Name: Solution Key Panther ID: ____________

Exam 1 Calculus II Spring 2016

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) (a) (4 pts) A particle moves on a straight line and let $v(t)$ represent the velocity (in ft/second) of the particle at time t (in seconds), where $t \in [0, 10]$. Fill in the blanks with appropriate words.

Then $v'(t)$ represents acceleration at time t,

while $\int_0^{10} |v(t)|\, dt$ represents distance traveled in the time interval [0, 10s]

(b) (4 pts) Suppose that oil is leaking into the ocean from a damaged tanker at a rate of $r(t)$ gallons per day, where t is the time in days since the accident occurred. In one sentence, explain what the integral $\int_2^3 r(t)\, dt$ represents.

The amount of oil that leaked in the ocean during day 3

(c) (4 pts) Simplify as much as possible the expression

$$\frac{d}{dx}\left(\int_e^{e^x} (\ln t)^2\, dt\right) \underset{\substack{\text{F.T.C part B} \\ \text{+ chain rule}}}{=} (\ln(e^x))^2 \cdot (e^x)' = \boxed{x^2 \cdot e^x}$$

2. (8 pts) Use summation notation and then find the value of the sum: 2+4+6+8+ ... +2014+2016 =

It's OK to leave your answer as a product.

Gauss' formula

$$= \sum_{k=1}^{1008} 2k = 2\sum_{k=1}^{1008} k = \cancel{2} \cdot \frac{1008 \cdot (1008+1)}{\cancel{2}}$$

$$= \boxed{1008 \cdot 1009}$$

3. (20 pts) True or False questions (4 pts each). In each case, circle your answer (2 pts) and briefly justify (2 pts).

(a) Any bounded sequence must be monotone. **True** **False**

Justification: $a_n = (-1)^n$ is bounded as $-1 \leq a_n \leq 1$ but is not monotone since it alternates between $1, -1$.

(b) If a sequence $\{a_n\}$ is monotone and satisfies $2 \leq a_n \leq 5$ for all $n \geq 1$, then $\{a_n\}$ is convergent. **True** **False**

Justification: If $2 \leq a_n \leq 5$ for all $n \geq 1$ then $\{a_n\}_n$ is bounded. Any sequence which is monotone and bounded is convergent (class theorem)

(c) The sequence $a_n = \frac{(-1)^n}{\sqrt{n}}$ is divergent. **True** **False**

Justification: $\lim_{n\to+\infty} a_n = 0$ (as $\lim_{n\to+\infty} |a_n| = 0$), so the sequence is convergent to 0

(d) The series $\sum_{k=1}^{+\infty} \frac{1}{k}$ is convergent. **True** **False**

Justification: Harmonic series diverges (as shown in class)

(e) If $\int_0^5 f(x)\,dx = 10$ and $\int_3^5 f(x)\,dx = -3$ then $\int_0^3 f(x)\,dx = 13$. **True** **False**

Justification: $\int_0^3 f(x)dx = \int_0^5 f(x)dx - \int_3^5 f(x)dx = 10 - (-3) = 13$

4. (8 pts) Show that the sequence $a_n = \frac{5^n}{n!}$ is eventually monotone (and specify the type of monotonicity you find).

Easiest is to consider the ratio

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{\cancel{5^{n+1}}\,5}{\cancel{5^n}} \cdot \frac{\cancel{n(n-1)\cdots 2\cdot 1}\;1}{(n+1)\cancel{n(n-1)\cdots 2\cdot 1}} = \frac{5}{n+1},$$

so we see that $\frac{a_{n+1}}{a_n} = \frac{5}{n+1} < 1$ for all $n \geq 5$.

Thus $a_{n+1} < a_n$ for any $n \geq 5$, so the sequence $\{a_n\}$ is eventually strictly decreasing.

5. (16 pts) Evaluate each of the following series or show it diverges:

(a) $\sum_{k=2}^{+\infty}(-1)^k\frac{2^{3k}}{3^{2k}}$ $= \sum_{k=2}^{\infty} (-1)^k \cdot \frac{(2^3)^k}{(3^2)^k} = \sum_{k=2}^{\infty}\left(-\frac{8}{9}\right)^k =$

convergent geometric series as ~~with~~ $r=-\frac{8}{9}$, so $|r|<1$

$= \left(-\frac{8}{9}\right)^2 + \left(-\frac{8}{9}\right)^3 + \left(-\frac{8}{9}\right)^4 + \ldots = \left(-\frac{8}{9}\right)^2\left[1 + \left(-\frac{8}{9}\right)^1 + \left(-\frac{8}{9}\right)^2 + \right.$

(b) $\ln\left(\frac{1}{3}\right) + \ln\left(\frac{3}{5}\right) + \ln\left(\frac{5}{7}\right) + \ln\left(\frac{7}{9}\right) + \ldots$ $= \sum_{k=1}^{\infty} \ln\left(\frac{2k-1}{2k+1}\right) = \sum_{k=1}^{\infty}\left(\ln(2k-1) - \ln(2k+1)\right)$

properties of logs

Realize that the series is telescopic.

