ECO 4933 Topics in Theory

Introduction to Economic Growth

Fall 2015

The Solow Model

Chapter 2

The Solow Growth Model

The Solow Model

Assumptions:

1. The world consists of countries that produce and consume only a single, homogenous good (GDP)
2. There is not international trade (closed economy)
3. Technology is *exogenous*
4. Saving rate is constant *s*
5. Fraction of time devoted to accumulation of skills is also constant.

The Solow Model

- Basic Solow Model: 2 equations
- 1st Equation: Aggregate Production function

$$Y = F\left(K, L\right) = K^{\alpha} L^{1-\alpha};\ 0 \leq \alpha \leq 1$$

- We assume perfect competition so that firms cannot influence w or r
- Firms have to solve this problem

$$\max_{K,L} F\left(K, L\right) - rK - wL$$

The Solow Model

- FOC

$$w = \frac{\partial F}{\partial L} = (1-\alpha)\frac{Y}{L}$$
$$r = \frac{\partial F}{\partial K} = \alpha \frac{Y}{K}$$

- Notice that: $wL + rK = Y$ ➔ there is no economic profit
- This is a general property of CRS PF

The Solow Model

- Shares of L & K

$$\frac{wL}{Y} = 1 - \alpha$$

$$\frac{rK}{Y} = \alpha$$

- Shares are constant over time (Fact #5)

The Solow Model

- Rewrite equations in per capita terms

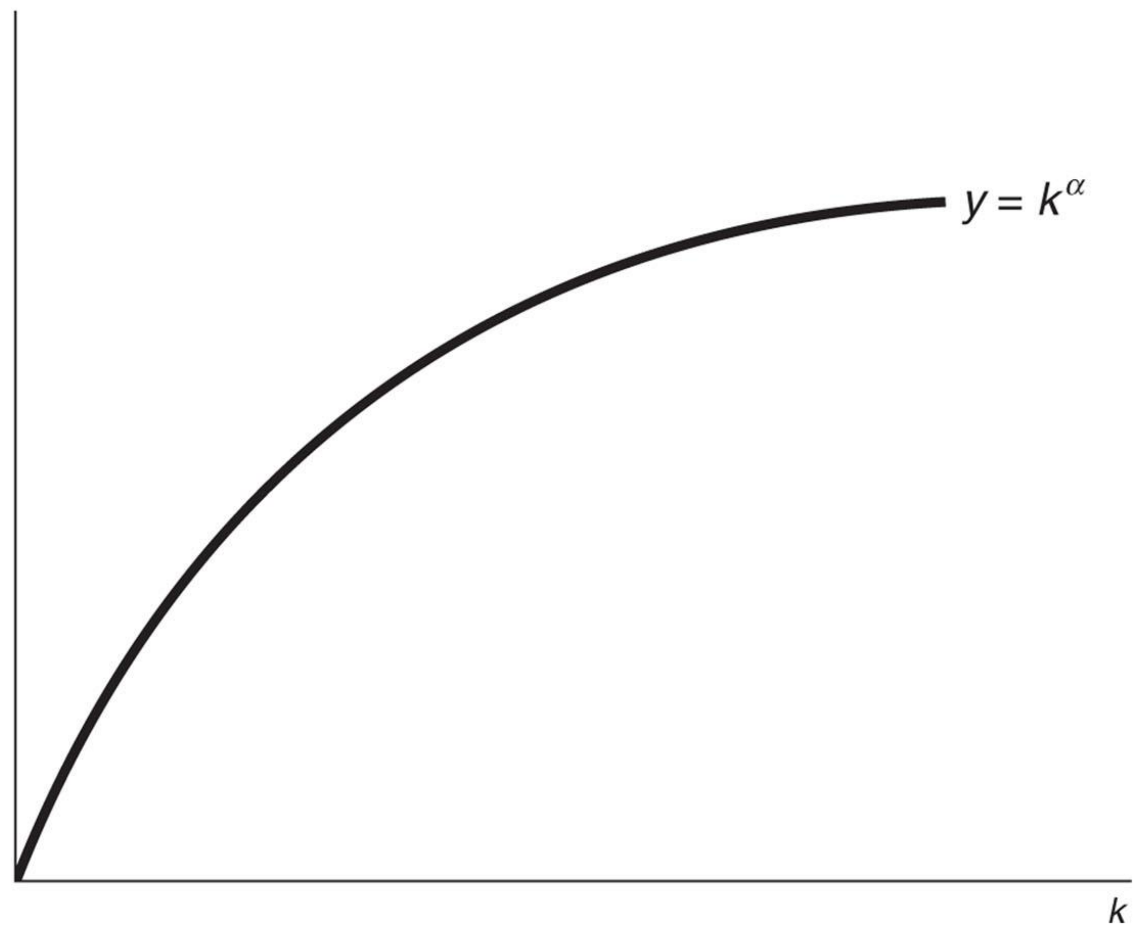
$$y \equiv \frac{Y}{L}; \quad k \equiv \frac{K}{L}$$

$$y = k^{\alpha}$$

- Diminishing return to capital per worker (k)

The Solow Model

FIGURE 2.1 **A COBB-DOUGLAS PRODUCTION FUNCTION**



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- 2^{nd} Equation: Capital Accumulation

$$\dot{K} = sY - \delta K \qquad (2.3)$$

$$\dot{K} = \frac{dK}{dt}$$

$$K_{t+1} - K_t \text{ (discrete time equivalent)}$$

- s is constant; closed economy ➔ $S = I$
- δ is also constant; depreciation rate

The Solow Model

- Capital Accumulation in per person terms

Example 1:

$$k \equiv \frac{K}{L} \Rightarrow \ln k = \ln K - \ln L$$

$$\Rightarrow \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L};\ \text{assume LFPR is constant. Rate of pop. growth} = n\ \&\ \frac{\dot{L}}{L} = n$$

Example 2:

$$y = k^{\alpha} \Rightarrow \ln y - \alpha \ln k$$

$$\Rightarrow \frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}$$

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- Combining (2.3) and Example 1:

$$\frac{\dot{k}}{k} = \frac{sY}{K} - n - \delta$$

$$\frac{\dot{k}}{k} = \frac{sy}{k} - n - \delta$$

$$\dot{k} = sy - (n+\delta)k; \quad \text{Capita accumulation in per worker terms}$$

