Name: Solution Key   **Panther ID:** ____________

Exam 3   Calculus II   Spring 2016

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) Circle the correct answer. No justification is necessary for this problem (3 pts each).

(a) The partial fraction decomposition for $\frac{2x+5}{x^4+4x^2}$ is of the form:

(i) $\frac{A}{x^2} + \frac{B}{x^2+4}$ (ii) $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$ (iii) $\frac{2x}{x^2} + \frac{5}{x^2+4}$

(iv) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4}$ (v) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$

(b) For the integral $\int \sqrt{4x^2-9}\,dx$, the following substitution is helpful:

(i) $x = 3\sin\theta$ (ii) $w = 4x^2-9$ (iii) $2x = 3\sec\theta$ (iv) $2x = 3\tan\theta$ (v) $w = (2x-3)^2$

(Don't spend time evaluating the integral. It is not required.)

(c) Let $f(x)$ be a positive, increasing, continuous function on $[a, b]$ and let R_4 be the right end point Riemann sum approximation with 4 subdivisions of the integral $\int_a^b f(x)\,dx$. Then compared with the integral, R_4 is an

(i) overestimate   (ii) underestimate   (iii) exact estimate   (iv) cannot tell (more should be known about f)

(d) Let $f(x)$ be a positive, concave down, continuous function on $[a, b]$ and let M_4 be the mid-point Riemann sum approximation with 4 subdivisions of the integral $\int_a^b f(x)\,dx$. Then compared with the integral, M_4 is an

(i) overestimate   (ii) underestimate   (iii) exact estimate   (iv) cannot tell (more should be known about f)

2. (10 pts) In each case answer True or False. No justification needed. (2 pts each)

(a) $\sum_{k=1}^{+\infty} \frac{1}{k} = 0$. **True** **False**

(b) If $0 < a_k < \frac{1}{k^2}$ for all $k \geq 1$, then $\sum_{k=1}^{\infty} a_k$ is convergent. **True** **False**

(c) If $\lim_{k \to +\infty} a_k = 0$ then $\sum_{k=1}^{\infty} a_k$ is convergent. **True** **False**

(d) If $\sum_{k=1}^{\infty} |a_k|$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is convergent. **True** **False**

(e) The series $2 - 1 - 1 + 2 - 1 - 1 + 2 - 1 - 1 + \ldots$ converges to 0. **True** **False**

3. (10 pts) Evaluate the improper integral or show it diverges $\int_0^{+\infty} e^{-2x}\, dx$

$$\int_0^{+\infty} e^{-2x}\,dx = \lim_{b \to +\infty} \left(\int_0^b e^{-2x}\,dx \right) = \lim_{b \to +\infty} \left(-\frac{1}{2} e^{-2x} \Big|_0^b \right)$$

$$= \lim_{b \to +\infty} \left(-\frac{1}{2} e^{-2b} + \frac{1}{2} e^{0} \right) =$$

$$= 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$$

so the improper integral converges to $\frac{1}{2}$.

4. (14 pts) Use a trigonometric substitution to evaluate $\int \frac{1}{(4-x^2)^{3/2}}\,dx = *$

<u>Sub.</u> $x = 2\sin\theta$ (3pts)

$$(4-x^2)^{\frac{3}{2}} = (4-4\sin^2\theta)^{\frac{3}{2}} = (4\cos^2\theta)^{\frac{3}{2}} = 8\cos^3\theta \quad (2pts)$$

(note also that $(4-x^2)^{\frac{1}{2}} = \sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$)

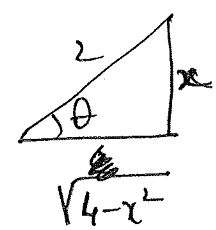
$dx = 2\cos\theta\,d\theta$ (1pt)

$$* = \int \frac{2\cos\theta\,d\theta}{8\cos^3\theta} = \overbrace{\frac{1}{4}\int \frac{1}{\cos^2\theta}\,d\theta}^{(2pts)} = \overbrace{\frac{1}{4}\int \sec^2\theta\,d\theta}^{(1pt)} =$$

$$= \overbrace{\frac{1}{4}\tan\theta + c}^{(2pts)} = \frac{1}{4}\frac{\sin\theta}{\cos\theta} + c = \boxed{\overbrace{\frac{1}{4}\frac{x}{\sqrt{4-x^2}} + c}^{(3pts)}}$$

This step can also be done with the triangle method

$\sin\theta = \frac{x}{2}$ so



$\Rightarrow \tan\theta = \frac{x}{\sqrt{4-x^2}}$

5. (24 pts) For each of the following series, determine if the series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Answer **and carefully** justify your answer. Very little credit will be given just for a guess. Most credit is given for the quality of the justification. (12 pts each)

(a) $\frac{1}{2} - \frac{3}{2^2} + \frac{5}{2^3} - \frac{7}{2^4} + \frac{9}{2^5} - \frac{11}{2^6} + \ldots \quad = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2k-1)}{2^k}$

Use Absolute Ratio test to check absolute convergence

$$\rho = \lim_{k\to+\infty} \frac{|b_{k+1}|}{|b_k|} = \lim_{k\to+\infty} \frac{2(k+1)-1}{\cancel{2^{k+1}}_{2}} \cdot \frac{\cancel{2^k}}{2k-1} = \lim_{k\to+\infty} \left(\frac{1}{2} \cdot \frac{2k+1}{2k-1}\right) = \frac{1}{2}$$

Since $\rho = \frac{1}{2} < 1$, the series is absolutely convergent $\underline{\underline{(AC)}}$.

