Name: Solution Key Panther ID: ______________

Exam 2 Calculus II Fall 2014

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (12 pts) Circle the correct answer (3 pts each):

(a) For the integral $\int \sqrt{4x^2 - 9}\, dx$, the following substitution is helpful:

(i) $3x = 2\cos\theta$ (ii) $2x = 3\sin\theta$ (iii) $w = 4x^2 - 9$ (iv) $2x = 3\sec\theta$ (v) $2x = 3\tan\theta$

(Don't spend time evaluating the integral. It is not required.)

(b) The partial fraction decomposition of $\dfrac{x+3}{x^4+9x^2}$ is of the form:

(i) $\dfrac{A}{x^2} + \dfrac{B}{x^2+9}$ (ii) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+3} + \dfrac{D}{(x+3)^2}$ (iii) $\dfrac{x+3}{x^4} + \dfrac{x+3}{9x^2}$

(iv) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+9}$ (v) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+3}$

(c) The function $f(x)$ is known to be continuous, positive and concave up when $x \in [-2, 2]$. Let M_4 be the mid-point approximation with 4 subdivisions of the integral $\int_{-2}^{2} f(x)\, dx$. Then compared with the integral, M_4 is an

(i) overestimate (ii) underestimate (iii) exact estimate (iv) cannot tell (more should be known about $f(x)$)

(d) The function $g(x)$ is known to be a quadratic function. Let S_4 be the Simpson approximation with 4 subdivisions of the integral $\int_{-2}^{2} g(x)\, dx$. Then compared with the integral, S_4 is an

(i) overestimate (ii) underestimate (iii) exact estimate (iv) cannot tell (more should be known about $g(x)$)

2. (20 pts) Compute each of the following:

(a) (8 pts) $\int x^3 \ln x\, dx =$

I.B.P. $u = \ln x$ $dv = x^3 dx$

$du = \frac{1}{x} dx$ $v = \frac{x^4}{4}$

$= \frac{x^4}{4}\ln x - \int \frac{1}{x} \cdot \frac{x^4}{4} dx =$

$= \frac{x^4}{4}\ln x - \frac{1}{4}\cdot\frac{x^4}{4} + C$

$$\boxed{= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}$$

Can also be done with the sub. $w = 4 - x^2$

(b) (12 pts) $\int \dfrac{x^3}{\sqrt{4-x^2}}\, dx = \int \dfrac{(2\sin\theta)^3\, \cancel{2\cos\theta}\, d\theta}{\cancel{2\cos\theta}}$

$x = 2\sin\theta$

$\sqrt{4-x^2} = \sqrt{4 - 4\sin^2\theta} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

$= 8\int \sin^3\theta\, d\theta = 8\int (1-\cos^2\theta)\sin\theta\, d\theta$

$w = \cos\theta$

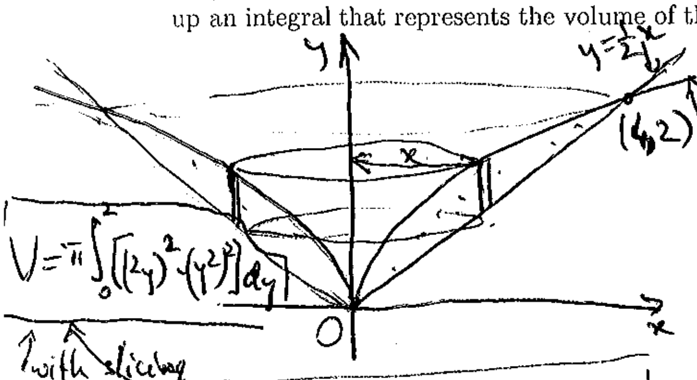
$dw = -\sin\theta\, d\theta$

$= 8\int (1-w^2)(-dw) = -8\left(w - \frac{w^3}{3}\right) + C =$

$= -8\cos\theta + \frac{8}{3}(\cos\theta)^3 + C$

$= -4\sqrt{4-x^2} + \frac{1}{3}\left(\sqrt{4-x^2}\right)^3 + C$

3. (10 pts) The region bounded between $y = \sqrt{x}$ and $y = \frac{1}{2}x$ is rotated around the y-axis. Sketch the solid and set up an integral that represents the volume of this solid. Just set up. The calculation of the integral is not required.



$V=\pi\int_0^2 [(2y)^2-(y^2)^2]\,dy$

↑ with slicing

Intersection $\begin{cases} y=\frac{1}{2}x \\ y=\sqrt{x} \end{cases} \Rightarrow \sqrt{x}=\frac{1}{2}x \Rightarrow$

$\Rightarrow x=\frac{1}{4}x^2$

$\Rightarrow x=0,\ x=4$

Solution with cyl. shells.

