningful only when $s$ is much larger than $n$.

Gegenbauer polynomials of large degree behave like Bessel functions, in the sense that

$$ \lim_{n \to \infty} \frac{P_n^{(s)}(\cos \frac{z}{n})}{P_n^{(s)}(1)} = \Gamma \left( s + \frac{1}{2} \right) \left( \frac{z}{2} \right)^{-s+\frac{1}{2}} J_{s-\frac{1}{2}}(z). $$

(1.2)

(1.2) easily follows from a well known Mehler-Heine type asymptotic formula for general Jacobi polynomials, (see [Sz], pg. 167).

However, $\tilde{P}_n^{(s)}(x)$ and $x^n$ have the same $L^\infty$ norm for every $s > 0$ and every $n \geq 0$ is. Indeed,

$$ \sup_{x \in [-1,1]} |\tilde{P}_n^{(s)}(x)| = \sup_{x \in [-1,1]} |x^n| = 1 $$

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because $|P_n(x)| \leq P_n^s(1)$, (see the next Section).

Also the ratio between the $L^2(d\mu_x)$ norm of $\tilde{P}_n^s(x)$ and the $L^2(d\mu_x)$ norm of $x^n$ can be estimated for every $n$ and $s$.

We prove the following

**Proposition 1.1** The function $N_2(n, s) = \left| \frac{\|\tilde{P}_n^s\|_{L^2(d\mu_x)}}{\|x^n\|_{L^2(d\mu_x)}} \right|$ is decreasing with $s$, and

$$2^{-\frac{n}{2}} \left( \frac{\sqrt{\pi} \Gamma(n + 1)}{\Gamma(n + \frac{1}{2})} \right)^\frac{1}{2} = \lim_{s \to \infty} N_2(n, s) \leq \lim_{s \to 0} N_2(n, s) \leq \left( \frac{\sqrt{\pi} \Gamma(n + 1)}{2 \Gamma(n + \frac{1}{2})} \right)^\frac{1}{2}. \quad (1.3)$$

Thus,

$$2^{-\frac{n}{2}} \pi^{\frac{1}{4}} n^{\frac{1}{4}} < N_2(n, s) < n^{\frac{1}{4}}. \quad (1.4)$$

It is interesting to observe that $N_2(n, \frac{1}{2}) = 1$. This follows from the explicit formula for $N_2(n, s)$ in Section 2. By Proposition 1.1, $N_2(n, s) = 1$ if and only if $s = \frac{1}{2}$.

Proposition 1.1 shows that while it is true that $\lim_{s \to \infty} \tilde{P}_n^s(x) = x^n$, and $\lim_{s \to \infty} \|\tilde{P}_n^s(x)\|_{L^\infty(d\mu_x)} = \|x^n\|_{L^\infty(d\mu_x)}$, it is not true in general that $\lim_{s \to \infty} ||\tilde{P}_n^s(x)||_{L^2(d\mu_x)} = ||x^n||_{L^2(d\mu_x)}$.

These consideration suggested us to investigate the ratio of the $L^r(d\mu_x)$ norms of $\tilde{P}_n^s(x)$ and $x^n$ for other values of $r$. We let

$$N_r(n, s) = \frac{||\tilde{P}_n^s||_{L^r(d\mu_x)}}{||x^n||_{L^r(d\mu_x)}}, \quad 1 \leq r \leq \infty.$$ 

Our next Lemma suggests that $N_r(n, s)$ can be bounded above by a power of $N_2(n, s)$.

**Lemma 1.2** For every $s > 0$, $n \geq 1$, and $r \geq 2$,

$$N_r(n, s) \leq N_2(n, s)^2 \left( r \left( \frac{r}{2} \right)^{\frac{1}{2}(s + \frac{1}{2})} \right), \quad (1.5)$$

and

$$N_r(n, s) \leq n^{\frac{1}{4}} \left( r \left( \frac{r}{2} \right)^{\frac{1}{2}(s + \frac{1}{2})} \right). \quad (1.6)$$

When $s \to 0$ this upper bound is sharp, in the sense that the power of $n$ in (1.6) cannot be replaced by a smaller power.

The proof of the Lemma is in Section 3.

Numerical evidence suggests that $N_r(n, s) \leq N_2(n, s)^2$ when $s \geq \frac{1}{2}$ and $1 \leq r \leq \infty$. When $0 \leq s < \frac{1}{2}$ we conjecture instead that $N_r(n, s) \geq N_2(n, s)^2$.

The upper bound in Lemma 1.2 can be improved if we restrict $\tilde{P}_n^s(x)$ to the intervals $\{1 \leq |x| \leq z_n\}$ and $(-z_n, z_n)$, where $z_n$ denotes the largest positive zero of $P_n^s(x)$.

Our main result is the following.
Theorem 1.3 For every \( n > 2, \ s > 0, \) and \( r \geq 1, \)
\[
\sin^2\left(\frac{\pi}{n + 1}\right) (1 - z_n^2)^{s+\frac{1}{2}} \leq \|\tilde{P}_n^{(s)}\|_{L^r([1 \leq |x| \leq z_n], \ d\mu_s)} \leq p(n, s)^{(s+\frac{1}{2})\frac{1}{n-1}}, \tag{1.7}
\]
where \( p(n, s) = \prod_{j=1}^{n} (1 - z_j) = \frac{\Gamma(s) \Gamma(n + 2s)}{2^n \Gamma(2s) \Gamma(n + s)} \) is as in (2.15).