(b) $\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^2+1}}$

$$\sum_{k=0}^{\infty} \left| \frac{(-1)^k}{\sqrt{k^2+1}} \right| = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k^2+1}} \quad \text{comparable with } \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2}} = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$$

harmonic series.

use limit comparison test

$$L = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{\frac{1}{\sqrt{k^2+1}}}{\frac{1}{k}} = \lim_{k\to+\infty} \frac{k}{\sqrt{k^2+1}} = 1$$

junk rule

Since $0 < L = 1 < \infty$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges $\Rightarrow \sum_{k=0}^{\infty} \frac{1}{\sqrt{k^2+1}}$ diverges, so the original series is <u>not A.C.</u>

But $\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^2+1}}$ converges by A.S.T. (as $\left\{\frac{1}{\sqrt{k^2+1}}\right\}$ is decreasing and $\lim_{k\to+\infty} \frac{1}{\sqrt{k^2+1}} = 0$)

Thus $\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^2+1}}$ is conditionally convergent (C.C.)

6. (14 pts) Use any method – definition or operations on familiar series – to find the Maclaurin series of the function $f(x) = \ln(1+2x)$. (Recall that the Maclaurin series is the same as the Taylor series at $x_0 = 0$.)

Easiest is to use the geometric series $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$,

but the definition and computation of $f^{(k)}(0)$ ~~can~~ also works.

So, with the first mentioned method, note

that $f'(x) = (\ln(1+2x))' = \frac{2}{1+2x} = 2\sum_{k=0}^{\infty}(-2x)^k$

$$2 \cdot \frac{1}{1-(-2x)}$$

Thus $f'(x) = 2 \cdot \sum_{k=0}^{\infty} (-1)^k \cdot 2^k \cdot x^k = \sum_{k=0}^{\infty} (-1)^k \cdot 2^{k+1} x^k$

Then

$$f(x) = \ln(1+2x) = \int f'(x)\,dx = \int \left(\sum_{k=0}^{\infty} (-1)^k \cdot 2^{k+1} x^k \right) dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot 2^{k+1} \int x^k\,dx = \sum_{k=0}^{\infty} (-1)^k \cdot 2^{k+1} \cdot \frac{x^{k+1}}{k+1} + c$$

Since $f(0) = \ln 1 = 0 \Rightarrow c = 0$

Thus, the Maclaurin series for $\ln(1+2x)$ is

$$\sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2^{k+1}}{k+1} x^{k+1} \quad \text{or} \quad \sum_{l=1}^{\infty} \frac{(-1)^{l-1} 2^l}{l} x^l$$

I did not ask for interval of convergence, but you can show that it would be $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

7. (14 pts) Find the interval of convergence (with endpoints) of the series $\sum_{k=1}^{\infty} \frac{1}{k \cdot 3^k} (x-1)^k$.

Apply Absolute Ratio Test

$$\rho = \lim_{k \to +\infty} \frac{|b_{k+1}|}{|b_k|} = \lim_{k \to +\infty} \left(\frac{|x-1|^{\cancel{k+1}}}{(k+1)\, 3^{\cancel{k+1}}} \cdot \frac{k \cdot \cancel{3^k}}{\cancel{|x-1|^k}} \right)$$

$$\rho = \lim_{k \to +\infty} \left(\frac{|x-1|}{3} \cdot \frac{k}{k+1} \right) = \frac{|x-1|}{3}$$

If $\rho = \frac{|x-1|}{3} < 1$ series is abs. convergent

If $\rho = \frac{|x-1|}{3} > 1$ series is divergent

$$\frac{|x-1|}{3} < 1 \iff |x-1| < 3 \iff -2 < x < 4$$

Test the end-points

$x=-2$ $\sum_{k=1}^{\infty} \frac{1}{k \cdot 3^k} (-2-1)^k = \sum_{k=1} \frac{(-1)^k \cdot \cancel{3^k}}{k \cdot \cancel{3^k}}$ alternating harmonic series, so convergent

$x=4$ $\sum_{k=1}^{\infty} \frac{1}{k \cdot 3^k} (4-1)^k = \sum_{k=1}^{\infty} \frac{1}{k}$ harmonic series, so divergent

Thus, the interval of convergence is

$$\boxed{I = [-2, 4)}$$

8. (12 pts) Choose ONE to prove:

(a) State and prove the k-th term divergence test for series. (You may ignore the inconclusive case.)

(b) State and prove the p-series test (using the integral test).

See notes or textbook