The Solow Model

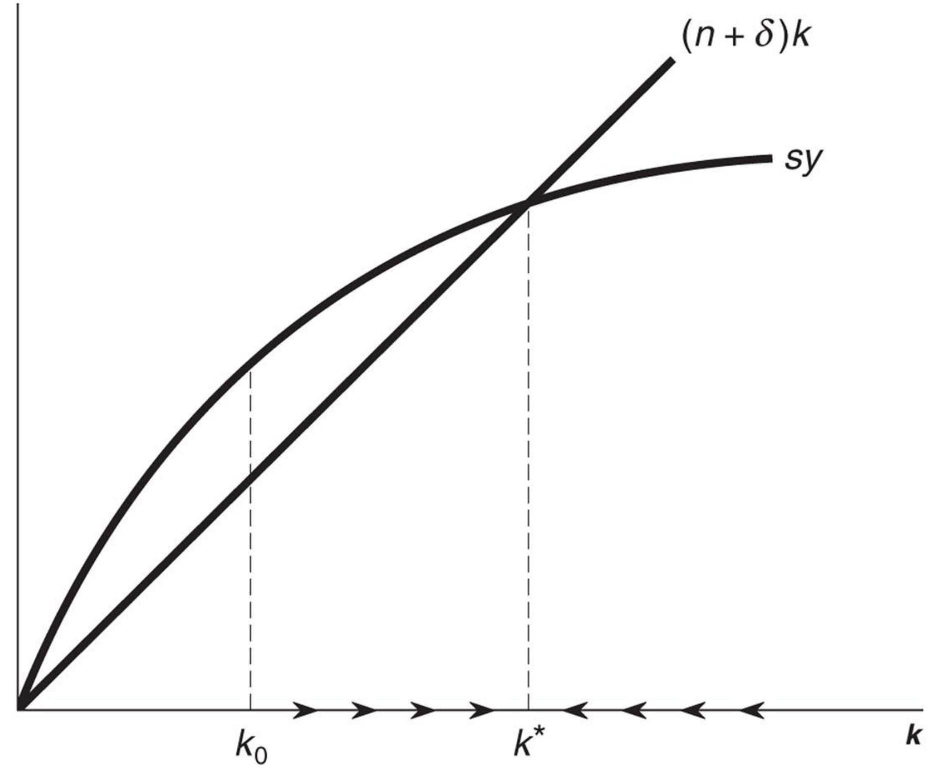
- Solving the model

1. Endogenous variables: Y, K, y, k
2. Exogenous variables: L
3. Parameters: α, δ, s, n, g
4. Solving the model = finding values of endogenous variables given 2. and 3.

The Solow Model

The Solow Diagram: $y = k^{\alpha}$ & $\dot{k} = sy - (n+\delta)k$

FIGURE 2.2 THE BASIC SOLOW DIAGRAM



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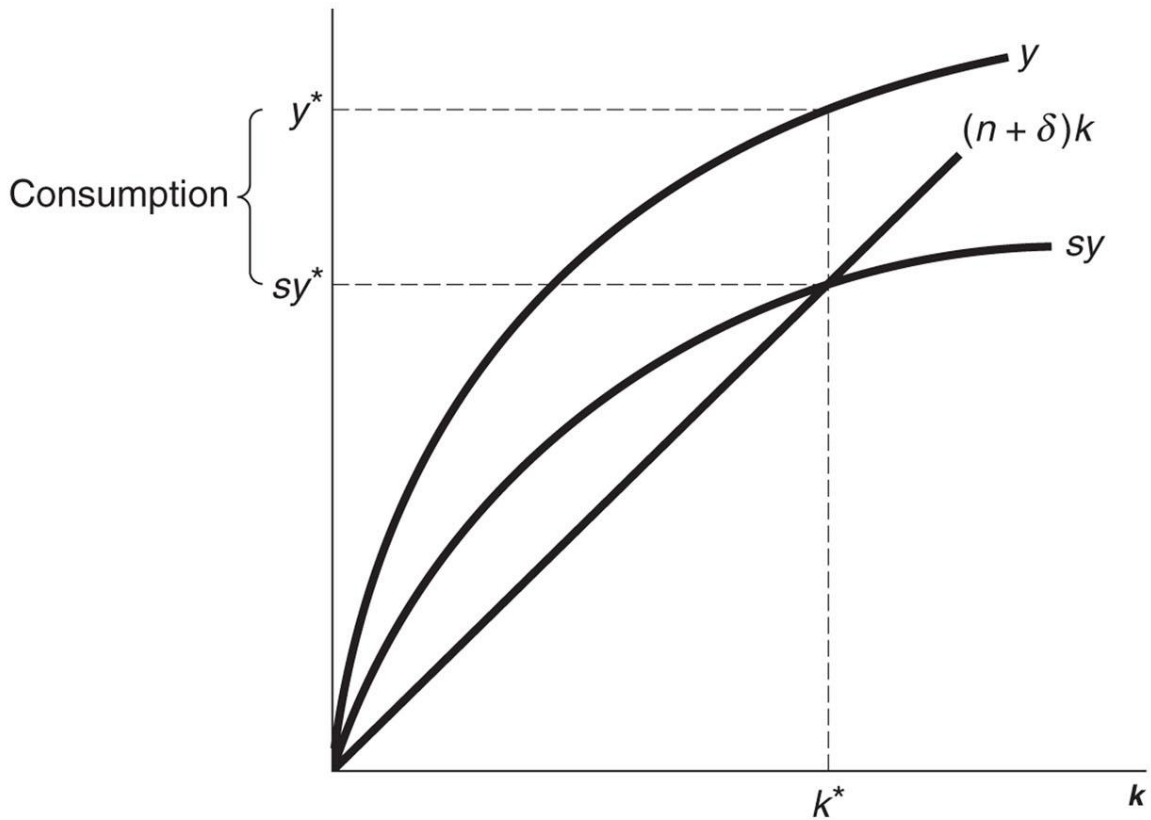
- Difference between sy & $(n + \delta)\, k$ is change in cap. per worker. When change is positive ➔ *capital deepening*
- When cap. per worker is zero but K is increasing ➔ only *capital widening*
- When $k = k_0$, $sy > (n + \delta)\, k$ ➔ k increases
- Deepening continues until $k = k^*$ ➔ $\dot{k} = 0$
- *Steady State*

The Solow Model

- When $k > k^*$, $sy < (n + \delta)\, k$ ➔ $\dot{k} < 0$ ➔ k decreases until $k = k^*$
- Solow diagram determines k^*, k^* determines y^*
- $c^* = y^*$ - sy^*

The Solow Model

FIGURE 2.3 **THE SOLOW DIAGRAM AND THE PRODUCTION FUNCTION**



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Comparative Statics:

- The economy begins in *steady state* and experiences a "*shock*"
- Possible shocks: increase in s or increase in n
- First: consider permanent increase in s, so that $s' > s$
- What happens to k and y?
- Now: at k^* $s'y > (n + \delta)$ ➔ Capital deepening
- k^* increases to k^{**}

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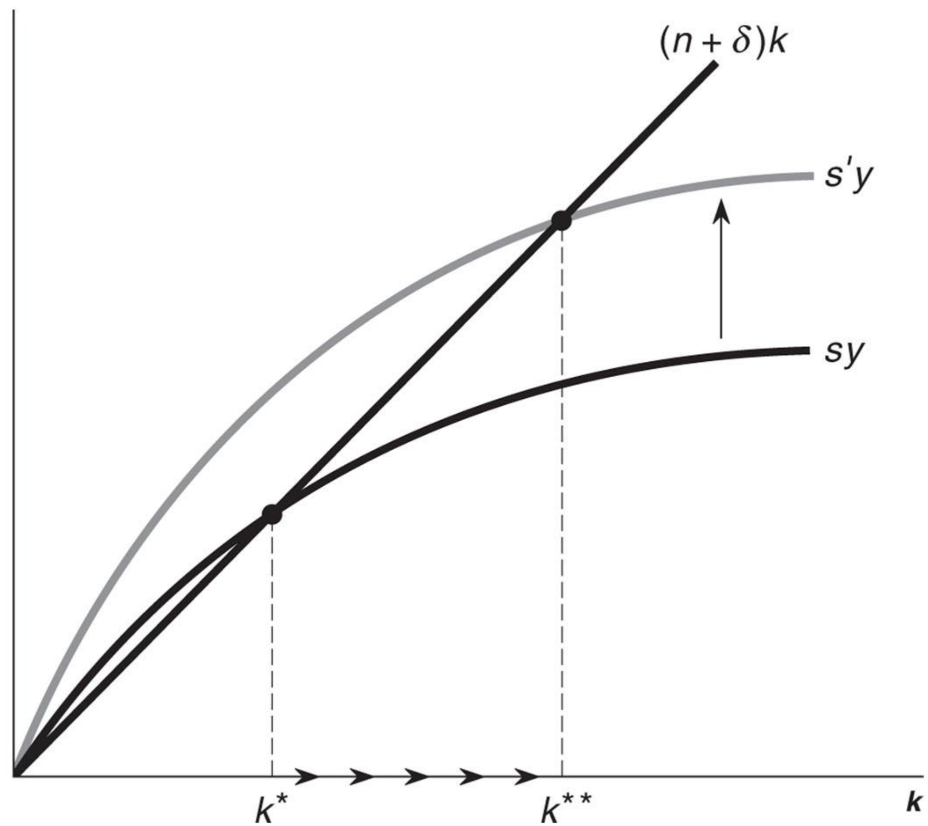
FIGURE 2.4 **AN INCREASE IN THE INVESTMENT RATE**