Using Stirling’s formula, it is possible to prove that \( \lim_{s \to \infty} p(n, s)^{(s+\frac{1}{2})} = e^{-\frac{n(n-1)}{4}}, \) and thus
\[
\lim_{s \to \infty} \frac{\|\tilde{P}_n^{(s)}\|_{L^r([1 \leq |x| \leq z_n], \ d\mu_s)}}{\|x^n\|_{L^r(d\mu_s)}} \leq \lim_{s \to \infty} p(n, s)^{(s+\frac{1}{2})\frac{1}{n-1}} = e^{-\frac{n-1}{2n}}.
\]

We have recalled in the next Section that \( z_n < \cos\left(\frac{\pi}{n + 1}\right) \sqrt{\frac{(n - 1)(n + 2s - 2)}{(n + s - 2)(n + s - 1)}}, \) (see (2.12)), and so
\[
\lim_{s \to \infty} \sin^2\left(\frac{\pi}{n + 1}\right) (1 - z_n^2)^{s+\frac{1}{2}} > \sin^2\left(\frac{\pi}{n + 1}\right) \lim_{s \to \infty} \left(1 - \frac{(n - 1)(n + 2s - 2) \cos^2\left(\frac{\pi}{n + 1}\right)}{(n + s - 2)(n + s - 1)}\right)^{s+\frac{1}{2}} = \sin^2\left(\frac{\pi}{n + 1}\right) e^{-\frac{n}{2}(n-1)\cos^2\left(\frac{\pi}{n + 1}\right)}.
\]

From the inequalities above and (1.7) follows that
\[
\sin^2\left(\frac{\pi}{n + 1}\right) e^{-\frac{n}{2}(n-1)\cos^2\left(\frac{\pi}{n + 1}\right)} < \lim_{s \to \infty} \frac{\|\tilde{P}_n^{(s)}\|_{L^r([1 \leq |x| \leq z_n], \ d\mu_s)}}{\|x^n\|_{L^r(d\mu_s)}} \leq e^{-\frac{n-1}{2n}}. \tag{1.8}
\]

This upper bound is not sharp; in fact we have proved in Proposition 1.1 that \( \lim_{s \to \infty} N_2(n, s) = (\pi n)^{\frac{1}{2}} 2^{-\frac{n}{2}}, \) while Lemma 1.3 yields \( \lim_{s \to \infty} \frac{\|\tilde{P}_n^{(s)}\|_{L^r([1 \leq |x| \leq z_n], \ d\mu_s)}}{\|x^n\|_{L^r(d\mu_s)}} \leq e^{-\frac{n-1}{4}}, \) and \( e^{-\frac{n-1}{4}} > (\pi n)^{\frac{1}{2}} 2^{-\frac{n}{2}} \) for every \( n \geq 2. \) However, Theorem (1.3) is interesting because it provides an upper and lower bound for the \( L^r([1 \leq |x| \leq z_n], \ d\mu_s) \) norm of \( \tilde{P}_n^{(s)}(x) \) and is valid for every \( r \geq 1. \)

Since \( \lim_{s \to \infty} z_n = 0, \) (see the next Section), it is natural to conjecture that \( N_2(n, s) \) is bounded above by a constant independent of \( s. \) In order to prove this conjecture we should prove that also the ratio of the \( L^r(d\mu_s) \) norm of \( \tilde{P}_n^{(s)}(x) \) in \( (-z_n, z_n) \) and \( ||x^n||_{L^r(d\mu_s)} \) is a bounded function of \( s. \)

In the next Theorem we estimate the \( L^r(d\mu_s) \) norm of \( \tilde{P}_n^{(s)}(x) \) in \( (-z_n, z_n) \) through interpolation.

Theorem 1.4 For every \( r \geq 2, \ s > 0 \) and \( n \geq 2, \)
\[
\frac{\|\tilde{P}_n^{(s)}\|_{L^r([1 \leq |x| \leq z_n], \ d\mu_s)}}{\|x^n\|_{L^r(d\mu_s)}} \leq N_2(n, s)^{\frac{1}{2}} \left(\frac{n(n + 2s)}{2s + 1}\right)^{1-\frac{1}{2}} \left(\frac{2(\frac{2s + 1}{n - \frac{1}{2}})}{z_n}\right)^{\frac{n}{2}-\frac{1}{2}} (\frac{n}{n - \frac{1}{2}})^{n(\frac{1}{2}-\frac{1}{4})}.
\]

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where \( z_n \) denotes the largest zero of \( P_n^{(s)} \). Furthermore,

\[
\lim_{s \to \infty} \frac{||P_n^{(s)}||_{L^r([-z_n, z_n], d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} \leq n^{1-\frac{2}{r}} 2^{n(\frac{n}{2} - \frac{r}{2})} N_2(n, s) \frac{2}{\pi}.
\]

From Theorems 1.3 and 1.4 and Proposition 1.1 we can easily prove the following

**Corollary 1.5** For every \( n \geq 2 \) and every \( r \geq 2 \), \( \lim_{s \to \infty} N_r(n, s) \) is finite. If \( r < 4 \) this limit is bounded above by a constant that does not depend on \( n \).

## 2. Preliminaries

Let us briefly review the main properties of the ultraspherical polynomials. For more details we refer to [Sz]. The ultraspherical polynomials can be defined through Rodrigues’ formula

\[
(1 - x^2)^{s - \frac{1}{2}} P_n^{(s)}(x) = \frac{(-1)^n \Gamma(s + \frac{1}{2}) \Gamma(n + 2s)}{\Gamma(2s) \Gamma(n + s + \frac{1}{2}) \Gamma(n + 1) 2^n} \left( \frac{d}{dx} \right)^n (1 - x^2)^n s - \frac{1}{2}.
\]

(2.1)

When \( s > 0 \) we have the following explicit expression

\[
P_n^{(s)}(x) = \sum_{m=0}^{\left[ \frac{2}{s} \right]} (-1)^m \frac{\Gamma(n - m + s)}{\Gamma(s) \Gamma(m + 1) \Gamma(n - 2m + 1)} (2x)^{n - 2m}.
\]

(2.2)

Note that \( P_n^{(s)}(x) \) is either even or odd.

