Worksheet - week 12 MAC-2312 - Spring 15 **Name:** Solution Key

1. Use the definition, to find the Taylor series at $x_0 = 1$ for the function $f(x) = \ln x$. Find the interval of convergence of the series that you obtained. It can be shown that the series converges to $\ln x$ for all the values of x in the interval of convergence. Accepting this, what do you obtain when $x = 2$?

Taylor series for $f(x)$ at x_0 is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad , \text{ where for us } f(x)=\ln x \text{ and } x_0=1$$

$f(x) = \ln x \qquad f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} = x^{-1}$

$f^{(2)}(x) = (-1)x^{-2}$

$f^{(3)}(x) = (-1)(-2)x^{-3}$

$f^{(4)}(x) = (-1)(-2)(-3)x^{-4} = (-1)^3 \cdot 3!\, x^{-4}$

$f^{(k)}(x) = (-1)^{k-1}(k-1)!\, x^{-k} \Rightarrow f^{(k)}(1) = (-1)^{k-1}(k-1)!$

So the Taylor series for $\ln x$ at $x_0=1$ is

$$\underbrace{\cancel{\ln} f(1)}_{0} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!}(x-1)^k = \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\cdot(x-1)^k}$$

For interval of convergence use absolute ratio test.

$$\rho = \lim_{k\to+\infty} \frac{1}{k+1}|x-1|^{k+1} \cdot \frac{k}{1} \cdot \frac{1}{|x-1|^k} = \lim_{k\to+\infty}\left(\frac{k}{k+1}\cdot|x-1|\right) = |x-1|$$

If $\rho = |x-1| < 1$ series absolutely convergent, so this happens when $0 < x < 2$.

Endpoints check! For $x=0$ series becomes $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\cdot(-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{2k-1}}{k} = -\sum_{k=1}^{\infty}\frac{1}{k} = -\infty$ diverges

For $x=2$, $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ which converges by A.S.T.

Thus $\boxed{I = (0,2]}$ and radius of convergence is $R=1$.

Accepting that $\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$ for all $x \in (0,2]$, plugging in $x=2$ in both sides, we get $\boxed{\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}}$ so the alt. harmonic series converges to $\ln 2$.