Name: Solution of Pb. 3 Panther ID: ____________

Worksheet week 7 Calculus II Fall 2014

1. The region bounded between the graph of $\sin x$ and the x-axis when $x \in [0, \pi]$ is rotated around the y-axis; the solid formed has volume V_1. Then the same region is rotated around the x-axis; the solid formed has volume V_2. Find V_1 and V_2 and observe that $V_1 = 4V_2$.

2. Evaluate (a) $\int \sin^2 x \cos^3 x \, dx$ (b) $\int \tan^2 x \sec^4 x \, dx$

3. (a) Derive a reduction formula for

$$\int sin^n x \, dx ,$$

where n is a positive integer. You may check formula (9) in 7.2 to confirm your result.

(b) Use part (a) to derive a recursion formula for

$$A_n = \int_0^{\pi/2} sin^n x \, dx .$$

(c) Find A_1 directly, then find A_3, A_5 using the recursion formula. Write a general formula for A_n when n is odd.

(d) Find A_0 directly, then find A_2, A_4 using the recursion formula. Write a general formula for A_n when n is even.

The general formulas for A_n are the so-called *Wallis sine formulas.*

(a)

Let $I = \int \sin^n x \, dx = \int \sin^{n-1} x \cdot \sin x \, dx \underset{I.B.P.}{=} -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$

$u = \sin^{n-1} x \qquad dv = \sin x \, dx$

$du = (n-1)\sin^{n-2} x \cos x \, dx \qquad v = \int \sin x \, dx = -\cos x$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x (1-\sin^2 x) \, dx =$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx \underbrace{- (n-1)\int \sin^n x \, dx}_{-(n-1)\cdot I}$$

Thus $\underbrace{(n-1) I + I}_{n \cdot I} = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx$

Thus $\boxed{\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{(n-1)}{n} \int \sin^{n-2} x \, dx}$

(b) The recursion formula for A_n follows from the reduction formula in part (a) but applied for <u>definite</u> integrals.

$$\underbrace{\int_0^{\frac{\pi}{2}} \sin^n x \, dx}_{A_n} = \underbrace{-\frac{1}{n} \sin^{n-1} x \cos x \Big|_{x=0}^{x=\frac{\pi}{2}}}_{0 \;\leftarrow \text{ since } \cos\frac{\pi}{2}=0,\ \sin 0 = 0} + \frac{n-1}{n} \underbrace{\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx}_{A_{n-2}}$$

Thus $A_n = \frac{n-1}{n} A_{n-2}$ for all $n \geq 2$.

(c) & (d) Directly

$$A_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = -\cos x \Big|_{x=0}^{x=\frac{\pi}{2}} = 1 \quad ; \quad A_0 = \int_0^{\frac{\pi}{2}} 1 \, dx = \frac{\pi}{2}$$

Next we look for a pattern. (It is better <u>not</u> to multiply when you look for patterns

$$A_3 = \frac{3-1}{3} A_1 = \frac{2}{3} \cdot 1 = \frac{2}{3}$$

$$A_5 = \frac{5-1}{5} A_3 = \frac{4}{5} \cdot \frac{2}{3}$$

$$A_7 = \frac{7-1}{7} A_5 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

Thus

$$\boxed{A_{2k+1} = \frac{(2k)\cdot(2k-2)\cdot \ldots \cdot 2}{(2k+1)(2k-1)\cdot \ldots \cdot 3}}$$

for every $k \geq 0$.

$$A_2 = \frac{2-1}{2} A_0 = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$A_4 = \frac{4-1}{4} A_2 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$A_6 = \frac{6-1}{6} A_4 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Thus

$$\boxed{A_{2k} = \frac{(2k-1)\cdot(2k-3)\cdot \ldots \cdot 1}{(2k)(2k-2)\cdot \ldots \cdot 2} \cdot \frac{\pi}{2}}$$

for $k \geq 0$.