The ultraspherical polynomials of order \( s = 0 \) are related to the Tchebicheff polynomials \( T_n(x) = \cos(n \cos^{-1}(x)) \) by the following limit relation.

\[
\lim_{s \to 0} s^{-1} P_n^{(s)}(x) = \frac{2}{n} T_n(x).
\]

(2.3)

The \( L^2 \) norm of \( P_n^{(s)}(x) \) with respect to the measure \( d\mu_s(x) = (1 - x^2)^{s - \frac{1}{2}} dx \) in \((-1, 1)\) can be explicitly computed. It is

\[
||P_n^{(s)}||_{L^2(d\mu_s)}^2 = \int_{-1}^{1} |P_n^{(s)}(x)|^2 (1 - x^2)^{s - \frac{1}{2}} dx = \frac{\pi 2^{1-2s} \Gamma(n + 2s)}{(n + s) (\Gamma(s))^2 \Gamma(n + 1)}.
\]

(2.4)

When \( s > 0 \) the maximum of \( |P_n^{(s)}(x)| \) in \([-1, 1]\) can be explicitly computed. We have:

\[
\sup_{-1 \leq x \leq 1} |P_n^{(s)}(x)| = P_n^{(s)}(1) = \frac{\Gamma(n + 2s)}{\Gamma(n + 1) \Gamma(2s)}, \quad s > 0.
\]

(2.5)

\( N_2(n, s) \) can be explicitly computed as well. Indeed, the \( L^r(d\mu_s) \) norm of \( x^n \) is

\[
||x^n||_{L^r(d\mu_s)} = \left( \int_{-1}^1 x^{nr} (1 - x^2)^{s - \frac{1}{2}} dx \right)^{\frac{1}{r}} = \beta^\frac{1}{r} \left( \frac{1}{2} (nr + 1), s + \frac{1}{2} \right) = \left( \frac{\Gamma \left( \frac{1}{2} (nr + 1) \right) \Gamma \left( s + \frac{1}{2} \right)}{\Gamma \left( \frac{nr}{2} + s + 1 \right)} \right)^{\frac{1}{r}},
\]

(2.6)
where $\beta(a, b)$ is the standard Beta function. Thus,

$$N_2(n, s) = \left(2^{-n} \sqrt{\pi} \Gamma(n + 1) \Gamma\left(s + \frac{1}{2}\right) \Gamma(n + s) \right) \left(\frac{1}{\Gamma\left(n + \frac{1}{2}\right) \Gamma\left(\frac{s}{2} + \frac{1}{2}\right)}\right)^{\frac{1}{2}}. \tag{2.7}$$

This expression has been simplified with the aid of the well known duplication formula for the Gamma function $\Gamma(2x) = \frac{2^{2x-1} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right)}{\sqrt{\pi}}$. It is interesting to note that $N_2(n, \frac{1}{2}) \equiv 1$.

The derivatives of ultraspherical polynomials are constant multiples of ultraspherical polynomials. From (2.2) easily follows that

$$\frac{d}{dx} P_n^{(s)}(x) = 2s P_{n-1}^{(s+1)}(x), \tag{2.8}$$

and if we let $\tilde{P}_n^{(s)}(x) = \frac{P_n^{(s)}(x)}{P_n^{(s)}(1)}$, from (2.8) and 2.5 follows that

$$\frac{d}{dx} \tilde{P}_n^{(s)}(x) = \frac{n(n + 2s)}{1 + 2s} \tilde{P}_{n-1}^{(s+1)}(x), \tag{2.9}$$

$$\frac{d^2}{dx^2} \tilde{P}_n^{(s)}(x) = \frac{n(n - 1)(n + 2s)(n + 2s + 1)}{(1 + 2s)(3 + 2s)} \tilde{P}_{n-2}^{(s+2)}(x). \tag{2.10}$$

$P_n^{(s)}(x)$ satisfies the following differential equation:

$$(1 - x^2)y'' - (2s + 1)xy' + n(n + 2s)y = 0. \tag{2.11}$$

The zeros of ultraspherical polynomials have important and well studied properties. The literature on the subject is extensive and we will not attempt to survey it. We refer to [E] and the references cited there.

The following properties are well known, and are shared also by other systems of orthogonal polynomials.

All zeros of $P_n^{(s)}(x)$ are real and simple and lie in $[-1, 1]$. Since $\frac{d}{dx} P_n^{(s)}(x) = 2s P_{n-1}^{(s+1)}(x)$, Rolle’s theorem implies that between any two zeros of $P_n^{(s)}(x)$ there is a zero of $P_{n-1}^{(s+1)}(x)$.

We will denote by $z_{n,k}(s)$, $k = 1, \ldots, n$, the zeros of $P_n^{(s)}(x)$ enumerated in increasing order. That is, $-1 < z_{n,1}(s) < \ldots < z_{n,n}(s) < 1$. When there is no risk of confusion, we will just let $z_{n,j}(s) = z_j$.