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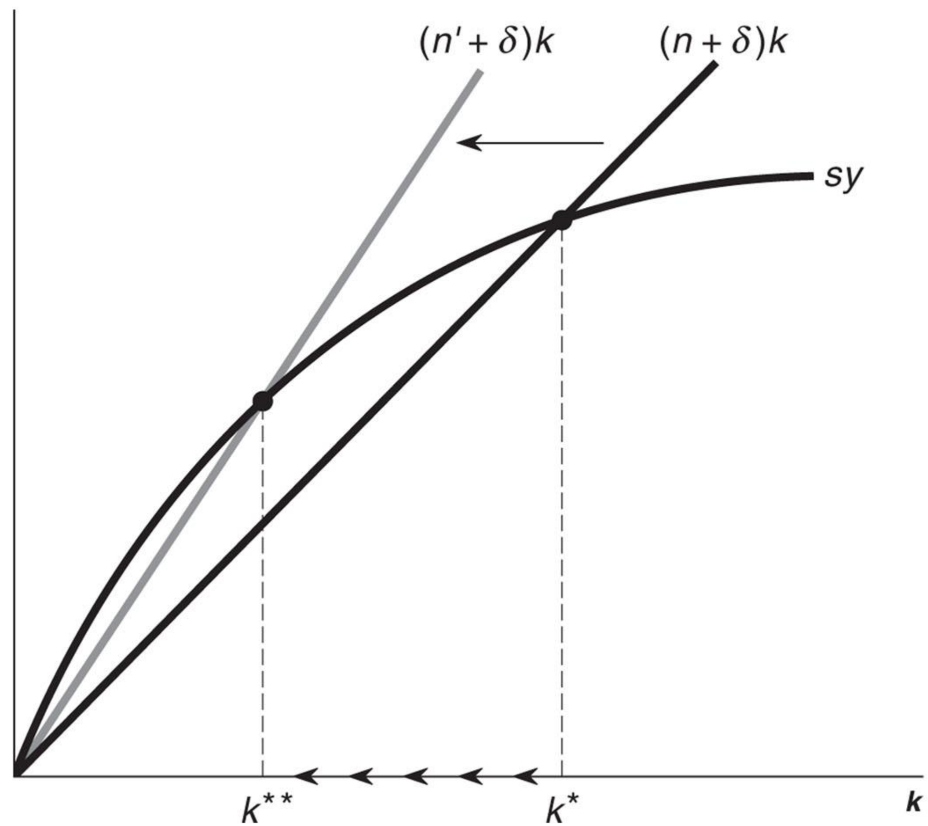
- Second: consider an increase in population growth n, so that $n' > n$
- What happens to k and y?
- The $(n+\delta)k$ line rotates to the left
- Now $sy < (n'+\delta)k$ ➔ k decreases from k^{**} to k^{*}

The Solow Model

FIGURE 2.5 **AN INCREASE IN POPULATION GROWTH**



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- Properties of the *Steady State*
- *SS* ➔ $\dot{k} = 0$
- From eq. (2.4) and (2.5)

$$\dot{k} = sk^{\alpha} - (n + \delta)k$$

- Setting this eq. = 0 yields

$$k^* = \left(\frac{s}{n+\delta}\right)^{1/(1-\alpha)} \qquad y^* = \left(\frac{s}{n+\delta}\right)^{\alpha/(1-\alpha)}$$

- This is the solution: endogenous variable expressed in terms of the parameters

The Solow Model

- The solution equation is Solow’s answer to the question “Why we are so rich and they so poor?”
- Countries with higher s tend to be richer. They accumulate more k ➔ more y
- Countries with higher n tend to be poorer. Higher portion of S has to go to keep k constant. Capital *widening* makes capital deepening more *difficult*. Accumulate less k