$V=\int 2\pi R_{shell} \cdot h_{shell} \cdot Th_{shell}$

$Th_{shell} = dx$

$h_{shell} = \sqrt{x}-x$

$R_{shell} = x$

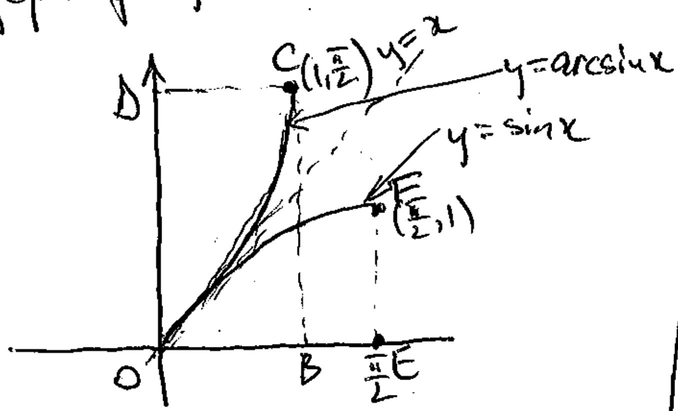
$$\boxed{V=\int_0^4 2\pi x(\sqrt{x}-x)\,dx}$$

Certainly, the problem can be done equally well with the slicing method. slices are washers of thickness dy so your integral should be in terms of y.

4. (22 pts) Compute each of the following:

(a) (10 pts) $\int_0^1 \arcsin x \, dx$

Standard solution would be to use I.B.P. Here is a quick solution using the graph of $y=\arcsin x$ and its relation to the graph of $y=\sin x$



$\int_0^1 \arcsin x\,dx$ = area of "curved" triangle" OBC

= area of rectangle OBCD −

− area of "curved triangle" OCD

= (area of ▭ OBCD) − (area of △OFE)

$= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{\pi}{2} + \cos x\Big|_{x=0}^{x=\frac{\pi}{2}} = \frac{\pi}{2} - 1$

(b) (12 pts) $\int \frac{x+2}{x(x^2+4)}\,dx = \int\left(\frac{A}{x} + \frac{Bx+C}{x^2+4}\right)dx =$

Partial fractions

$\frac{x+2}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

$x+2 = A(x^2+4) + x(Bx+C)$

$\Rightarrow \begin{cases} A+B=0 \\ C=1 \\ 4A=2 \end{cases} \Rightarrow \begin{matrix} A=\frac{1}{2} \\ B=-\frac{1}{2} \\ C=1 \end{matrix}$

$* = A\int \frac{1}{x}\,dx + \int\left(\frac{Bx}{x^2+4} + \frac{C}{x^2+4}\right)dx$

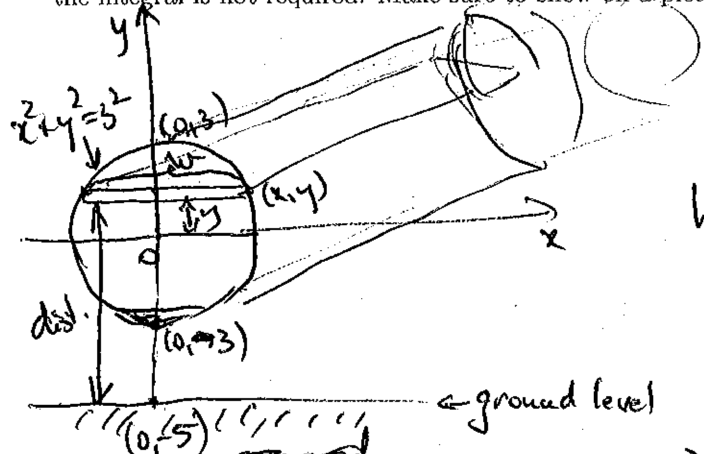
$= A\ln x + \frac{B}{2}\ln(x^2+4) + \frac{C}{2}\arctan\left(\frac{x}{2}\right)$

$= \frac{1}{2}\ln x - \frac{1}{4}\ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

plug in A, B, C.

Trig. sub. $x=2\tan\theta$ also works for this problem

5. (10 pts) The tank of a fuel truck is a cylinder of radius 3 ft and length 30 ft. The tank sits horizontally with the lower side at an altitude of 2 ft above the ground (wheels of the truck are 2 ft high). Assuming that the tank is initially half-full, set up an integral that represents the total work required to completely fill up the tank by pumping up gasoline from ground level. The density of gasoline is $\rho = 45$ lb/ft^3. (Just set up. The calculation of the integral is not required. Make sure to show on a picture what variable(s) you are using.)



