Exam 3- MAC 2311, Spring 16 **NAME:** Solution Key **Panther ID:** ____________

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (6 pts) Solve the initial value problem:

$$\frac{dy}{dx} = 3\sin x + \sec x \tan x, \quad y(0) = 0$$

$y(x) = \int (3\sin x + \sec x \tan x)\,dx$

$y(x) = -3\cos x + \sec x + c$

$0 = y(0) = -3\cos 0 + \sec 0 + c$

$0 = -3 \cdot 1 + 1 + c$

so $c = 2$

so $\boxed{y(x) = -3\cos x + \sec x + 2}$

2. (24 pts) Find each indicated antiderivative:

(a) $\int \left(3+3^x-\frac{3}{x^3}\right) dx =$

$= 3x + \frac{3^x}{\ln 3} - \frac{3}{(-2)} x^{-2} + c$

$= \boxed{3x + \frac{3^x}{\ln 3} + \frac{3}{2x^2} + c}$

(b) $\int x^2\sqrt{x^3+9}\,dx = \int \sqrt{w}\,\frac{1}{3}dw$

let $w = x^3 + 9$

$dw = 3x^2 dx$

$\frac{1}{3}dw = x^2 dx$

$= \frac{1}{3}\int w^{\frac{1}{2}} dw$

$= \frac{1}{3}\cdot\frac{2}{3} w^{\frac{3}{2}} + c$

$= \boxed{\frac{2}{9}(x^3+9)^{\frac{3}{2}} + c}$

(c) $\int e^{\sin(3x)} \cos(3x)\,dx = \int e^u \frac{1}{3} du =$

let $u = \sin(3x)$

$du = \cos(3x)\cdot 3\,dx$

$\frac{1}{3}du = \cos(3x)\,dx$

$= \frac{1}{3}e^u + c = \boxed{\frac{1}{3}e^{\sin(3x)} + c}$

(d) $\int \frac{1}{x(1+(\ln x)^2)}\,dx$

let $u = \ln x$

$du = \frac{1}{x}dx$

$\int \frac{1}{1+u^2} du = \arctan u + c =$

$= \boxed{\arctan(\ln x) + c}$

since $f(x)$ is decreasing on $[a,b]$

3. (8 pts) True or False questions. No justification needed. 2 points each.

(a) If $f'(x) < 0$ for all $x \in [a, b]$, then $x = a$ is an absolute maximum for $f(x)$ on the interval $[a, b]$. **True** **False**

(b) If $f'(2) = 0$ and $f''(2) < 0$ then f has a relative minimum at $x = 2$. **True** **False** f has a rel. max at $x=2$

(c) A continuous function $f(x)$ with domain all reals always has an absolute maximum and an absolute minimum. ~~**True**~~ **False**

Exp: $f(x) = x$ has no abs. max nor min on $\mathbb{R}$.

(d) If $f''(x) = 0$ for all $x \in \mathbf{R}$, then $f(x) = mx + b$ for some constants m and b. **True** **False**

$f''(x) = 0 \Rightarrow f'(x) = m \leftarrow$ constant $\Rightarrow f(x) = mx + b$

4. (10 pts) Use an appropriate local linear approximation to estimate $\sqrt[5]{0.95}$ without a calculator. Be sure to specify the function and the point you are using for the local linear approximation.

Let $f(x) = \sqrt[5]{x}$, we'll use the local linear approximation at $x_0 = 1$ (since $f(x_0) = f(1) = \sqrt[5]{1} = 1$)

$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$ $\leftarrow$ Local lin. approx. formula

$f'(x) = \left(x^{\frac{1}{5}}\right)' = \frac{1}{5} x^{-\frac{4}{5}}$

$f'(x_0) = f'(1) = \frac{1}{5} 1^{-\frac{4}{5}} = \frac{1}{5}$

so $\sqrt[5]{x} \approx 1 + \frac{1}{5}(x - 1)$ for x close to 1

Take $x = 0.95$ so

$\sqrt[5]{0.95} \approx 1 + \frac{1}{5}(0.95 - 1)$ so

$\sqrt[5]{0.95} \approx 1 - \frac{0.05}{5}$ so

$\sqrt[5]{0.95} \approx 0,99$

5. (10 pts) Find a number in the closed interval $[\frac{1}{3}, 2]$ so that that the sum of the number and its reciprocal is

(a) as small as possible
(b) as large as possible.