The Solow Model

FIGURE 2.6 GDP PER WORKER VERSUS THE INVESTMENT RATE

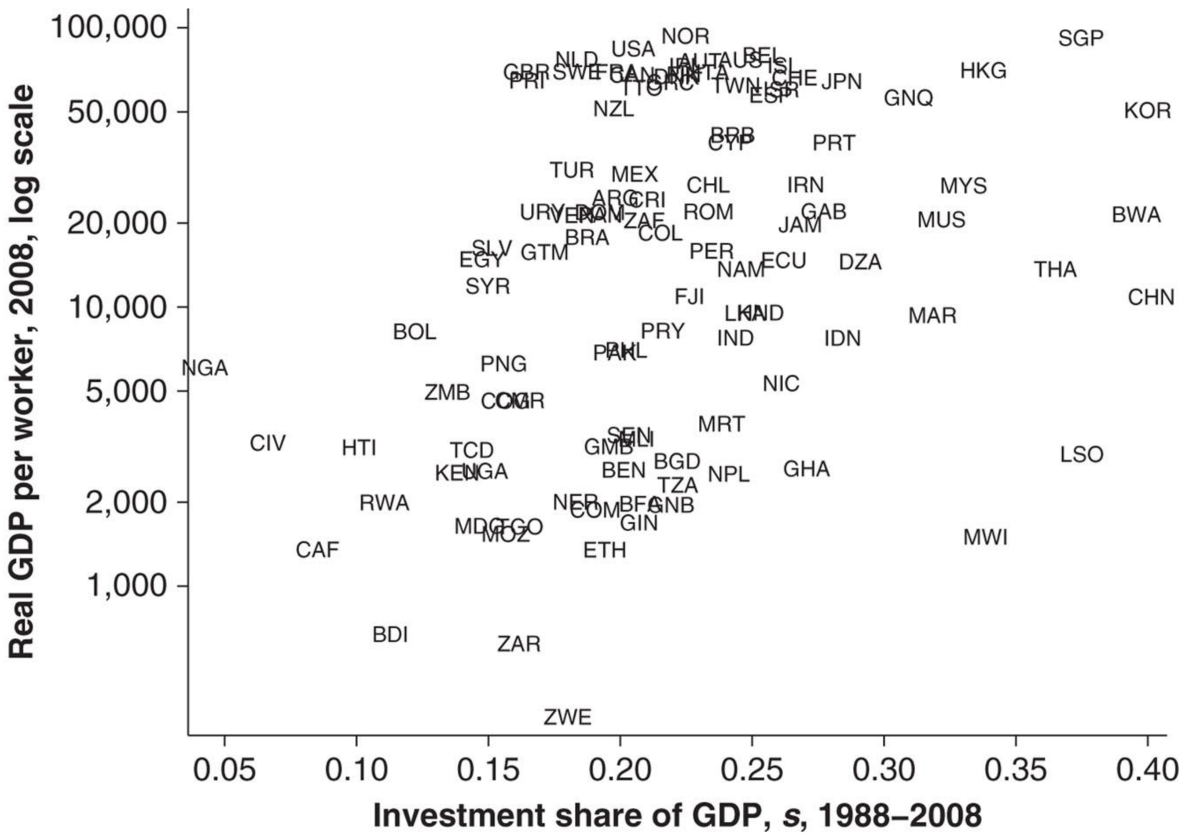

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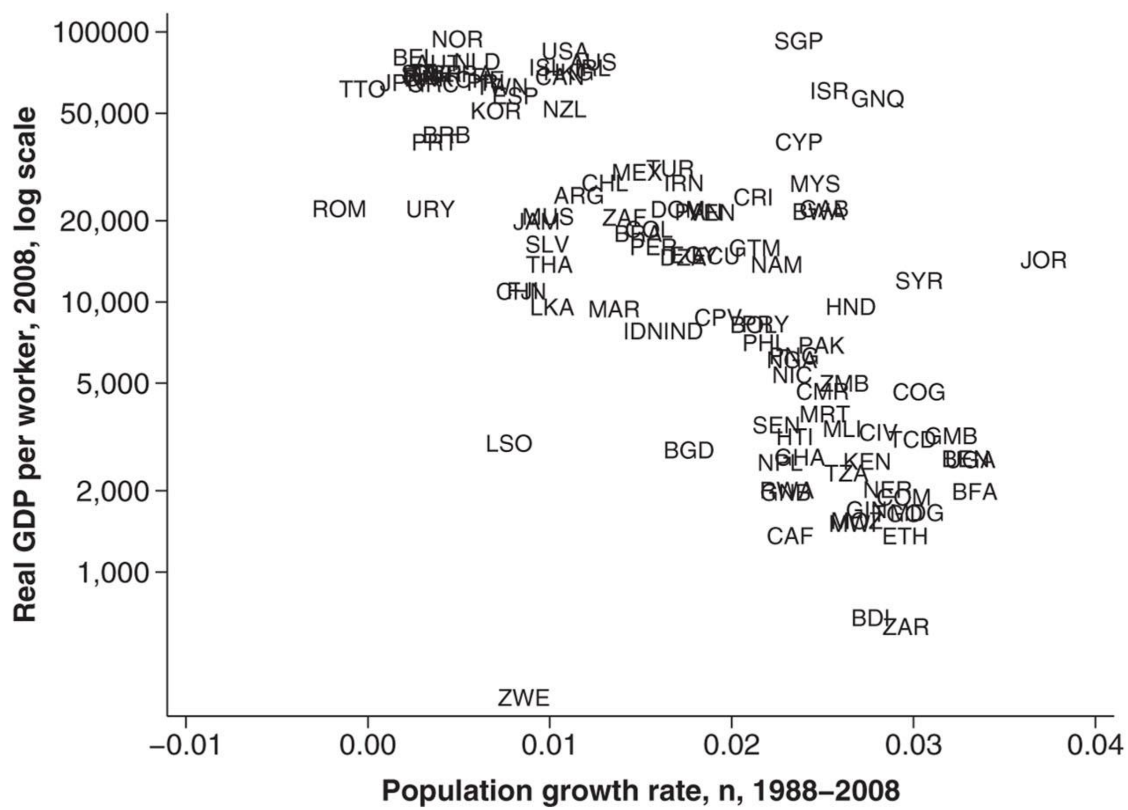
FIGURE 2.7 GDP PER WORKER VERSUS POPULATION GROWTH RATES



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Economic Growth in the Simple Model

- There is <u>no per capita growth</u> in this model.
- Output per person (worker) is constant in the Steady State
- Output (Y) growths but only at rate n
- This version of the model fits some of the stylized fact in Chapter 1
- It fails to predict that economies exhibit sustained per capita income growth
- Economies may grow for a while but not forever

The Solow Model

Economic Growth in the Simple Model (cont.)

- An economy that begins with $k < k^*$ will experience growth in k & y along the transition path to the SS
- Over time, growth slows down as the economy approaches its SS, and eventually it stops.

The Solow Model

- From the capital acculation eq. dividing both sides by k

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (n+\delta) \qquad (2.6)$$

- $\alpha < 1$, as k rises, the growth rate of k gradually declines
- The growth rate of y is proportional to the growth rate of k
- Transition dynamics in Figure 2.8

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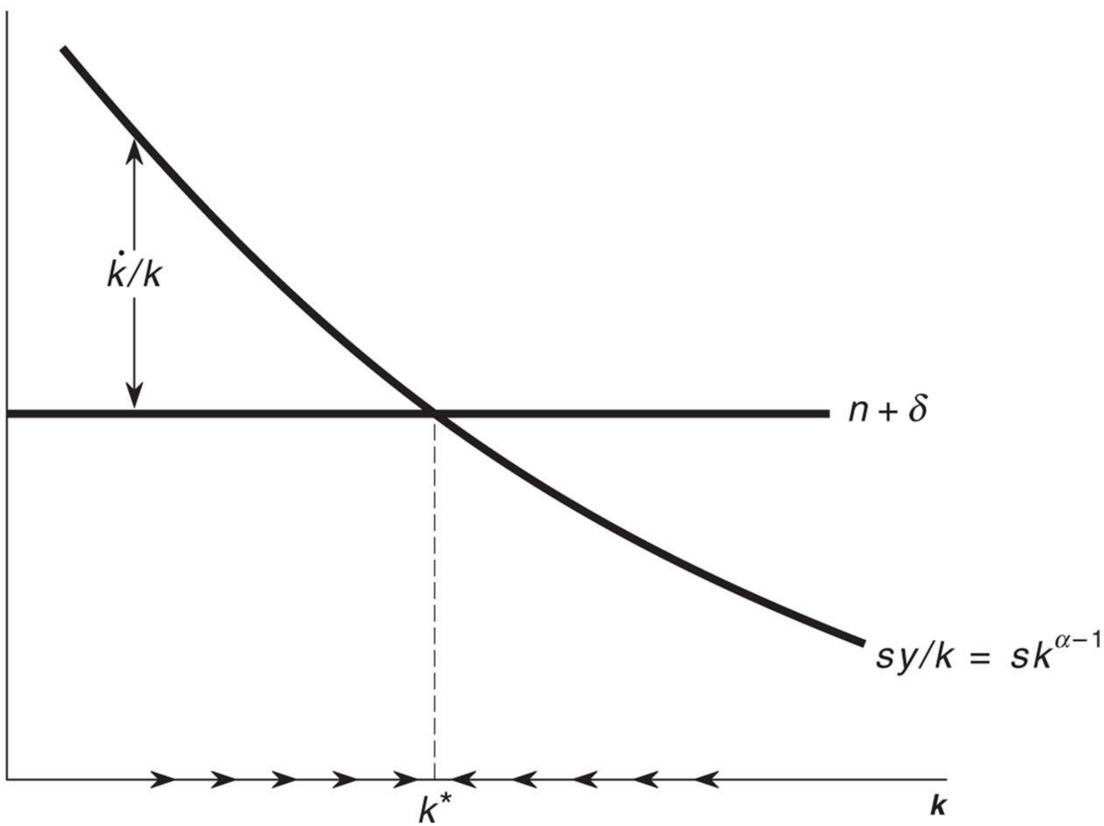
FIGURE 2.8 **TRANSITION DYNAMICS**