To the best of our knowledge, the best available upper bound for $z_n$ is in [ADGR].

$$z_n < \sqrt{\frac{(n - 1)(n + 2s - 2)}{(n + s - 2)(n + s - 1)}} \cos \left(\frac{\pi}{n + 1}\right), \quad n \geq 1. \tag{2.12}$$

The inequality (2.12) improves the following inequality due to Elbert, (see [E]).

$$z_n < \sqrt{\frac{(n - 1)(n + 2s + 1)}{n + s}} = \sqrt{1 - \left(\frac{s + 1}{n + s}\right)^2}. \tag{2.13}$$
2.1 Four useful Lemmas

Lemma 2.1 Let \( z_1 \ldots z_n \) be the zeros of \( P_n^{(s)}(x) \) arranged in increasing order. Then

\[
P_n^{(s)}(x) = \prod_{k=[n+1]/2}^{n} \frac{x^2 - z_k^2}{1 - z_k^2},
\]

(2.14)

and

\[
\prod_{k=[n+1]/2}^{n} (1 - z_k^2) = \prod_{k=1}^{n} (1 - z_k) = \frac{\Gamma(s) \Gamma(n + 2s)}{2^n \Gamma(2s) \Gamma(n + s)}.
\]

(2.15)

Furthermore

\[
P_n^{(s)}(x) \leq x^n \text{ for } x \geq z_n
\]

(2.16)

Proof. We have already observed that the zeros of \( P_n^{(s)}(x) \) are symmetric with respect to \( x = 0 \). When \( n \) is odd, \( P_n^{(s)}(x) \) vanishes also at \( x = 0 \). Therefore, if we let

\[
M(n, s) = 2^n \frac{\Gamma(n + 2s)}{\Gamma(2s) \Gamma(n + s)},
\]

we can factorize \( P_n^{(s)}(x) \) as follows:

\[
P_n^{(s)}(x) = M(n, s) \prod_{k=1}^{n} (x - z_k) = M(n, s) \left\{ \begin{array}{ll}
\prod_{k=[n+1]/2}^{n} (x^2 - z_k^2) & \text{if } n \text{ is even,} \\
x \prod_{k=[n-1]/2}^{n} (x^2 - z_k^2) & \text{if } n \text{ is odd.}
\end{array} \right.
\]

(2.17)

Thus,

\[
\tilde{P}_n^{(s)}(x) = \frac{P_n^{(s)}(x)}{P_n^{(s)}(1)} = \left\{ \begin{array}{ll}
\prod_{k=[n+1]/2}^{n} \frac{x^2 - z_k^2}{1 - z_k^2} & \text{if } n \text{ is even,} \\
x \prod_{k=[n-1]/2}^{n} \frac{x^2 - z_k^2}{1 - z_k^2} & \text{if } n \text{ is odd}
\end{array} \right.
\]

which is (2.14).

Since \( \frac{x^2 - z_k^2}{1 - z_k^2} \leq x^2 \), (2.16) follows.

Let \( p(n, s) = \prod_{k=1}^{n} (1 - z_k) \). Note that \( p(n, s) = \frac{P_n^{(s)}(1)}{M(n, s)} \), and since \( P_n^{(s)}(1) \) is as in (2.5),

\[
p(n, s) = \frac{\Gamma(s) \Gamma(n + 2s)}{2^n \Gamma(2s) \Gamma(n + s)}
\]

as required.

The inequality (2.16) can also be proved from the following Lemma.

Lemma 2.2 for every \( n > 1 \) and \( s > 0 \), \( \tilde{P}_n^{(s)}(x) \leq x \tilde{P}_{n-1}^{(s+1)}(x) \) in \([z_n, 1]\).
Proof. Our key tool is the differential equation (2.11). That is,
\[ n(n + 2s)y = (2s + 1)xy' - (1 - x^2)y'', \] (2.18)
where \( y = P^{(s)}_n(x). \)

We divide both sides of (2.18) by \( P^{(s)}_n(1) \) and recall that, by (2.9) and (2.10),
\[ \frac{d}{dx} \tilde{P}^{(s)}_n(x) = \frac{n(n + 2s)}{1 + 2s} \tilde{P}^{(s+1)}_{n-1}(x) \]
and\[ \frac{d^2}{dx^2} \tilde{P}^{(s)}_n(x) = \frac{n(n - 1)(n + 2s)(n + 2s + 1)}{(1 + 2s)(3 + 2s)} \tilde{P}^{(s+2)}_{n-2}(x). \]
We obtain the following three term relation.
\[ \frac{(n - 1)(n + 2s + 1)}{(1 + 2s)(3 + 2s)}(1 - x^2) \tilde{P}^{(s+2)}_{n-2}(x) - x \tilde{P}^{(s+1)}_{n-1}(x) + \tilde{P}^{(s)}_n(x) = 0, \]
and since \( \tilde{P}^{(s+2)}_{n-2}(x) \) is positive in \([z_n, 1], \) we gather
\[ \tilde{P}^{(s)}_n(x) < x \tilde{P}^{(s+1)}_{n-1}(x), \]
as required.

The following Lemma improves a Lemma in [DC]

Lemma 2.3 For every \( 0 \leq |x| \leq z_k, \ s > 0 \) and \( n \geq 2, \)
\[ |\tilde{P}^{(s)}_n(x)| \leq \frac{n(n + 2s)}{2s + 1} \xi^{n-1}z_n. \] (2.19)

Proof. It is well known, (see e.g. [Sz]), that the local maxima of \( |P^{(s)}_n(x)| \) are increasing. The critical points of \( P^{(s)}_n(x) \) are the zeros of \( P^{(s+1)}_{n-1}(x), \) and hence \( |P^{(s)}_n(x)|, \) restricted to the interval \([0, z_n], \) attains its maximum at the largest zero of \( P^{(s+1)}_{n-1}(x), \) which we can denote by \( \xi_{n-1}. \) Thus, for every \( x \in [-z_n, z_n], \)
\[ \tilde{P}^{(s)}_n(x) \leq \tilde{P}^{(s)}_n(\xi_{n-1}). \]

