Worksheet week 5 - MAC 2312, Fall 2014 NAME: Solution

1. Find the area of the region bounded between $x = y^2 - 1$ and $x + y = 1$. Sketch and computation are required.

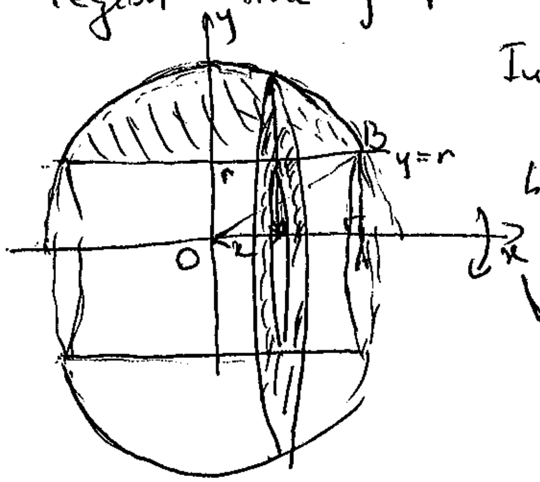
2. Set up an integral that gives the volume of the solid obtained when the region in problem 1 is rotated around the line $x = 3$. You are not required to evaluate the integral, but you should sketch the solid.

3. (Volume of a sphere) Consider the region above the x-axis bounded by the semi-circle $y = \sqrt{R^2 - x^2}$ and rotate this region around the x-axis. Use the slicing method to find the formula for the volume of the sphere of radius R. Full computation is required now.

4. (Precise drilling) Suppose you have a solid sphere of radius R. A cylindrical hole with radius r, $r < R$, is to be drilled symmetrically through the center of the sphere, so that the volume of the solid left be exactly half of the volume of the original solid sphere. Find the exact value of the ratio r/R for this to happen.

Note: You should finish Problems 1, 2, 3 in class during the problem solving session. Problem 4 is for take-home (possibly for some bonus), but you could start it in class if you have time.

Solution for Pb. 4: The solid can be obtained by rotating the region inside $y = \sqrt{R^2 - x^2}$, but above $y = r$, around the x-axis.



In this case, the slicing method or the cyl. shells method are equally effective, but we'll compute the volume by slicing here

$$V_{solid} = \int_?^? A_{slice} \cdot Th_{slice}$$

$$Th_{slice} = dx$$

$$A_{slice} = \pi R_{outer}^2 - \pi r_{inner}^2$$

but $R_{outer} = y = \sqrt{R^2 - x^2}$ (note $R_{outer} \neq R$)

$r_{inner} = r$

For the limits of integration, we need to find the x-coordinate of the point A from the picture above. That is obtained from ΔOAB. The hypotenuse $|OB| = R$, side $|AB| = r$, so by Pythagora

$$|OA| = \sqrt{R^2 - r^2}.$$

Thus $$V_{solid} \overset{symmetry}{=} 2\int_0^{\sqrt{R^2-r^2}} \pi\left(\left(\sqrt{R^2-x^2}\right)^2 - r^2\right) dx = 2\pi \int_0^{\sqrt{R^2-r^2}} (R^2 - r^2 - x^2)\, dx$$

$$V_{solid} = 2\pi\left[(R^2-r^2)x - \frac{x^3}{3}\right]\Big|_0^{\sqrt{R^2-r^2}} = 2\pi\sqrt{R^2-r^2}\left[(R^2-r^2) - \frac{R^2-r^2}{3}\right] = \frac{4\pi}{3}(R^2-r^2)^{\frac{3}{2}}$$

We want

$$\frac{4\pi}{3}\left(R^2-r^2\right)^{\frac{3}{2}} = \frac{1}{2}\,\frac{4\pi}{3}R^3 \Rightarrow$$

$$\Rightarrow \left(R^2-r^2\right)^{\frac{3}{2}} = \frac{1}{2}R^3 \Rightarrow$$

$$\Rightarrow \frac{\left(\sqrt{R^2-r^2}\right)^3}{R^3} = \frac{1}{2} \Rightarrow \left(\frac{\sqrt{R^2-r^2}}{R}\right)^3 = \frac{1}{2} \Rightarrow$$

$$\Rightarrow \frac{\sqrt{R^2-r^2}}{R} = \frac{1}{\sqrt[3]{2}} \Rightarrow$$

$$\Rightarrow \sqrt{\frac{R^2-r^2}{R^2}} = \frac{1}{\sqrt[3]{2}} \Rightarrow \sqrt{1-\left(\frac{r}{R}\right)^2} = \frac{1}{\sqrt[3]{2}} \Rightarrow$$

$$\Rightarrow 1-\left(\frac{r}{R}\right)^2 = \frac{1}{\sqrt[3]{4}} \Rightarrow$$

$$\Rightarrow \left(\frac{r}{R}\right)^2 = 1-\frac{1}{\sqrt[3]{4}} \Rightarrow \boxed{\frac{r}{R} = \sqrt{1-\frac{1}{\sqrt[3]{4}}}}$$