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- The first term in RHS of eq. (2.6) is
$sk^{\alpha-1} = \frac{sy}{k}$
- The higher is k, the lower is the average product of k (y/k) because of diminishing returns to capital accumulation ($\alpha < 1$). This curve slops downward
- The second term in RHS of eq. (2.6) is ($n + \delta$). Does not depend on k ➔ horizontal line
- The difference is $\dot{k}/k$ (growth rate of K stock)

The Solow Model

Technology and the Solow Model

- To generate sustained growth in y we need technological progress

$$Y = F\left(K, AL\right) = K^{\alpha}(AL)^{1-\alpha} \qquad (2.7)$$

- A is the technology variable; it is said to be "labor augmenting" or "Harrod-neutral"
- Tech. progress happens when A growth over time

The Solow Model

- An important assumption of Solow is that A is *exogenous*
- We assume it grows at a constant rate g

$$\frac{\dot{A}}{A} = g \Leftrightarrow A = A_0 e^{gt}$$

- The capital accumulation equation is now

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta \qquad (2.8)$$

The Solow Model

- To see the growth implication of the model with technology rewrite (2.7) in terms of output per worker

$$y = k^{\alpha} A^{1-\alpha}$$

- Taking logs and differentiating

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} + (1-\alpha)\frac{\dot{A}}{A} \qquad (2.9)$$

- From (2.8) the growth rate of K will be constant iff Y/K is constant

The Solow Model

- If Y/K is constant, y/k is also constant and y and k grow at the same rate
- When capital, output, consumption and population are growing at constant rates we call it a *balanced growth path (b.g.p.)*
- Let g_x denote the growth rate of x along a b.g.p.
- Along a b.g.p. $g_y = g_k$. Substituting in 2.9
- $g_y = g_k = g$ ➔ Tech. progress is the source of sustained per capita growth

The Solow Model

- Solow Model with technology
- k is no longer constant in the long run
- The new *state* variable is $\tilde{k} \equiv \frac{K}{AL} = \frac{k}{A}$
- It is constant along a b.g.p. b/c $g_y = g_A = g$
- $\tilde{k}$ represent the ratio of k to technology
- We refer to this a "capita-technology" ratio
- The new PF is $\tilde{y} = \tilde{k}^{\alpha}$ where $\tilde{y} \equiv \frac{Y}{AL} = \frac{y}{A};$

$\tilde{y}$ is "output-technology ratio"

The Solow Model

- Note that

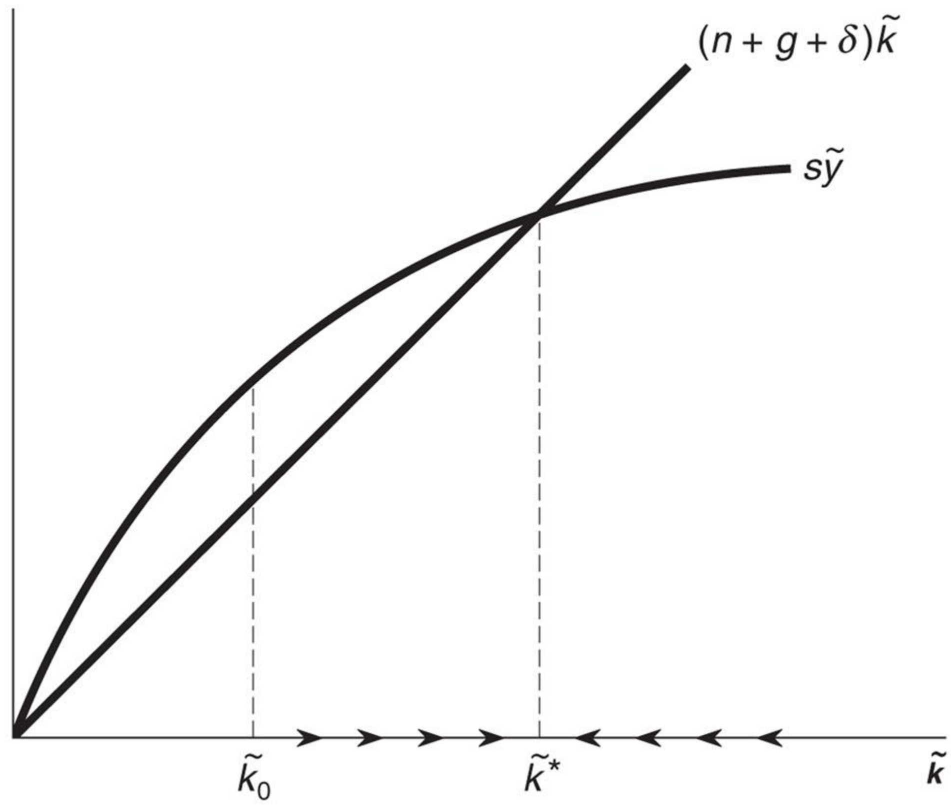
$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}$$

$$\dot{\tilde{k}} = s\tilde{y} - (n + g + \delta)\tilde{k} \qquad (2.12)$$

- The Solow diagram with technological progress is Figure 2.9

The Solow Model

FIGURE 2.9 **THE SOLOW DIAGRAM WITH TECHNOLOGICAL PROGRESS**



The Solow Model

- The analysis of the diagram is similar to the case w/o tech. progress
- If the economy starts at $\tilde{k}_0$ the capital-tech. ratio will increase over time. Investment is more than is needed to keep capital-tech. ratio constant. This is true until
  $s\tilde{y} = (n + g + \delta)\tilde{k}$ at point $\tilde{k}^*$
- At that point the econ. is in SS and growths along a b.g.p.

The Solow Model

- Solving for the Steady State

$$\tilde{k}^* = \left(\frac{s}{n+g+\delta} \right)^{1/(1-\alpha)}$$

$$\tilde{y}^* = \left(\frac{s}{n+g+\delta} \right)^{\alpha/(1-\alpha)}$$

- $\tilde{y}$ and $\tilde{k}$ are called "output per effective unit of labor" and "capital per effective unit of labor"

The Solow Model

- To see what this implies about y we rewrite the equation as

$$y^*(t) = A(t)\left(\frac{s}{n+g+\delta}\right)^{\alpha/1-\alpha} \qquad (2.13)$$

- both y and A depend on time
- From eq. (2.13) we see that y along the b.g.p. is determined by g, s, and n. When $g = 0$ and $A_0 = 1$ is the model w/o tech. progress

The Solow Model

- Changes in s or n affect the long-run *level* of output per worker but not the long-run *growth rate* of output per worker
- Economy in SS with s that permanently increases to s' (See Figure 2.10)
- At $\tilde{k}^*$ investment exceeds the amount needed to keep capital-tech. ratio constant so $\tilde{k}$ begins to rise