By the mean value theorem,
\[ \tilde{P}^{(s)}_n(z_n) - \tilde{P}^{(s)}_n(\xi_{n-1}) = (z_n - \xi_{n-1}) \frac{\partial}{\partial x} \tilde{P}^{(s)}_n(\xi) \]
where \( \xi_{n-1} < \xi < z_n. \) By (2.9),
\[ -\tilde{P}^{(s)}_n(\xi_{n-1}) = (z_n - \xi_{n-1}) \frac{n(n + 2s)}{2s + 1} \tilde{P}^{(s+1)}_{n-1}(\xi) \]
and since \( \tilde{P}^{(s+1)}_{n-1}(x) \leq x^{n-1} \) in \([\xi_{n-1}, 1], \) and \( -\tilde{P}^{(s)}_n(\xi_{n-1}) = |\tilde{P}^{(s)}_n(\xi_{n-1})|, \) we can infer that
\[ \tilde{P}^{(s)}_n(x) \leq \xi^{n-1}(z_n - \xi_{n-1}) \frac{n(n + 2s)}{2s + 1} < \xi_{n-1}^{n-1} \frac{n(n + 2s)}{2s + 1}, \]
as required.

The following Lemma concerns the monotonicity of ratios of Gamma functions.
Lemma 2.4  a) The function

\[ x \to \frac{\Gamma(x)x^y}{\Gamma(x+y)}, \quad x > 0, \]

is decreasing when \( 0 < y \leq 1 \) and is increasing when \( y > 1 \). Therefore,

\[ \frac{\Gamma(x)x^y}{\Gamma(x+y)} \leq \lim_{x \to \infty} \frac{\Gamma(x)x^y}{\Gamma(x+y)} = 1 \tag{2.20} \]

when \( y > 1 \), and the inequality reverses when \( y < 1 \).

b) The function

\[ x \to \frac{\Gamma(x)(x+y)^y}{\Gamma(x+y)}, \quad x > 0, \]

is decreasing for every \( y > 0 \), and

\[ \frac{\Gamma(x)x^y}{\Gamma(x+y)} \geq \lim_{x \to \infty} \frac{\Gamma(x)x^y}{\Gamma(x+y)} = 1. \tag{2.21} \]

Proof. We prove only a) since the proof of b) is almost the same. Let \( g_y(x) = \frac{\Gamma(x)x^y}{\Gamma(x+y)} \). When \( y = 0 \) and \( y = 1 \), then \( g_y(x) \equiv 1 \), so we assume either \( y > 1 \) or \( 0 < y < 1 \).

To investigate the monotonicity of \( g_y(x) \) we study the sign of the derivative of

\[ \ln g_y(x) = y \ln x + \ln(\Gamma(x)) - \ln(\Gamma(x+y)). \]

The logarithmic derivative of \( \Gamma(z) \) is

\[ \frac{\Gamma'(z)}{\Gamma(z)} = \gamma - \frac{1}{z} - \sum_{m=1}^{\infty} \left( \frac{1}{z+m} - \frac{1}{m} \right) \]

where \( \gamma \) is Euler’s constant. Therefore,

\[ (\ln g_y(x))' = \frac{y}{x} - \sum_{m=0}^{\infty} \frac{1}{x+m} - \frac{1}{x+y+m} \]

\[ = y \left( \frac{1}{x} - \sum_{m=0}^{\infty} \frac{1}{(x+m)(x+m+y)} \right). \]

Note that

\[ \frac{1}{x} = \sum_{m=0}^{\infty} \frac{1}{x+m} - \frac{1}{x+m+1} = \sum_{m=0}^{\infty} \frac{1}{(x+m)(x+m+1)}. \]

Thus,

\[ (\ln g_y(x))' = y \left( \sum_{m=0}^{\infty} \frac{1}{(x+m)(x+m+y)} - \sum_{m=0}^{\infty} \frac{1}{(x+m)(x+m+y)} \right) \]

\[ = y \sum_{m=0}^{\infty} \frac{y-1}{(x+m+1)(x+m+y)}. \tag{2.22} \]

When \( y > 1 \) the function in (2.22) is positive and when \( y < 1 \) it is negative. Therefore, \( \ln g_y(x) \) is increasing whenever \( y > 1 \) and is decreasing whenever \( 0 \leq y < 1 \), as required.

(2.20) follows by Stirling’s formula.
3 Most of the proofs

Proof of Theorem 1.3. We use the factorization in (2.14). Suppose that $n$ is even, since the proof is similar in the other case. By Hölder inequality,

$$||\tilde{P}_n^{(s)}||_{L^r((z_n, 1), \, d\mu_s)} = \left( \int_{z_n}^{1} \prod_{j=\frac{n}{2}}^{n} \left( \frac{t^2 - z_j^2}{1 - z_j^2} \right)^{\frac{r}{2}} (1 - t^2)^{s - \frac{1}{2}} \right)^{\frac{1}{r}}$$

$$\leq \prod_{j=\frac{n}{2}}^{n} \left( \int_{z_n}^{1} \left( \frac{t^2 - z_j^2}{1 - z_j^2} \right)^{\frac{r}{2}} (1 - t^2)^{s - \frac{1}{2}} \right)^{\frac{2}{r}} = \prod_{j=\frac{n}{2}}^{n} J(z_j),$$

where we have let $J(z_j) = \left( \int_{z_n}^{1} \left( \frac{t^2 - z_j^2}{1 - z_j^2} \right)^{\frac{r}{2}} (1 - t^2)^{s - \frac{1}{2}} \right)^{\frac{2}{r}} dt$.

In order to compare $J(z_j)$ with $||x^n||_{L^r(d\mu_s)}$ we let $\frac{t^2 - z_j^2}{1 - z_j^2} = x^2$, so that $t = \sqrt{x^2(1 - z_j^2) + z_j^2}$ and $dt = \frac{x(1 - z_j^2)}{\sqrt{x^2(1 - z_j^2) + z_j^2}} dx$. Note that $x \leq \frac{x}{\sqrt{x^2(1 - z_j^2) + z_j^2}} \leq 1$.

