Name: Solution Key Panther ID: ____________

Exam 1 Calculus II Spring 2017

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) The first four terms of a sequence are

$$a_1 = \frac{2}{3},\ a_2 = -\frac{4}{9},\ a_3 = \frac{8}{27},\ a_4 = -\frac{16}{81},\ \ldots$$

(a) (4 pts) Assuming that the pattern continues, find the formula for the general term a_n.

$$a_n = (-1)^{n+1}\left(\frac{2}{3}\right)^n$$

(b) (4 pts) Is the sequence a_n eventually monotone? Justify your answer.

No, the sequence is oscillating, so it is not eventually monotone

(c) (4 pts) Is the sequence a_n convergent? Justify your answer.

Yes, the sequence converges to 0 (even though it is oscillating)

Note that $\lim\limits_{n\to+\infty} |a_n| = \lim\limits_{n\to+\infty} \left(\frac{2}{3}\right)^n = 0$ since $\frac{2}{3} < 1$

so $\lim\limits_{n\to+\infty} (-1)^{n+1}\left(\frac{2}{3}\right)^n = 0$ by a result we proved in class

You could also invoke the squeeze thm $-\left(\frac{2}{3}\right)^n \leq (-1)^{n+1}\left(\frac{2}{3}\right)^n \leq \left(\frac{2}{3}\right)^n$

$n\to+\infty$ $n\to+\infty$ $n\to+\infty$

0

2. (10 pts) For each of the statement below decide if it is True or False. No justification necessary. (2 pts each)

(a) If a sequence is increasing and bounded from above then it must be convergent. **True** **False**

(b) Any convergent sequence must be bounded. **True** **False**

(c) $2 + 4 + 6 + \dots + (2n-2) + (2n) = n(n+1)$. **True** **False**

(d) The series $\sum_{k=1}^{+\infty} \frac{1}{k}$ is convergent to 0. **True** **False**

(e) The function $f(x) = \frac{1}{\sqrt{x}}$ is integrable on the interval $[0,1]$. **True** **False**

3. (8 pts) A particle moves along the x-axis with a velocity of $v(t) = 60\sin(\pi t)$. Assume the time t is measured in hours and the velocity is measured in miles per hour. Find the total distance traveled by the particle over the time interval $0 \le t \le 1$ hours.

Total dist. traveled $= \int_{t=0}^{t=1} |v(t)|\, dt = \int_0^1 |60\sin(\pi t)|\, dt$

But when $t \in [0,1] \Rightarrow \pi t \in [0,\pi]$, so 1st & 2nd quadrant

so $\sin(\pi t) \geq 0$ for all $t \in [0,1]$

Thus, ~~that~~ we can drop the absolute value.

Tot. dist. traveled $= \int_0^1 60\sin(\pi t)\, dt =$

$$= -60 \cdot \frac{1}{\pi}\cos(\pi t)\Big|_{t=0}^{t=1}$$

$$= -\frac{60}{\pi}\cos\pi + \frac{60}{\pi}\cos 0$$

$$= -\frac{60}{\pi}\cdot(-1) + \frac{60}{\pi}\cdot 1 = \boxed{\frac{120}{\pi} \text{ miles}}$$

4. (21 pts) Evaluate each of the following or show it diverges:

(a) $\lim_{n\to+\infty} (-1)^n \frac{2n^3}{n^3+1}$

If $a_n = (-1)^n \frac{2n^3}{n^3+1}$ note that $|a_n| = \frac{2n^3}{n^3+1}$

and $\lim_{n\to+\infty} |a_n| = \lim_{n\to+\infty} \frac{2n^3}{n^3+1} = 2$ (by jumh rule or l'H)

However, our sequence $\{a_n\}_n$ <u>diverges</u>

since $\lim_{k\to+\infty} a_{2k} = 2$ and $\lim_{k\to+\infty} a_{2k+1} = -2$

Thus, the required limit <u>does not exist</u>!

(b) $\sum_{k=2}^{+\infty} \frac{2^{3k}}{3^{2k}}$

$$= \sum_{k=2}^{+\infty} \frac{(2^3)^k}{(3^2)^k} = \sum_{k=2}^{\infty} \left(\frac{8}{9}\right)^k$$

geometric series with $-1 < r = \frac{8}{9} < 1$ so the series is convergent

$$= \left(\frac{8}{9}\right)^2 \sum_{k=2}^{+\infty} \left(\frac{8}{9}\right)^{k-2} \underset{l=k-2}{=} \left(\frac{8}{9}\right)^2 \cdot \sum_{l=0}^{\infty} \left(\frac{8}{9}\right)^l = \left(\frac{8}{9}\right)^2 \cdot \frac{1}{1-\frac{8}{9}} =$$

$$= \frac{64}{81} \cdot \frac{9}{1} = \boxed{\frac{64}{9}}$$

The series converges to $\frac{64}{9}$.