If x is the number, the sum of the number and its reciprocal is given by

$$S(x) = x + \frac{1}{x}$$

We have to find the extrema (abs. max and min) for $S(x)$ when $x \in [\frac{1}{3}, 2]$

$$S'(x) = \left(x + x^{-1}\right)' = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

Critical pt(s) $S'(x) = 0 \iff 1 - \frac{1}{x^2} = 0 \iff x^2 = 1 \iff x = \pm 1$

$x = -1$ is not in the interval $[\frac{1}{3}, 2]$, but $x = 1$ is.

So the "candidates" are $x = \frac{1}{3}$, $x = 1$ and $x = 2$.

$$S\left(\tfrac{1}{3}\right) = \frac{1}{3} + \frac{1}{\frac{1}{3}} = 3 + \frac{1}{3} = \frac{10}{3}$$

$$S(1) = 1 + \frac{1}{1} = 2$$

$$S(2) = 2 + \frac{1}{2} = \frac{5}{2}$$

As $2 < \frac{5}{2} < \frac{10}{3}$, it means that for $x = 1$ the sum $S(x)$ is minimum and for $x = \frac{1}{3}$ the sum $S(x)$ is maximum in the interval $[\frac{1}{3}, 2]$.

6. (20 pts) This problem is about two moving "things", call them Thing1 and Thing2. At the initial moment $t = 0$, Thing1 is at rest at the point $(-2, 0)$ on the x-axis, but starts moving to the right on the x-axis with a constant acceleration of 2 cm$^{\cancel{2}}$/second^2. At the same initial moment $t = 0$, Thing2 is at rest at the point $(0, -8)$ on the y-axis, but starts moving up on the y-axis with a constant acceleration of 4 cm$^{\cancel{2}}$/second^2. (Assume that the units of length on both the x and the y-axis are centimeters.)

(a) (5 pts) Write the equations for $x(t)$, respectively $y(t)$, giving the position at time t for Thing1, respectively Thing2.

The equation for position when acceleration is constant is:

$$s(t) = \frac{at^2}{2} + v_0 t + s_0.$$

In our case, $s(t) = x(t)$ for Thing 1, $s(t) = y(t)$ for Thing 2.

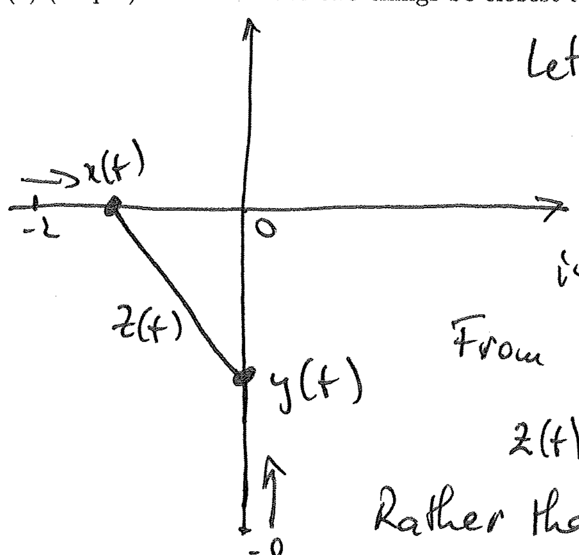
Since both start from rest, $v_0 = 0$ for both.

Using initial positions for each we get
$$x(t) = \frac{2t^2}{2} - 2 = t^2 - 2$$
$$y(t) = \frac{4t^2}{2} - 8 = 2t^2 - 8$$

(b) (5 pts) Will the two things collide at the origin? Justify your answer.