The Solow Model

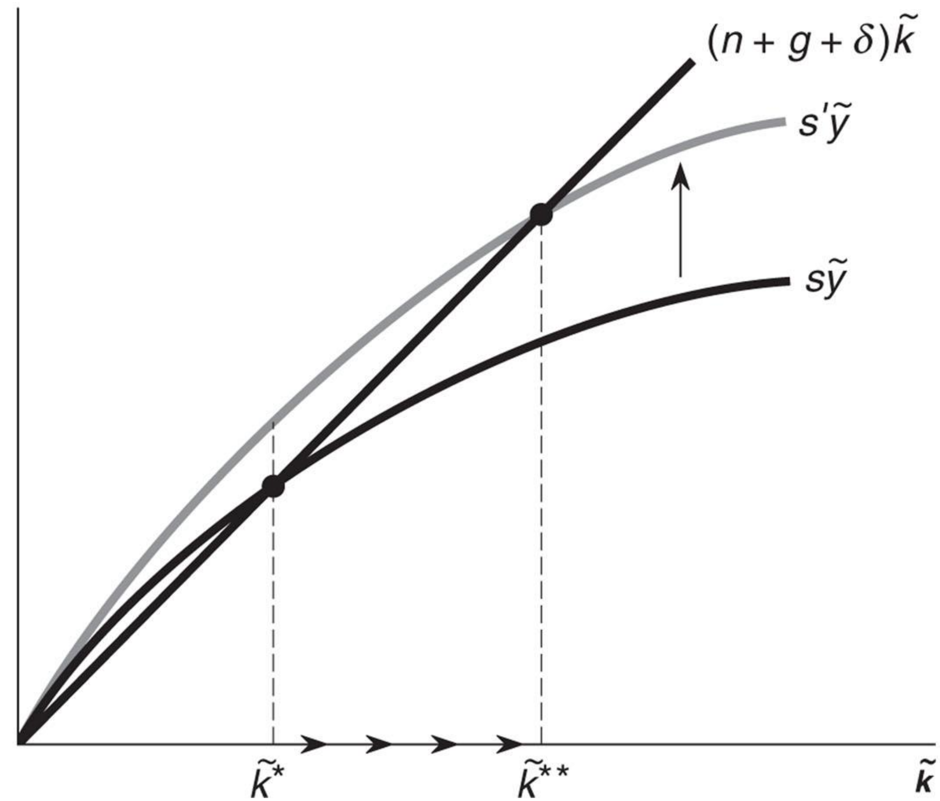
- To see the effect on growth rewrite (2.12) as

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = s\frac{\tilde{y}}{\tilde{k}} - (n + g + \delta); \quad \text{note that } \frac{\tilde{y}}{\tilde{k}} = \tilde{k}^{\alpha-1}$$

- Figure 2.11 illustrates the transition dynamics

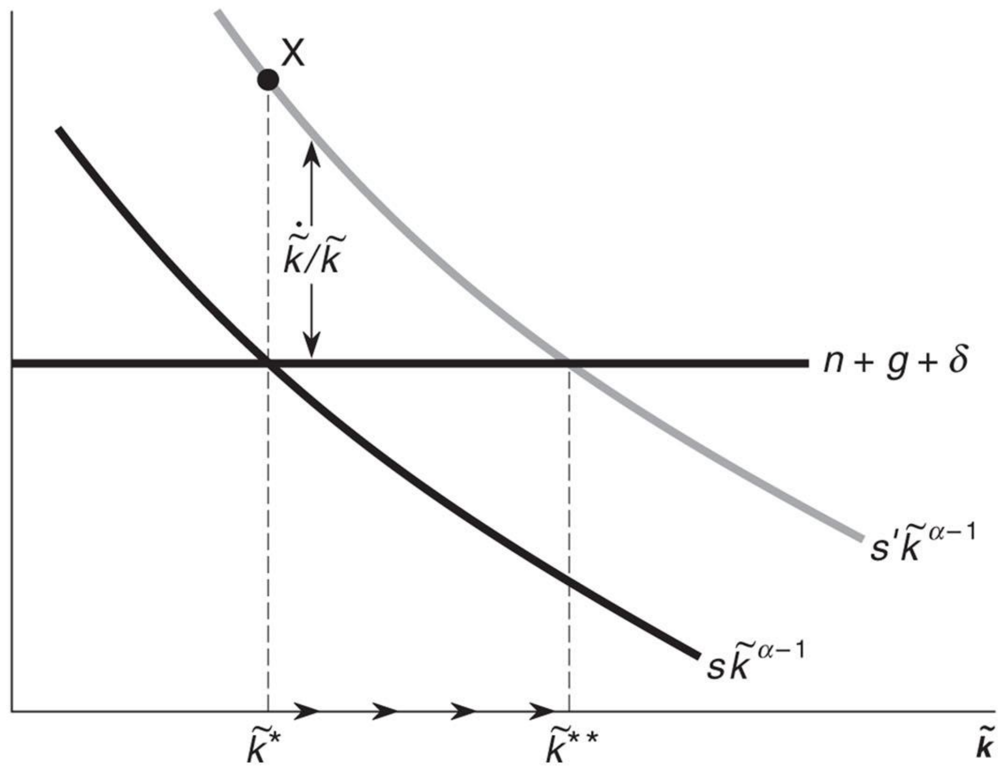
The Solow Model

FIGURE 2.10 **AN INCREASE IN THE INVESTMENT RATE**



The Solow Model

FIGURE 2.11 **AN INCREASE IN THE INVESTMENT RATE: TRANSITION DYNAMICS**



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- The increase of s to s' raises the growth rate temporarily as the economy goes from $\tilde{k}^*$ to $\tilde{k}^{**}$
- Since g is constant, faster growth in $\tilde{k}$ along the transition path implies that $\dot{k}/k > g$ (see Figure 2.12)
- Figure 2.13 cumulates the effect on growth to show what happens to the log level of y over time

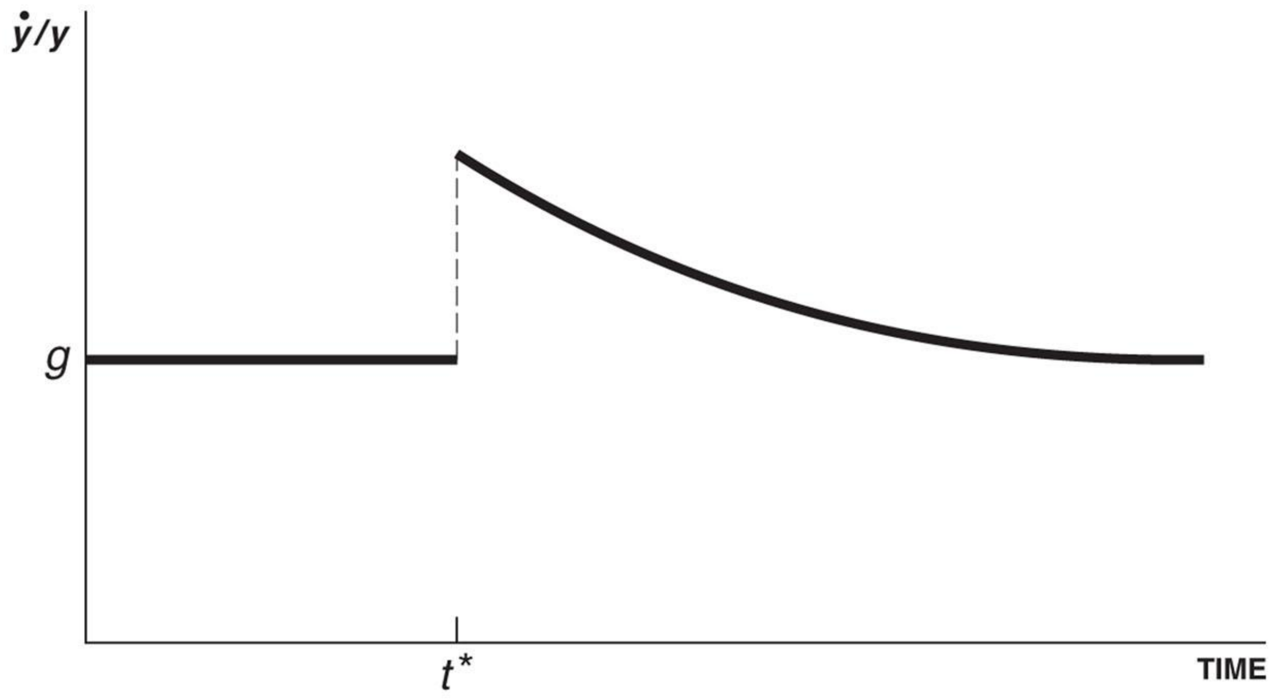
The Solow Model

- Prior to the policy change y is growing at the rate g, so the log y rises linearly
- At time t^*, y begins to grow more rapidly. Rapid growth continues until the output-technology ratio reaches its new SS. At this point growth returns to long run level g

