Name: Solution Key Panther ID: ____________

Exam 3 Calculus II Fall 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) In each case answer True or False. No justification needed. (2 pts each)

(a) The curve $r = 1 - 2\sin\theta$ has a graph symmetrical with respect to the y-axis.

True (since $\sin(\pi-\theta) = \sin\theta$)

(b) If $\lim_{k\to+\infty} a_k = 0$ then the series $\sum_{k=1}^{\infty} a_k$ is convergent.

False (Example: $\lim_{k\to\infty} \frac{1}{k} = 0$ and $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$)

(c) If $0 < a_k < \frac{1}{k^2}$ for all $k \geq 1$, then $\sum_{k=1}^{\infty} a_k$ is convergent.

True (simple comparison test)

(d) The alternating harmonic series is absolutely convergent.

False (alternating harmonic series is conditionally convergent)

(e) $1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \ldots = 1$

False (seq. of partial sums oscillates $S_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ \frac{3}{2} & \text{if } n \text{ even} \end{cases}$

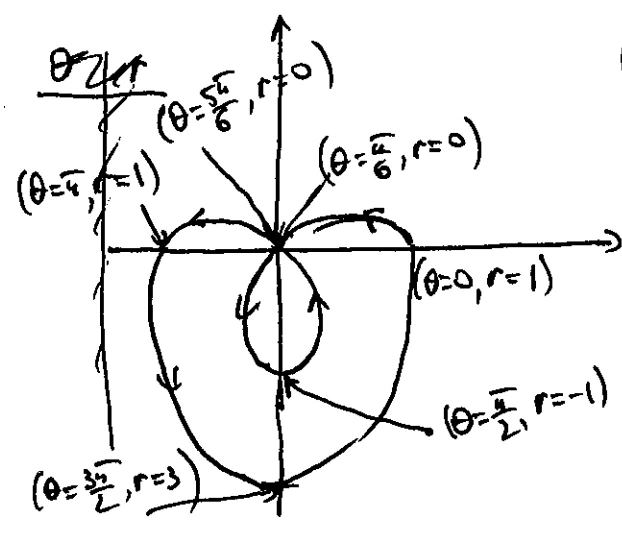
(f) Suppose $a_k > 0$, $b_k > 0$, and $\lim_{k\to+\infty} \frac{a_k}{b_k} = 0$. Then if $\sum_{k=1}^{\infty} b_k$ converges, $\sum_{k=1}^{\infty} a_k$ also converges.

True (see proof in Pb. 7)

2. (12 pts) (a) (6 pts) Sketch the graph of the limacon with inner loop $r = 1 - 2\sin\theta$. Be sure to give the polar coordinates of all points where the graph intersects the x-axis, the y-axis, or passes through the origin.

(b) (6 pts) Set up, but do **not** evaluate, an integral that represents the area inside the inner loop of the limacon $r = 1 - 2\sin\theta$.

θ	r
0	1
$\frac{\pi}{6}$	0
$\frac{\pi}{3}$	$1-\sqrt{3}$
$\frac{\pi}{2}$	-1
$\frac{2\pi}{3}$	$1-\sqrt{3}$
$\frac{5\pi}{6}$	0
π	1
$\frac{3\pi}{2}$	3



(b) $A = \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{5\pi}{6}} \frac{1}{2}(1-2\sin\theta)^2\,d\theta$

or $A = 2\cdot\frac{1}{2}\int_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{2}} (1-2\sin\theta)^2\,d\theta$

3. (20 pts) Determine whether each of the following series converges or diverges. Full justification is required.

(a) $\sum_{k=1}^{\infty} \frac{k}{2k+1}$

$\lim_{k\to+\infty} \frac{k}{2k+1} = \frac{1}{2} \neq 0$

So the series diverges by the k^{th}-term divergence test

(b) $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

The function $f(x) = \frac{1}{x(\ln x)^2}$ is continuous, positive and decreasing so we can apply the integral test:

$$\int_2^{+\infty} \frac{1}{x(\ln x)^2}\,dx \underset{\substack{u=\ln x \\ du=\frac{1}{x}dx}}{=} \int_{u=\ln 2}^{u=+\infty} \frac{1}{u^2}\,du$$

$$= -\frac{1}{u}\Big|_{u=\ln 2}^{u=+\infty} = \frac{1}{\ln 2}$$

Since the integral is convergent, the series is also convergent.

4. (20 pts) For each of the following series, determine if the series is divergent (D), conditionally convergent (CC), or absolutely convergent (AC). Answer **and carefully** justify your answer. Very little credit will be given just for a guess. Most credit is given for the quality of the justification. (10 pts each)

(a) $\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$

This series is $\underline{\underline{\text{not}}}$ (A.C.) since

$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{\sqrt{k}} \right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ is divergent p-series with $p = \frac{1}{2} < 1$

But $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ is convergent by A.S.T.,

since $a_k = \frac{1}{\sqrt{k}}$ is decreasing and

$\lim_{k\to+\infty} a_k = 0$

Thus, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ is (C.C.)

(b) $\sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k}}{(2k)!}$

Apply the Absolute Ratio Test

$$\rho = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|} =$$

$$= \lim_{k\to+\infty} \frac{\pi^{2(k+1)}}{(2(k+1))!} \cdot \frac{(2k)!}{\pi^{2k}} =$$

$$= \lim_{k\to+\infty} \frac{\pi^2}{(2k+2)(2k+1)} = 0$$

Since $\rho = 0 < 1$, it follows that the given series is (A.C.)

Additional note: As $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$, observe that $\sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k}}{(2k)!} = \cos \pi = -1$.