Then $S_n = \sum_{k=1}^{n}\left(\ln(2k-1) - \ln(2k+1)\right) =$

$= (\ln 1 - \cancel{\ln 3}) + (\cancel{\ln 3} - \cancel{\ln 5}) + (\cancel{\ln 5} - \cancel{\ln 7}) + \ldots + (\cancel{\ln(2n-1)} - \ln(2n+1))$

so $S_n = \ln 1 - \ln(2n+1) = -\ln(2n+1)$

Thus $\sum_{k=1}^{\infty} \ln\left(\frac{2k-1}{2k+1}\right) = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(-\ln(2n+1)\right) = -\infty$

so the series diverges (to $-\infty$).

6. (8 pts) Find the average value of $f(x) = \sqrt{x}$ on the interval $[1, 9]$.

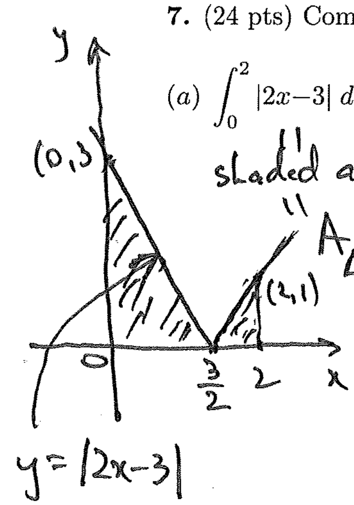
In general, $f_{ave} = \dfrac{\int_a^b f(x)\,dx}{b-a}$

In our case $f_{ave} = \dfrac{\int_1^9 \sqrt{x}\,dx}{9-1} = \dfrac{\int_1^9 x^{\frac12}\,dx}{8} = \dfrac{\frac{\cancel{2}}{3}x^{\frac32}\Big|_{x=1}^{x=9}}{\cancel{8}_4} =$

$= \dfrac{1}{12}\left(9^{\frac32} - 1^{\frac32}\right) = \dfrac{1}{12}(27-1) = \dfrac{26}{12} = \boxed{\dfrac{13}{6}}$

7. (24 pts) Compute each integral and simplify your answer when possible (6 pts each):

(a) $\displaystyle\int_0^2 |2x-3|\,dx$ Easiest with geometry



$=$ shaded area $= A_{\Delta_1} + A_{\Delta_2} = \dfrac{3\cdot\frac32}{2} + \dfrac{1\cdot\frac12}{2} = \dfrac94 + \dfrac14 = \dfrac{10}{4} = \boxed{\dfrac52}$

(b) $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x\Big|_{x=0}^{x=\frac12}$

$= \arcsin\frac12 - \arcsin 0$

$= \dfrac{\pi}{6} - 0 = \boxed{\dfrac{\pi}{6}}$

(c) $\displaystyle\int_0^{\pi/4} 4\sin(2x)(1+\cos(2x))^3\,dx =$

sub. $u = 1+\cos(2x)$ $\left(\begin{matrix} x=0 \to u=2 \\ x=\frac{\pi}{4} \to u=1 \end{matrix}\right)$

$du = -2\sin(2x)\,dx$

$\Rightarrow -\frac12 du = \sin(2x)\,dx$

$= \displaystyle\int_{u=2}^{u=1} 4u^3\cdot\left(-\tfrac12 du\right) = -2\int_{u=2}^{u=1} u^3\,du$

$= 2\displaystyle\int_{u=1}^{u=2} u^3\,du = \cancel{2}\frac{u^4}{\cancel{4}_2}\Big|_{u=1}^{u=2} =$

$= \dfrac12(2^4-1) = \boxed{\dfrac{15}{2}}$

(d) $\displaystyle\int_0^1 \frac{x}{1+3x^2}\,dx =$

sub $w = 1+3x^2$ $\left(\begin{matrix} x=0 \to w=1 \\ x=1 \to w=4 \end{matrix}\right)$

$dw = 6x\,dx$

$\frac16 dw = x\,dx$

$= \displaystyle\int_{w=1}^{w=4} \frac{\frac16 dw}{w} = \frac16 \ln w\Big|_{w=1}^{w=4}$

$= \frac16\ln 4 - \frac16\ln 1 = \boxed{\frac16 \ln 4}$

8. (12 pts) Choose ONE to prove. If possible, use sentences or formulas with complete justifications. The grading will be based on the clarity of your logic and explanations, as much as on any calculations involved.

(a) State and prove the geometric series theorem.

(b) State FTC, both parts. Prove the part of FTC about $\frac{d}{dx}(\int_a^x \ldots)$. You may use without proof MVT for integrals.

See notes or text