$W = \int_?^? dW =$

$W = \int_?^? \rho \cdot A_{slice} \cdot dist \cdot Th_{slice}$

$Th_{slice} = dy$

$dist. = 5+y$

$x = \sqrt{9-y^2}$

$A_{slice} = l \cdot w = 30 \times 2x = 60\sqrt{9-y^2}$

$$W = \int_{y=0}^{y=3} 45 \times 60\sqrt{9-y^2}\,(5+y)\,dy$$

Best choice for origin is at the center of the cross-section circle at one of the ends of the tank.

6. (16 pts) Compute each of the following improper integrals. Specify if they are convergent or divergent. Why is the second integral improper?

(a) $\int_0^{+\infty} e^{-3x}\,dx =$

$= \lim_{B\to\infty} \int_0^B e^{-3x}\,dx$

$= \lim_{B\to\infty} \left(-\frac{1}{3} e^{-3x} \Big|_{x=0}^{x=B} \right)$

$= \lim_{B\to+\infty} \left(-\frac{1}{3} e^{-3B} + \frac{1}{3} e^{0} \right)$

$= 0 + \frac{1}{3} = \frac{1}{3}$

integral converges to $\frac{1}{3}$

(b) $\int_0^2 \frac{1}{(t-1)^2}\,dt$ $= \int_0^1 \frac{1}{(t-1)^2}\,dt + \int_1^2 \frac{1}{(t-1)^2}\,dt$

improper because function goes to $+\infty$ at $t \to 1$.

$\int \frac{1}{(t-1)^2}\,dt = \int (t-1)^{-2}\,dt = -(t-1)^{-1} + C$

$= -\frac{1}{t-1} + C$

$\int_0^1 \frac{1}{(t-1)^2}\,dt = \lim_{B\to 1^-} \int_0^B \frac{1}{(t-1)^2}\,dt =$

$= \lim_{B\to 1^-} \left(-\frac{1}{t-1} \Big|_{t=0}^{t=B} \right) =$

$= \lim_{B\to 1^-} \left(-\frac{1}{B-1} - 1 \right) = +\infty - 1 = \boxed{+\infty}$

Similarly $\int_1^2 \frac{1}{(t-1)^2}\,dt = +\infty$

So the given integral diverges to $+\infty$.

7. (24 pts) Choose TWO out of the following THREE (12 pts each):

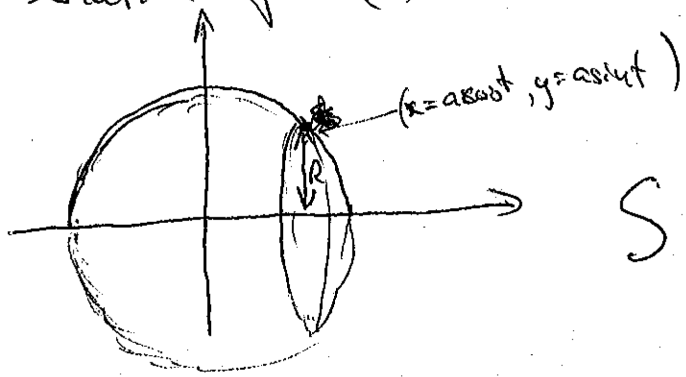
(a) Using the slicing method, prove the formula for the volume of a pyramid. (if needed, you may assume that the base of the pyramid is a square).

(b) Find (with proof) a reduction formula for $\int \tan^n x\, dx$.

(c) Find the formula for surface area of a sphere of radius a, by rotating the semi-circle $x = a\cos t$, $y = a\sin t$, $t \in [0, \pi]$, around the x-axis. Full computation is required.

For (a) or (b) see your notes. I did both in class.

Solution for (c).



$$S = \int_?^? 2\pi R \cdot ds$$

where $ds = \sqrt{(dx)^2 + (dy)^2}$

but $dx = x'(t)\,dt = -a\sin t\,dt$

$dy = y'(t)\,dt = a\cos t\,dt$

$$ds = \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = \sqrt{a^2}\,dt = a\,dt$$

$$R = y = a\sin t$$

$$S = \int_{t=0}^{t=\pi} 2\pi a \sin t \cdot a\,dt = 2\pi a^2 \int_{t=0}^{t=\pi} \sin t\,dt$$

$$= 2\pi a^2 \left(-\cos t\right)\Big|_{t=0}^{t=\pi} = 2\pi a^2\left(-(-1) + 1\right) = \boxed{4\pi a^2}$$