5. Suppose that a tour service offers the following rates:

- \$200 per person if 50 people go on the tour (this is also the minimum number of people needed to book the tour).
- For each additional person over 50, up to a maximum of 80 total people, the rate per person is reduced by \$2.

It costs \$6000 (a fixed cost) plus \$32 per person to conduct the tour. What number of people will maximize the profit per tour?

6. You are (unjustly) sent to jail. The prison is a tall building surrounded by an 8ft tall fence situated 4ft away from the building. Your buddies are organizing an escape for you. The main tool is a straight ladder (but made from an expensive material). What is the shortest ladder that touches the ground at one end, passes over the fence (touching the fence to deactivate the prison alarm system!) and then touches the building at the other end?

7. James Bond is at the southernmost point of a circular lake with diameter of two miles. He needs to get to the northernmost point (diametrically opposite) of the lake as quickly as possible. It is known that James Bond runs twice as fast as he can swim. He can swim directly across the lake, he can run all the way around the lake, or he can try a combination of swimming and running. What path should James Bond take in order to minimize the time of the trip?

Pb. 5 x = # of people going on the tour $\underline{50 \le x \le 80}$ (0.5pts)
Domain

cost per person $= 200 - 2\cdot(x-50) = 200 - 2x + 100 = 300 - 2x$ (0.5pts)

Revenue = total income $= x\cdot(300-2x) = 300x - 2x^2$ (0.5pts)

total cost $= 6000 + 32x$ (0.5pts)

profit $= P(x) = \underbrace{(300x - 2x^2)}_{\text{Revenue}} - \underbrace{(6000 + 32x)}_{\text{Cost}} = -2x^2 + 268x - 6000$ (1pt)

We need to maximize $P(x) = -2x^2 + 268x - 6000$
when $x \in [50, 80]$

$P'(x) = -4x + 268$

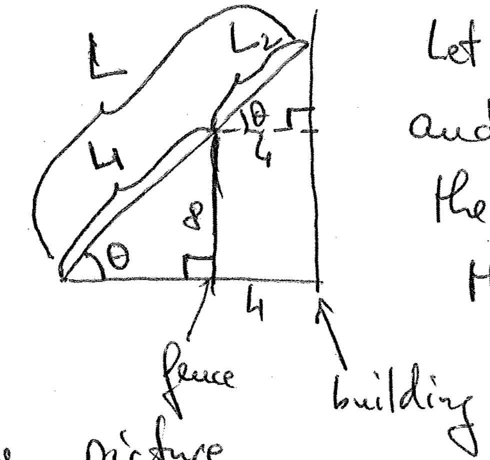
$P'(x) = 0 \iff -4x + 268 = 0 \Rightarrow 4x = 268 \Rightarrow \underline{x = 67}$ (1pt)
critical point

Since $P''(x) = -4 < 0$, the function is always concave down (you could also argue this by just mentioning that the graph is a parabola concave down), thus the critical point is an absolute maximum for the function in the interval. (0.5pts)

Thus, to maximize profit 67 people should go on the tour. (0.5pts)

6.[6] Solutions for the jail problem:

Solution 1:



Let L the length of the ladder and let θ be the angle that the ladder is making with the horizontal

From the picture

$L = L_1 + L_2$.

Using trigonometry,

$\sin\theta = \frac{8}{L_1}$, so $L_1 = \frac{8}{\sin\theta} = 8\csc\theta$

$\cos\theta = \frac{4}{L_2}$, so $L_2 = \frac{4}{\cos\theta} = 4\sec\theta$

Thus $L = L(\theta) = 8\csc\theta + 4\sec\theta$, with $\theta \in (0, \frac{\pi}{2})$

We want to minimize this function on the interval $(0, \frac{\pi}{2})$

$L'(\theta) = -8\csc\theta\cot\theta + 4\sec\theta\tan\theta$

$L'(\theta) = 0 \iff 8\csc\theta\cot\theta = 4\sec\theta\tan\theta \iff$

$\iff 8\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta} = 4\cdot\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} \iff$

$\iff 2\cos^3\theta = \sin^3\theta \iff 2 = \tan^3\theta \iff$

$\iff \tan\theta = \sqrt[3]{2} \iff \boxed{\theta = \arctan\sqrt[3]{2}}$ critical point

Since $\lim_{\theta\to 0^+} L(\theta) = +\infty$ and $\lim_{\theta\to\frac{\pi}{2}} L(\theta) = +\infty$

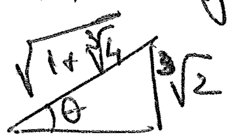
the critical point $\theta = \arctan\sqrt[3]{2}$ will yield the absolute minimum

To find the lenght of the shortest ladder we should evaluate, $L(\theta)$ when $\theta = \arctan(\sqrt[3]{2})$.