1. Policy changes increase growth rates but only *temporarily* (no *growth effect*)
2. Policy changes can have *level effects*

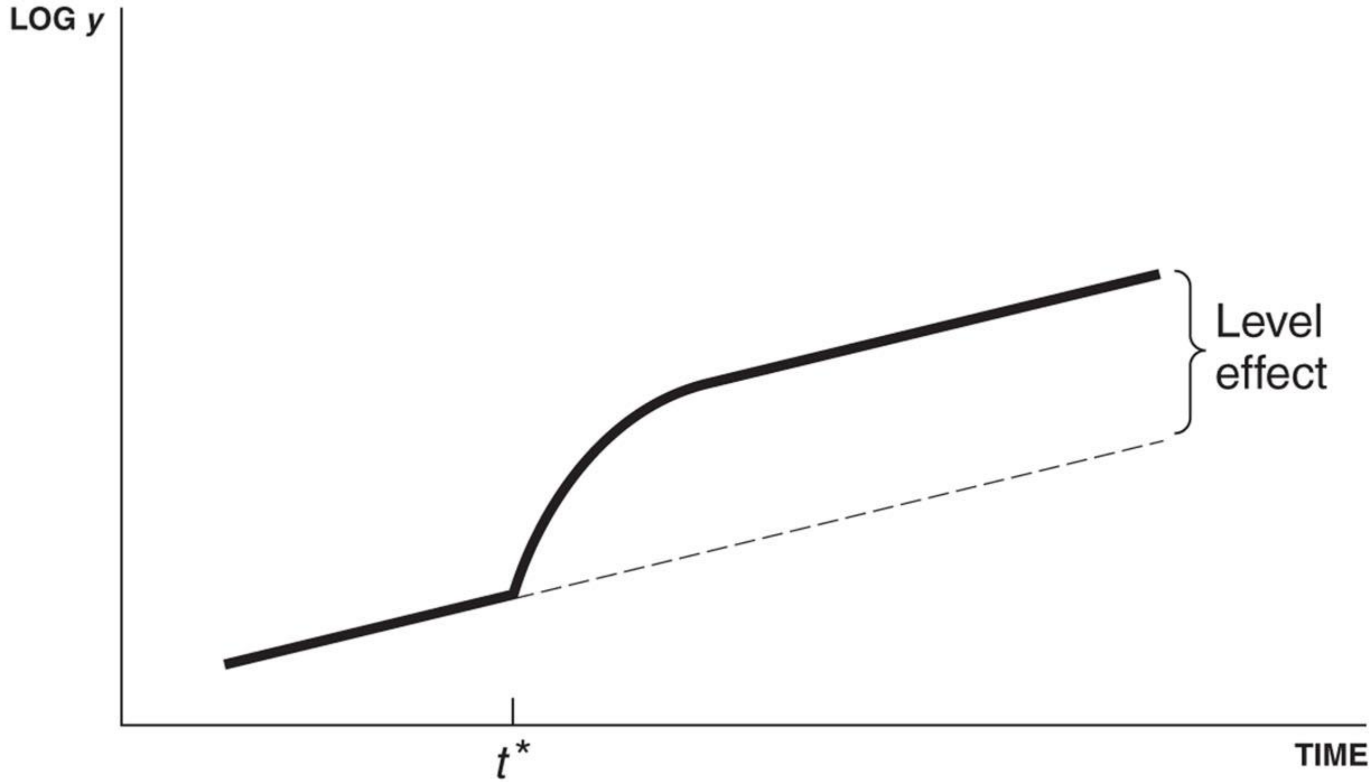
The Solow Model

FIGURE 2.12 THE EFFECT OF AN INCREASE IN INVESTMENT ON GROWTH



The Solow Model

FIGURE 2.13 **THE EFFECT OF AN INCREASE IN INVESTMENT ON y**



The Solow Model

Evaluating the Solow Model

How does the SM answers the key questions of growth?

1. Differences in s and n (and g) explain difference in per y. Why are we so rich: we invest more & lower pop. growth ➔ can accumulate more k and increase labor prod.
2. Why sustained growth? Tech. progress

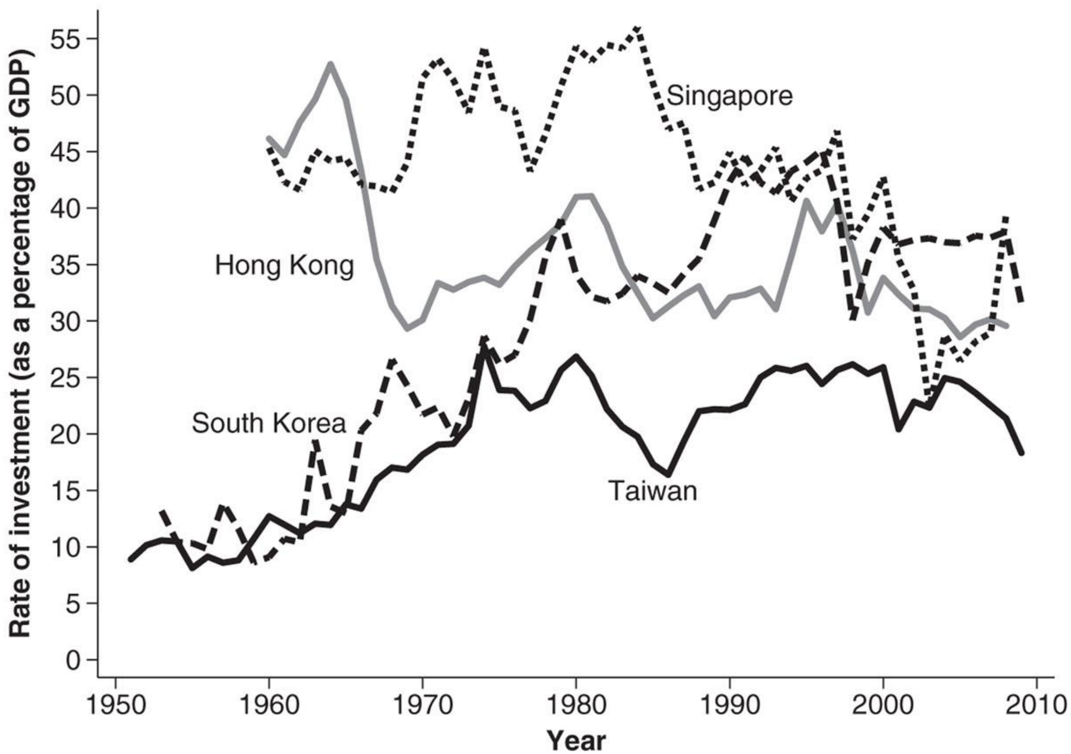
The Solow Model

Evaluating the Solow Model (cont.)