No, Thing1 reaches the origin after $\underline{\underline{\sqrt{2}\text{ s}}}$ (solve $0 = t^2 - 2$), whereas Thing 2 reaches the origin after $\underline{\underline{2\text{ s}}}$ (solve $0 = 2t^2 - 8$)

(c) (10 pts) When will the two things be closest to one another? Justify your answer.



Let $z(t)$ be the distance between them at time t.

We need to find t so that $z(t)$ is minimum. So it's an optimization pb.!

From Pythagora

$$z(t) = \sqrt{(x(t))^2 + (y(t))^2} = \sqrt{(t^2-2)^2 + (2t^2-8)^2}$$

Rather than minimizing $z(t)$ we'll minimize

$$g(t) = z(t)^2 = (t^2-2)^2 + (2t^2-8)^2$$ (the critical pts are the same, but we got rid of the $\sqrt{\ }$)

when $t \in [0, +\infty)$

$$g'(t) = 2(t^2-2)\cdot 2t + 2(2t^2-8)\cdot 4t = 4t[t^2 - 2 + 4t^2 - 16] = 4t(5t^2 - 18)$$

$$g'(t) = 0 \iff t = 0 \text{ or } t = \sqrt{\frac{18}{5}} \approx 1.9\text{s}$$

Since $g'(t) < 0$ for $0 < t < \sqrt{\frac{18}{5}}$ and $g'(t) > 0$ for $t > \sqrt{\frac{18}{5}}$ $\Rightarrow$ $\boxed{t = \sqrt{\frac{18}{5}}\text{ s}}$ is the moment when they are closest to each other

7. (18 pts) Give a complete graph of the function $f(x) = \frac{3x^2-1}{(x-2)^2}$. Your work should include: the domain of the function, equations of eventual asymptotes (vertical or/and horizontal), coordinates of the axis intercepts, a sign chart for the derivative and the second derivative, the location and nature of the critical points (if any), location of inflection points (if any). To save you time, here are the first and the second derivatives $f'(x) = \frac{2(1-6x)}{(x-2)^3}$, $f''(x) = \frac{6(4x+3)}{(x-2)^4}$.

Domain: all reals, except $x=2$.

$x=2$ is a vertical asymptote for the graph. $\lim_{x\to 2} \frac{3x^2-1}{(x-2)^2} = \frac{11}{0^+} = +\infty$

The graph has horizontal asymptote $y=3$ for both $x\to+\infty$ and $x\to-\infty$.

$$\lim_{x\to\pm\infty} \frac{3x^2-1}{(x-2)^2} = \lim_{x\to\pm\infty} \frac{3x^2}{x^2} = 3$$

junk rule

Critical pt.
$f'(x)=0 \iff 1-6x=0$ so $\boxed{x=\frac{1}{6}}$
$x=2$ will also act like a critical point, but is not in the domain of f

Sign-chart

Test value

x	$-\infty$	$-\frac{3}{4}$	0	$\frac{1}{6}$		2		$+\infty$
$f'(x)$	$-\,-\,-\,-\,-\,-\,-\,-$			0	$+\,+\,+\,+$		$-\,-\,-\,-\,-$	
$f(x)$	3 $\searrow$			$f(\frac{1}{6})$ ← negative	$\nearrow +\infty$		$+\infty \searrow$	3
$f''(x)$	$-\,-\,-\,-$	0	$+\,+\,+\,+\,+\,+$		$+$		$+\,+\,+\,+\,+\,+\,+\,+$	