(c) $\sum_{k=1}^{+\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right)$

It is a telescopic series

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+2}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \dots + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$$

$$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

So $$\sum_{k=1}^{+\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right) = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \boxed{\frac{3}{2}}$$

5. (8 pts) In each case, circle the correct answer:

(a) The expression

$$\frac{d}{dx}\left(\int_0^{x^2} e^{t^2}\,dt\right) \text{ is equivalent to}$$

(i) e^{x^2} (ii) $x^2e^{t^2}$ (iii) $2xe^{x^4}$ (iv) e^{x^4} (v) $e^{x^4}-1$

(b) A water-tank, initially full, starts being drained at the moment $t = 0$. Suppose that $r(t)$ (in gals/min) is the rate of the water flow out of the reservoir at the moment t. The equality $\int_0^{30} r(t)dt = 1000$ says that:

(i) The reservoir initially contains 1000 gallons of water.

(ii) After the first half an hour there are 1000 gallons of water left in the tank.

(iii) In the first half an hour 1000 of gallons of water were drained from the tank.

(iv) At 30 minutes, the water is flowing out of the tank at the rate of 1000 gals/min.

6. (12 pts) (a) (8 pts) Find the area under the curve $y = \frac{1}{x^2+1}$ over the interval $[0, 1]$.

$$A = \int_0^1 \frac{1}{x^2+1}\,dx = \arctan x \Big|_{x=0}^{x=1} = \arctan 1 - \arctan 0$$

$$\boxed{\frac{\pi}{4}}$$

(b) (4 pts) John claims that the area under the curve $y = \frac{1}{x^2+1}$ over the interval $[0, 1]$ is the same as the average value of the function $y = \frac{1}{x^2+1}$ on the interval $[0, 1]$. Is he right or wrong? Briefly justify.

He is right, because the interval has length 1

$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$ so if $b-a=1$

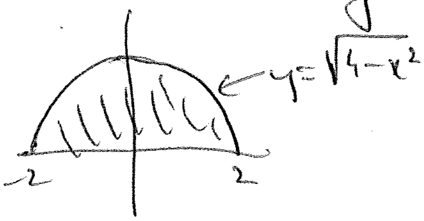
$f_{ave} = \frac{1}{1}\int_a^b f(x)dx$ = area under the curve

(we also use that $f(x) \geq 0$, so integral is exactly the area under the curve)

7. (28 pts) Compute each integral and simplify your answer when possible (7 pts each):

(a) $\int_{-2}^{2} \sqrt{4-x^2}\, dx = \frac{1}{2}\pi \cdot 2^2 = \boxed{2\pi}$

use geometry



(b) $\int_{1}^{e} \frac{2+x}{x^2}\, dx = \int_1^e \left(\frac{2}{x^2} + \frac{1}{x}\right) dx$

$= \left(-2x^{-1} + \ln x\right)\Big|_{x=1}^{x=e}$

$= -\frac{2}{e} + \ln e - \left(-\frac{2}{1} + \ln 1\right)$

$= -\frac{2}{e} + 1 + 2 = \boxed{3 - \frac{2}{e}}$

(c) $\int_{0}^{2} x^2\sqrt{1+x^3}\, dx =$

sub $w = 1+x^3$

$dw = 3x^2 dx$

$\frac{1}{3} dw = x^2 dx$

$= \int_{w=1}^{w=9} \sqrt{w}\, \frac{1}{3} dw = \frac{1}{3}\int_1^9 w^{\frac{1}{2}} dw$

$= \frac{1}{3}\cdot\frac{2}{3} w^{\frac{3}{2}}\Big|_{w=1}^{w=9} = \frac{2}{9}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$

$= \frac{2}{9}(27-1) = \boxed{\frac{52}{9}}$

(d) $\int_{0}^{\pi/4} \frac{\cos(2x)}{3+\sin(2x)}\, dx =$

sub $u = 3 + \sin(2x)$

$du = 2\cos(2x) dx$

$\frac{1}{2} du = \cos(2x) dx$

$x = 0 \Rightarrow u = 3 + \sin(0) = 3$

$x = \frac{\pi}{4} \Rightarrow u = 3 + \sin\left(2\cdot\frac{\pi}{4}\right) = 4$

$= \int_{u=3}^{u=4} \frac{\frac{1}{2} du}{u} = \frac{1}{2} \ln|u|\Big|_{u=3}^{u=4}$

$= \frac{1}{2}\ln 4 - \frac{1}{2}\ln 3 = \boxed{\frac{1}{2}\ln\left(\frac{4}{3}\right)}$

8. (12 pts) Choose ONE to prove. If possible, use sentences or formulas with complete justifications. The grading will be based on the clarity of your logic and explanations, as much as on any calculations involved.

(a) State and prove the geometric series theorem.

(b) State and prove the part of FTC about $\frac{d}{dx}(\int_a^x \ldots)$. You may use without proof MVT for integrals.

See notes or text