With this substitution, $(1 - t^2)^{s - \frac{1}{2}} = (1 - z_j^2)(1 - x^2)^{s - \frac{1}{2}}$, and

$$J(z_j)^{\frac{r}{2}} = (1 - z_j^2)^{s + \frac{1}{2}} \int_{\frac{z_j^2}{1 - z_j^2}}^{1} x^{nr}(1 - x^2)^{s - \frac{1}{2}} \frac{xdx}{\sqrt{x^2(1 - z_j^2) + z_j^2}} \leq (1 - z_j^2)^{s + \frac{1}{2}} \int_{0}^{1} x^{nr}(1 - x^2)^{s - \frac{1}{2}} dx = (1 - z_j^2)^{s + \frac{1}{2}} ||x^n||_{L^r(d\mu_s)}$$

and

$$||\tilde{P}_n^{(s)}||_{L^r((z_n, 1), \, d\mu_s)} \leq \prod_{j=1}^{n} J(z_j) \leq \prod_{j=\frac{n}{2}}^{n} \left( (1 - z_j^2)^{s + \frac{1}{2}} \right)^{2\frac{r}{2}} = ||x^n||_{L^r(d\mu_s)} \mathcal{P}(n, s)^{(s + \frac{1}{2})\frac{2}{nr}},$$

as required.

To prove the other inequality we observe that

$$\frac{x^2 - z_j^2}{1 - z_j^2} \geq \frac{x^2 - z_n^2}{1 - z_n^2}$$

whenever $j \leq \frac{n}{2}$. Therefore,

$$||\tilde{P}_n^{(s)}||_{L^r((z_n, 1), \, d\mu_s)} \geq \left( \int_{z_n}^{1} \left( \frac{t^2 - z_n^2}{1 - z_n^2} \right)^{\frac{r}{2}} d\mu_s(t) \right)^{\frac{1}{r}}.$$
We use again the substitution \( \frac{t^2 - z_n^2}{1 - z_n^2} = x^2 \), so that and

\[
\|\tilde{P}_n^{(s)}\|_{L^r((z_n, 1), d\mu_s)} \geq (1 - z_n^2)^{s+\frac{1}{2}} \int_0^1 \frac{x^{nr+1}}{\sqrt{2(1 - z_n^2) + z_n^2}} (1 - x^2)^{s-\frac{1}{2}} dx
\]

\[= (1 - z_n^2)^{s+\frac{1}{2}} \int_0^1 x^{nr} \psi(x, z_n^2)(1 - x^2)^{s-\frac{1}{2}} dx, \quad (3.23)\]

where we have let \( \psi(x, t) = \frac{x}{\sqrt{2(1 - t) + t}} \).

The easy inequality \( \psi(x, t) \geq x \) is not enough to prove (1.7). We use the elementary inequality \( a^2 + b^2 - 2ab \geq 0 \), with \( a = \psi(x, t)^\frac{1}{2} \) and \( b \in \mathbb{R} \), to infer that

\( \psi(x, t) \geq 2b(\psi(x, t))^\frac{1}{2} - b^2 \geq 0 \) for every \( b \in \mathbb{R} \). From (3.23) follows that

\[
(1 - z_n^2)^{-(s+\frac{1}{2})} \|\tilde{P}_n^{(s)}\|_{L^r((z_n, 1), d\mu_s)}
\]

\[\geq \left( \int_0^1 2b(\psi(x, z_n^2))^\frac{1}{2} x^{nr}(1 - x^2)^{s-\frac{1}{2}} dx - b^2 \|x^n\|_{L^r(d\mu_s)} \right)^2, \quad (3.24)\]

Our next task is to choose \( b \) so to maximize the function in (3.24).

It is easy to verify that \( (\psi(x, t))^\frac{1}{2} \) is a convex whenever \(-1 < x < 1\), and thus, by Taylor formula,

\[
\psi^\frac{1}{2}(x, t) \geq (\psi(x, 0))^\frac{1}{2} + t \frac{\partial}{\partial t}(\psi(x, 0))^\frac{1}{2} = 1 - t \frac{1 - x^2}{2x^2},
\]

and

\[
\|\tilde{P}_n^{(s)}\|_{L^r((z_n, 1), d\mu_s)} (1 - z_n^2)^{-(s+\frac{1}{2})}
\]

\[\geq 2b \int_0^1 \left( 1 - z_n^2 \frac{1 - x^2}{2x^2} \right) x^{nr}(1 - x^2)^{s-\frac{1}{2}} dx - b^2 \|x^n\|_{L^r(d\mu_s)}
\]

\[= \|x^n\|_{L^r(d\mu_s)} \left( 2b - b z_n^2 \int_0^1 x^{nr-2}(1 - x^2)^{s-\frac{1}{2}} dx \right) - b^2 \|x^n\|_{L^r(d\mu_s)}
\]

\[= b \left( 2 - b - z_n^2 \left( \frac{2s + 1}{2(nr - 1)} \right) \right) \|x^n\|_{L^r(d\mu_s)}.
\]

The function \( b \to b \left( 2 - z_n^2 \left( \frac{2s + 1}{2(nr - 1)} \right) - b \right) \) attains its maximum when

\[b = \frac{1}{2} \left( 2 - z_n^2 \frac{2s + 1}{2(nr - 1)} \right),\]

and so

\[
\|\tilde{P}_n^{(s)}\|_{L^r((z_n, 1), d\mu_s)} \geq (1 - z_n^2)^{s+\frac{1}{2}} \left( 1 - z_n^2 \frac{2s + 1}{4(nr - 1)} \right)^2 \|x^n\|_{L^r(d\mu_s)}.
\]