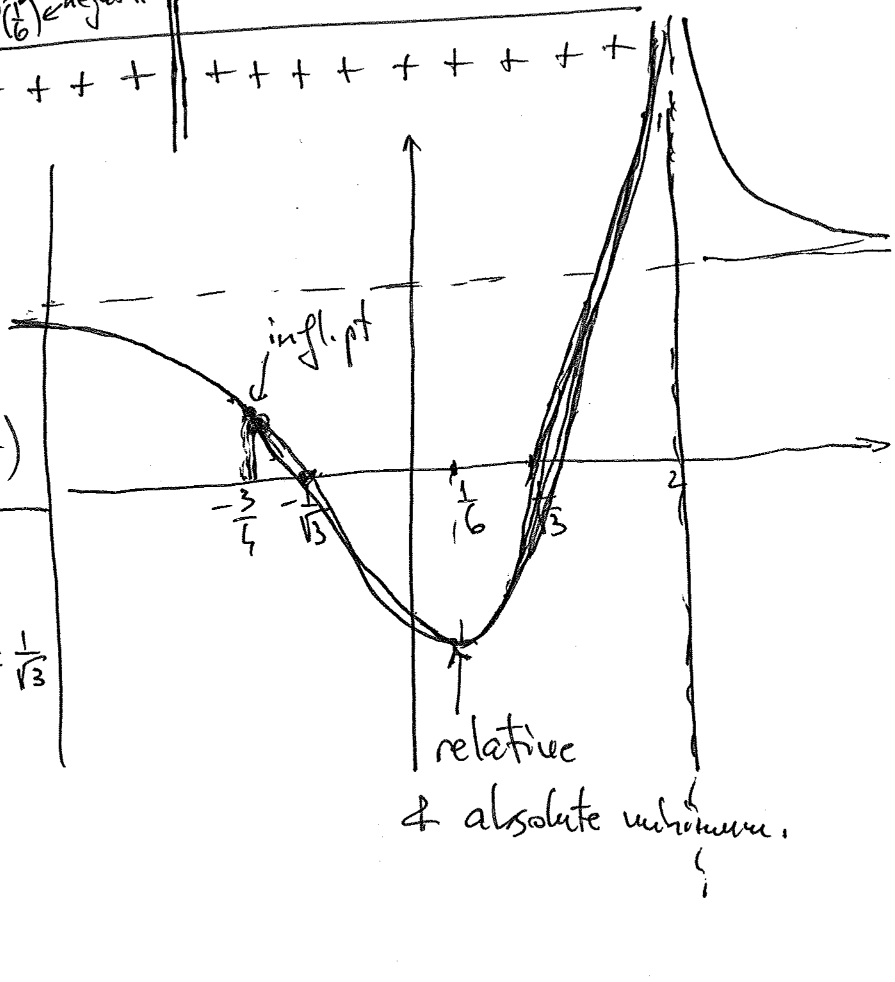
From the sign chart we see that:

$x=\frac{1}{6}$ is a relative minimum
$\left(f(\frac{1}{6}) = \frac{3\cdot(\frac{1}{6})^2-1}{(\frac{1}{6}-1)^2} = \text{negative \#} \right)$

$x=-\frac{3}{4}$ is an inflection point
(since concavity changes at this pt)

y-intercept $f(0) = \frac{-1}{4}$ so $(0,-\frac{1}{4})$

x-intercepts $0=f(x) \iff 0=3x^2-1 \iff x=\pm\frac{1}{\sqrt{3}}$



8. (14 pts) Compute each of the following limits (7 pts each):

(a) $\lim_{x\to 0} \dfrac{e^{5x}-1}{\arcsin(2x)}$ $\underset{\ell'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0} \dfrac{5e^{5x}}{\frac{1}{\sqrt{1-(2x)^2}}\cdot 2} = \boxed{\dfrac{5}{2}}$

(b) $\lim_{x\to+\infty} x^{1/x}$ $\overset{\infty^0}{=} \lim_{x\to+\infty} e^{\ln\left(x^{\frac{1}{x}}\right)} = e^{\lim_{x\to+\infty}\frac{\ln x}{x}} = \boxed{e^0 = 1}$

but $\lim_{x\to+\infty} \dfrac{\ln x}{x} \underset{\ell'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to+\infty} \dfrac{\frac{1}{x}}{1} = 0$