5. (14 pts) (a) (8 pts) Find the Taylor polynomial of degree 2 of $f(x) = \tan x$ at $x_0 = \pi/4$.

(b) (6 pts) Use this Taylor polynomial to get an approximation of $\tan(101\pi/400)$.

(a) $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$. In our case, $n=2$, $x_0 = \frac{\pi}{4}$ so

$T_2(x) = f(\frac{\pi}{4}) + \frac{f'(\frac{\pi}{4})}{1!}(x-\frac{\pi}{4}) + \frac{f''(\frac{\pi}{4})}{2!}(x-\frac{\pi}{4})^2$

$f(x) = \tan x \qquad f(\frac{\pi}{4}) = \tan(\frac{\pi}{4}) = 1$

$f'(x) = \sec^2 x \qquad f'(\frac{\pi}{4}) = \sec^2(\frac{\pi}{4}) = 2$

$f''(x) = 2\sec x \cdot \sec x \cdot \tan x = 2\sec^2 x \tan x$

$f''(\frac{\pi}{4}) = 2 \cdot 2 \cdot 1 = 4$

$\Rightarrow \boxed{T_2(x) = 1 + 2(x-\frac{\pi}{4}) + 2(x-\frac{\pi}{4})^2}$

(b) $\tan\left(\frac{101\pi}{400}\right) \approx 1 + 2\left(\frac{101\pi}{400} - \frac{\pi}{4}\right) + 2\left(\frac{101\pi}{400} - \frac{\pi}{4}\right)^2 = 1 + 2\cdot\frac{\pi}{400} + 2\cdot\left(\frac{\pi}{400}\right)^2$

6. (a) (12 pts) Find the interval of convergence (with endpoints) of the power series $\sum_{k=1}^{\infty} \frac{k(x-2)^{k-1}}{3^k}$.

(b) (6 pts) The series in part (a) is the Taylor series at $x_0 = 2$ of a certain function $f(x)$. Can you find the function $f(x)$? (*Hint:* Integrate, to first find $\int f(x)\,dx$.)

(a) By Abs. Ratio Test $\rho = \lim_{k\to+\infty} \frac{(k+1)|x-2|^k}{3^{k+1}} \cdot \frac{3^k}{k|x-2|^{k-1}}$

$\rho = \lim_{k\to+\infty}\left(\frac{k+1}{k} \cdot \frac{|x-2|}{3}\right) = \frac{|x-2|}{3}$

If $\rho = \frac{|x-2|}{3} < 1$ series absolutely converges

$\Updownarrow$

$|x-2| < 3 \Leftrightarrow -1 < x < 5$

Endpoints:

$x=-1 \quad \sum \frac{k(-3)^{k-1}}{3^k} = \sum \frac{(-1)^{k-1}\cdot k}{3}$ diverges by k^{th} term test

$x=5 \quad \sum \frac{k\cdot 3^{k-1}}{3^k} = \sum \frac{k}{3}$ diverges (k^{th} term test)

Thus, $I = (-1, 5)$

(b) If $f(x) = \sum \frac{k(x-2)^{k-1}}{3^k} \Rightarrow \int f(x)\,dx = \sum_{k=1}^{\infty} \frac{(x-2)^k}{3^k} \underset{\text{geometric series}}{=} \frac{1}{1-\frac{x-2}{3}} - 1$

Thus $f(x) = \left(\frac{1}{1-\frac{x-2}{3}} - 1\right)' = (-1)\frac{1}{(1-\frac{x-2}{3})^2}\cdot\left(-\frac{1}{3}\right) = \ldots = \boxed{\frac{3}{(5-x)^2}}$

7. (12 pts) Choose ONE to prove:

(a) State and prove the p-series test (using the integral test).

(b) State and prove the area formula for polar coordinates. Be sure to have in your proof a picture, a sum, and a limit.

(c) Prove or disprove the statement in Pb. 1 (f): Suppose $a_k > 0$, $b_k > 0$, for all $k \geq 1$, and suppose $\lim_{k\to+\infty} \frac{a_k}{b_k} = 0$.

$$\text{Then if } \sum_{k=1}^{\infty} b_k \text{ converges, the series } \sum_{k=1}^{\infty} a_k \text{ also converges .}$$

For (a) and (b) see your notes or textbook.

For (c): The statement is true:

Proof: $\lim_{k\to+\infty} \frac{a_k}{b_k} = 0 \Rightarrow$ there is a rank k_0, so that $\frac{a_k}{b_k} \leq 1$ for all $k \geq k_0$

Thus $a_k \leq b_k$ for all $k \geq k_0$, so by simple comparison, since $\sum_{k=1}^{\infty} b_k$ converges, it follows that $\sum_{k=1}^{\infty} a_k$ also converges

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Note 1: This is a strengthening of the limit comparison test:

If $a_k > 0$, $b_k > 0$ and $\lim_{k\to+\infty} \frac{a_k}{b_k} = L$, where $0 < L < +\infty$

then the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ have same nature

Note 2: However, if $\lim_{k\to+\infty} \frac{a_k}{b_k} = 0$, we cannot say that the series $\sum a_k$, $\sum b_k$ must have the same nature

$\boxed{Ex}$ Give an example with $\lim_{k\to+\infty} \frac{a_k}{b_k} = 0$,

$\sum_{k=1}^{\infty} b_k$ divergent and $\sum_{k=1}^{\infty} a_k$ convergent,

Note 3: The case $\lim_{k\to+\infty} \frac{a_k}{b_k} = +\infty$ is similar.