We are left to prove that

\[c(n, s, r) = 1 - z_n^2 \frac{2s + 1}{4(nr - 1)} \]

is always positive. We use the upper bound for \( z_n \) in (2.13), so to obtain

\[1 - z_n^2 \frac{2s + 1}{4(nr - 1)} \geq 1 - \frac{(n - 1)(2s + 1)(n + 2s - 2) \cos^2 \left( \frac{\pi}{n+1} \right)}{4(nr - 1)(n + s - 2)(n + s - 1)}
\]

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\[
\geq 1 - \frac{(n-1)(2s+1)(n+2s-2) \cos^2 \left( \frac{\pi(n+1)}{2(n-1)} \right)}{4(n-1)(n+s-2)(n+s-1)}.
\]
It is easy to verify that the function above decreases with \(s\) and hence,
\[
1 - z_n^2 \frac{2s+1}{4nr-1} \geq \lim_{s \to \infty} 1 - \frac{(2s+1)(n+2s-2) \cos^2 \left( \frac{\pi(n+1)}{2(n-1)} \right)}{4(n+s-2)(n+s-1)}
= 1 - \cos^2 \left( \frac{\pi}{n+1} \right) = \sin^2 \left( \frac{\pi}{n+1} \right)
\]
and from that (1.7) follows.

Proof of Proposition 1.1 To prove that \(N_2(n, s)\) decreases with \(s\) we study the function \(s \to \log(N_2(n, s))\); \(N_2(n, s)\) is decreasing in \(s\) if and only \(\frac{\partial}{\partial s}\log(N_2(n, s))\) is negative.

We recall that \(N_2(n, s) = \left( \frac{2^{n-1} \sqrt{\pi} \Gamma(n+1) \Gamma \left( s + \frac{1}{2} \right) \Gamma(n+s)}{\Gamma \left( n + \frac{1}{2} \right) \Gamma \left( \frac{n}{2} + s \right) \Gamma \left( \frac{n}{2} + \frac{1}{2} \right)} \right)^{\frac{1}{2}}\).

The partial derivative of \(\log(N_2(n, s))\) with respect to \(s\) is
\[
\frac{\partial}{\partial s}\log(N_2(n, s)) = -\frac{1}{2} \sum_{m=0}^{\infty} \left( \frac{1}{\frac{n}{2} + s + m} + \frac{1}{\frac{n}{2} + s + \frac{1}{2} + m} - \frac{1}{\frac{1}{2} + s + m} - \frac{1}{n + s + m} \right)
= -\frac{n(n+1)}{2} \sum_{m=0}^{\infty} \frac{4m+2n+4s+1}{(m+n+s)(2m+2s+1)(2m+n+2s)(2m+n+2s+1)}
\]
which is negative, as required.

(1.4) follows by Stirling formula.

Proof of Lemma 1.2. We use Riesz interpolation theorem. When \(r \geq 2\),
\[
||P_n^{(s)}||_{L^r(d\mu_s)} \leq ||P_n^{(s)}||_{L^2(d\mu_s)}^{\frac{2}{r}} ||P_n^{(s)}||_{L^\infty(d\mu_s)}^{1-\frac{2}{r}},
\]
or equivalently
\[
||\tilde{P}_n^{(s)}||_{L^r(d\mu_s)} \leq ||\tilde{P}_n^{(s)}||_{L^2(d\mu_s)}^{\frac{2}{r}} ||\tilde{P}_n^{(s)}||_{L^\infty(d\mu_s)}^{1-\frac{2}{r}}
\]
since \(||\tilde{P}_n^{(s)}||_{L^\infty(d\mu_s)} = 1\). From the inequality above follows that
\[
\frac{||\tilde{P}_n^{(s)}||_{L^r(d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} \leq \left( \frac{||\tilde{P}_n^{(s)}||_{L^2(d\mu_s)}}{||x^n||_{L^2(d\mu_s)}} \right)^{\frac{2}{r}} \frac{||x^n||_{L^2(d\mu_s)}}{||x^n||_{L^r(d\mu_s)}}
= N_2(n, s)^{\frac{2}{r}} \left( \frac{\Gamma \left( n + \frac{1}{2} \right) \Gamma \left( \frac{nr}{2} + s + 1 \right)}{\Gamma \left( \frac{1}{2}(nr+1) \right) \Gamma(n+s+1)} \right)^{\frac{1}{r}}.
\]
(3.25)
We can argue as in Lemma 1.1 to show that the function

$$n \to \frac{\Gamma \left( n + \frac{1}{2} \right) \Gamma \left( \frac{nr}{2} + s + 1 \right)}{\Gamma \left( \frac{1}{2}(nr + 1) \right) \Gamma(n + s + 1)}$$

is increasing, and is then bounded above by its limit at $n \to \infty$, which is $\left( \frac{r}{2} \right)^{s+\frac{1}{2}}$.

(2.20) follows from Lemma 1.1.

We are left to prove that the upper bound in (2.21) is actually sharp when $s = 0$.

Recalling that $\lim_{s \to 0} s^{-1} P_n^{(s)}(x) = \frac{2}{n} \cos(nx)$, (see Section ), we can see that

$$\lim_{s \to 0} ||\tilde{P}_n^{(s)}||_{L^r(d\mu_s)} = \lim_{s \to 0} \int_{-1}^{1} |\tilde{P}_n^{(s)}(x)|^r (1 - x^2)^{s-\frac{1}{2}} dx$$

(3.26)

We have used the change of variable $x = \cos t$ in the integral in (3.26). Therefore,

$$N_r(n, 0) = \lim_{s \to 0} \frac{||\tilde{P}_n^{(s)}||_{L^r(d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} = \left( \frac{\Gamma \left( \frac{r+1}{2} \right) \Gamma \left( \frac{nr}{2} + 1 \right)}{\Gamma \left( \frac{r}{2} + 1 \right) \Gamma \left( \frac{1}{2}(nr + 1) \right)} \right)^{\frac{1}{r}}.$$

By Lemma 2.4, the function $n^{-\frac{1}{2}} N_r(n, 0)$ is increasing, and its limit is $\left( \frac{(\frac{r}{2})^2 \Gamma \left( \frac{r+1}{2} \right)}{\Gamma \left( \frac{r}{2} + 1 \right)} \right)^{\frac{1}{r}}$.