We can do this better (and without calculator – not allowed in jail!)

finding $\csc\theta = ?$ and $\sec\theta = ?$ if $\tan\theta = \sqrt[3]{2}$

For this, use triangle method:



so $\csc\theta = \dfrac{\sqrt{1+\sqrt[3]{4}}}{\sqrt[3]{2}}$, $\sec\theta = \dfrac{\sqrt{1+\sqrt[3]{4}}}{1}$

So shortest ladder has a length of

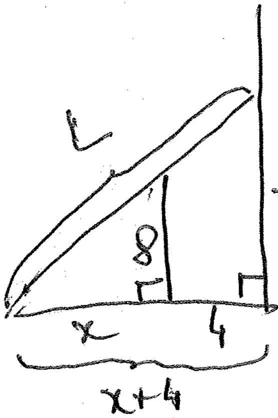
$$L_{min} = 8\,\frac{\sqrt{1+\sqrt[3]{4}}}{\sqrt[3]{2}} + 4\sqrt{1+\sqrt[3]{4}} = 4\sqrt{1+\sqrt[3]{4}}\left(1+\frac{2}{\sqrt[3]{2}}\right)$$

or, since $\dfrac{2}{\sqrt[3]{2}} = \dfrac{\sqrt[3]{8}}{\sqrt[3]{2}} = \sqrt[3]{4}$ changes to

$$L_{min} = 4\sqrt{1+\sqrt[3]{4}}\left(1+\sqrt[3]{4}\right) = 4\left(1+\sqrt[3]{4}\right)^{\frac{3}{2}} = \boxed{4\left(1+2^{\frac{2}{3}}\right)^{\frac{3}{2}}\text{ ft}}$$

Isn't it that this expression is much nicer looking than the decimal approximation $L_{min} = 16.6478$ ft?

Pb. 6 Solution 2



↳ we'll use similar triangles in this solution, and we choose the distance between the foot of the ladder and the fence be our variable (x). As before, let L be the length of the ladder and let h be the height of the point on the building where the ladder touches.

From Pythagora, $L^2 = (x+4)^2 + h^2$

We need a relation between x & h. That is given by the similarity of the two right angle triangles in the picture. From ratio of corresponding sides we get

$$\frac{x}{x+4} = \frac{8}{h} \quad \text{so} \quad h = \frac{8(x+4)}{x}$$

Thus $$L^2 = (x+4)^2 + \frac{64(x+4)^2}{x^2} = (x+4)^2\left(1+\frac{64}{x^2}\right)$$

An idea to simplify computations is to minimize L^2 instead of minimizing L.

So consider the function $g(x) = (x+4)^2\left(1+\frac{64}{x^2}\right)$ with domain $x \in (0, +\infty)$ and try to find the absolute minimum (if any) on this domain.

$$g'(x) = 2(x+4)\left(1+\frac{64}{x^2}\right) + (x+4)^2\left(-64 \cdot 2x^{-3}\right)$$

$$g'(x) = (x+4)\left[2 + \frac{128}{x^2} - \frac{128(x+4)}{x^3}\right] = (x+4)\cdot\frac{2x^3 + \cancel{128x} - \cancel{128x} - 512}{x^3}$$

Thus $$g'(x) = \frac{2(x+4)}{x^3}\left(x^3 - 256\right)$$

There is only one critical point in $(0, +\infty)$

$$x = \sqrt[3]{256} = \sqrt[3]{2^8} = 2^{\frac{8}{3}} = 4\sqrt[3]{4}$$

We evaluate ~~$g(x)$~~ the limit of $g(x)$ towards the end points

$$\lim_{x \to 0^+} \left[(x+4)^2 \left(1 + \frac{64}{x^2}\right) \right] = +\infty \qquad \lim_{x \to +\infty} \left[(x+4)^2 \left(1 + \frac{64}{x^2}\right) \right] = +\infty$$

Thus, at $x = 4\sqrt[3]{4}$, the function $g(x)$ will attain the absolute minimum. Thus

$$L_{min}^2 = g(4\sqrt[3]{4}) = (4\sqrt[3]{4} + 4)^2 \left(1 + \frac{64}{(4\sqrt[3]{4})^2}\right)$$

so $L_{min} = (4\sqrt[3]{4} + 4)\sqrt{1 + \frac{16}{\sqrt[3]{16}}} \approx 16.6478$ ft