3. How to account for differences in growth rates across countries? It may seem that SM can't. ➔ Diff. in (unmodeled) tech. progress?
4. Other option: Transition dynamics. During the transition period countries grow faster that the long run growth rate. Examples: Japan & Germany (low k); South Korea & Taiwan (increase in s). Less so for Hong Kong & Singapore

The Solow Model

FIGURE 2.14 INVESTMENT RATES IN SOME NEWLY INDUSTRIALIZING ECONOMIES



The Solow Model

Growth Accounting

- In the SM sustained growth occurs only with tech progress
- Improvements in tech offset diminishing returns to capital; labor prod. growth (more tech & and more K)
- Solow (1957): Growth Accounting

The Solow Model

Growth Accounting (cont.)

- Break down growth in output in: 1) growth in K; growth in L and 3) growth in tech. change
- New form of PF:

$Y = BK^{\alpha} L^{1-\alpha}$; B is "Hicks-neutral productivity term

taking logs and differenciating

$$\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L} + \frac{\dot{B}}{B} \quad (2.14)$$

The Solow Model

Growth Accounting (cont.)

- Eq. (2.14) says that output growth is a weighted average of K & L growth plus the growth of B. The term $\dot{B}/B$ is referred to as *total factor productivity growth* or *multifactor productivity growth*
- Solow and other have used this eq. to understand the sources of growth in output

The Solow Model

Growth Accounting (cont.)

- It is useful to rewrite eq. (2.14) as

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} + \frac{\dot{B}}{B} \quad (2.15)$$

- Growth rate of output per worker is decomposed into the contribution of capital per worker and cont. of multifactor productivity

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Growth Accounting (cont.)

- The BLS provides an accounting of US growth using a generalization of eq. (2.15)
- BLS measure labor in terms of hours
- BLS includes a term to adjust for changes in the composition of labor force (e.g. more education)
- Result are listed in Table 2.1

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TABLE 2.1 GROWTH ACCOUNTING FOR THE UNITED STATES

	1948–2010	1948–73	1973–95	1995–2000	2000–2010
Output per hour	2.6	3.3	1.5	2.9	2.7
Contributions from:					
Capital per hour worked	1.0	1.0	0.7	1.2	1.2
Information technology	0.2	0.1	0.4	0.9	0.5
Other capital services	0.8	0.9	0.3	0.3	0.7
Labor composition	0.2	0.2	0.2	0.2	0.3
Multifactor productivity	1.4	2.1	0.6	1.5	1.3

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Growth Accounting (cont.)

- For example in 1948/2010 about ½ of US growth was to factor accumulation and the other ½ to improvements in technology
- Because the way is calculated, the second ½ is called "*residual*" or the "*measure of our ignorance*"

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Productivity Slowdown

- Happened between 1973 and 1995
- Various explanations:

1. Oil shocks in 1973 & 1979. But oil prices were low in the 1980s
2. Change in composition of LF and sectoral shifts
3. Maybe growth in the 1950s and 1960s was too high

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The New Economy

- Rise in productivity growth in 1995-2000
- Both growth in output per capita ant TFP rose substantially
- The increase in growth rate is partially associated with increase use of inf. tech.
- Solow (1987): "You can see the computer age everywhere but in the productivity statistics.”

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The New Economy

- Some economists suggest that the information-tech revolution can explain both the prod slowdown and the new economy
- Growth slowed down temporarily while firms adapted their factories to the new tech and workers learned to take advantage of new tech
- The upsurge in prod growth reflects the widespread use of the new technology

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- Growth accounting has also been used to analyze growth in other countries.
- One interesting application is to NICs (average growth rates > 4% since 1960s)
- Young (1995) shows that a large part of the growth is the result of factor accumulation: increased investment in physical and human K; increased in LFPR; and shift from agriculture to industry. Figure 2.15

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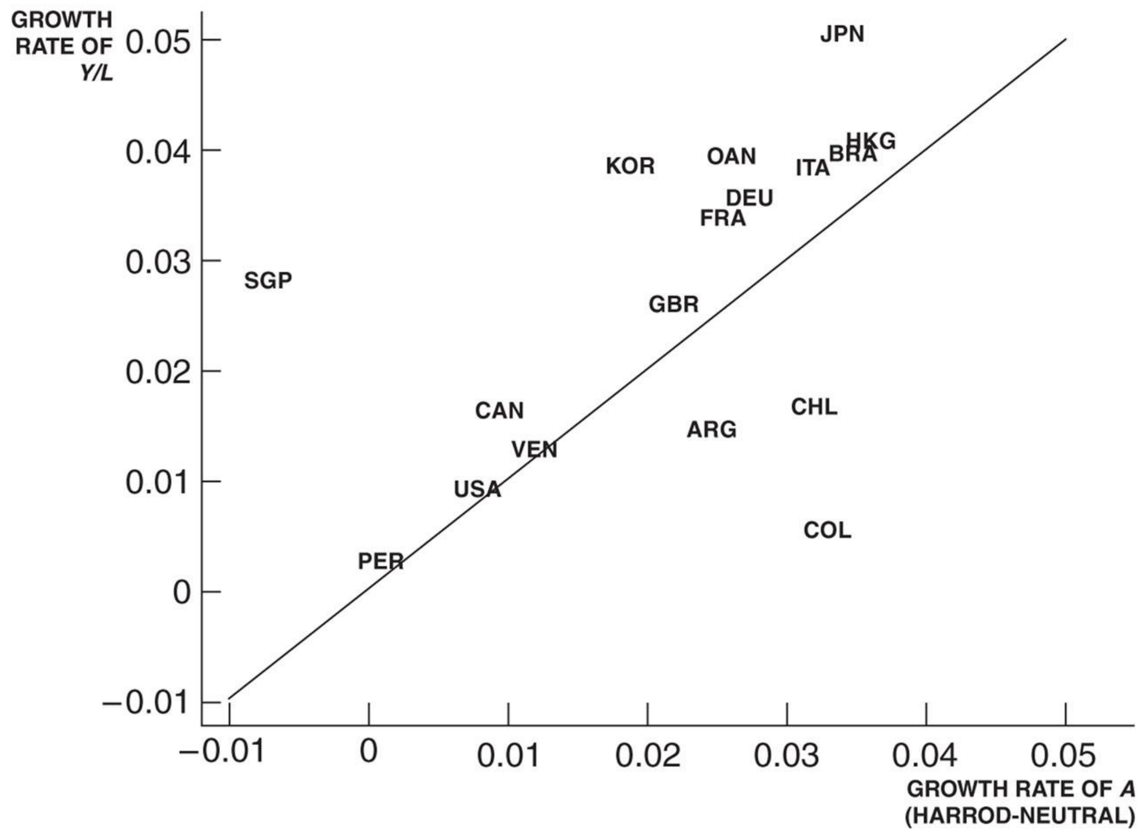
FIGURE 2.15 **GROWTH ACCOUNTING**



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- Growth rates in y are remarkable; growth in TPF less so. TFP growth, while typically higher than in US was not exceptional in Asian countries
- Growth along a balanced path should lie in the 45º line
- Asian countries are far above the 45º line. This means growth in y is much higher than TFP growth would suggest