Proof of Theorem 1.4. We use Lemma 2.3 and interpolation. When $r \geq 2$,

$$||P_n^{(s)}||_{L^r((-z_n, z_n), d\mu_s)} \leq ||P_n^{(s)}||_{L^2(d\mu_s)}^{\frac{r}{2}} ||P_n^{(s)}||_{L^{\infty}((-z_n, z_n), d\mu_s)}^{1-\frac{r}{2}}$$

or equivalently

$$||\tilde{P}_n^{(s)}||_{L^r((-z_n, z_n), d\mu_s)} \leq \left( \frac{n(n + 2s)}{2s + 1} \zeta_n^{n-1} z_n \right)^{1-\frac{2}{r}} ||\tilde{P}_n^{(s)}||_{L^2(d\mu_s)}^{\frac{2}{r}}.$$

From the inequality above follows that

$$\frac{||\tilde{P}_n^{(s)}||_{L^r((-z_n, z_n), d\mu_s)}}{||x^n||_{L^r(d\mu_s)}} 
\leq \left( \frac{n(n + 2s)}{2s + 1} \zeta_n^{n-1} z_n \right)^{1-\frac{2}{r}} \left( \frac{||\tilde{P}_n^{(s)}||_{L^2(d\mu_s)}}{||x^n||_{L^2(d\mu_s)}} \right)^{\frac{2}{r}} 
\leq \left( \frac{n(n + 2s)}{2s + 1} \zeta_n^{n} \right)^{1-\frac{2}{r}} N_2(n, s)^{\frac{2}{r}} \left( \frac{\Gamma \left( n + \frac{1}{2} \right) \Gamma \left( \frac{nr}{2} + s + 1 \right)}{\Gamma \left( \frac{1}{2}(nr + 1) \right) \Gamma(n + s + 1)} \right)^{\frac{1}{r}}.$$
We use Lemma 2.4 to estimate the ratio of the Gamma functions in the inequality above.

First we apply the Lemma to the ratio

\[
\frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}(nr + 1)\right)} = (n - \frac{1}{2})^{-\frac{nr}{2}+n-1} \frac{\Gamma\left(n - \frac{1}{2}\right)(n - \frac{1}{2})^{\frac{nr}{2}-n+1}}{\Gamma\left(\frac{1}{2}(nr + 1)\right)} = (n - \frac{1}{2})^{-\frac{nr}{2}+n} \frac{\Gamma(x) x^y}{\Gamma(x+y)}
\]

with \(x = n - \frac{1}{2}\) and \(y = \frac{nr}{2} - n + 1\). Since \(y > 1\), \(\frac{\Gamma(x) x^y}{\Gamma(x+y)} < 1\).

Then we apply the Lemma to the ratio

\[
\frac{\Gamma\left(\frac{nr}{2} + s + 1\right)}{\Gamma(n + s + 1)} = \left(\frac{nr}{2} + s + 1\right)^{\frac{nr}{2}-n} \frac{\Gamma(x+y)}{(x+y)^y \Gamma(x)}
\]

\(x = n + s + 1\) and \(y = \frac{nr}{2} - n\). The ratio \(\frac{\Gamma(x+y)}{(x+y)^y \Gamma(x)}\) is always increasing, and so it is < 1. Therefore,

\[
\left(\frac{\Gamma\left(n + \frac{1}{2}\right) \Gamma\left(\frac{nr}{2} + s + 1\right)}{\Gamma\left(\frac{1}{2}(nr + 1)\right) \Gamma(n + s + 1)}\right)^\frac{1}{r} \leq \left(\frac{nr}{2} + s + 1\right)^{\frac{nr}{2}-n} \Gamma(x)\]

and

\[
\frac{||\tilde{P}_n^{(s)}||_{L^r((-z_n, z_n), d\mu_n)}}{||x^n||_{L^r(d\mu_n)}} \leq \left(n(n + 2s)\right)^{1 - \frac{2}{r}} \left(\frac{nr}{2} + s + 1\right)^{\frac{nr}{2}-n} \Gamma(x)
\]

as required.

By (2.12), \(z_n^2 < \frac{(n - 1)(n + 2s - 2) \cos^2\left(\frac{\pi}{n+1}\right)}{(n + s - 2)(n + s - 1)}\), and so

\[
\frac{||\tilde{P}_n^{(s)}||_{L^r((-z_n, z_n), d\mu_n)}}{||x^n||_{L^r(d\mu_n)}} \leq \left(n(n + 2s)\right)^{1 - \frac{2}{r}} \left(\frac{(n - 1)(n + 2s - 2) \cos^2\left(\frac{\pi}{n+1}\right)}{(n + s - 2)(n + s - 1)}\right)^{\frac{nr}{2}-n} \Gamma(x)
\]

When \(s \rightarrow \infty\) the right hand side tends to \(n^{1 - \frac{2}{r}} \left(\frac{4(n - 1) \cos^2\left(\frac{\pi}{n+1}\right)}{2n - 1}\right)^{\frac{nr}{2}-n} \Gamma(x)\). It is easy to prove that the function in parenthesis is an increasing function of \(n\), and its limit is 2. This concludes the proof of the Theorem.

References


