(c) ϕ^5

(d) ϕ^6

(e) What do you notice?

21. Find the twelfth decimal in the decimal expansion of $\sqrt{3}$.

22. The continued fraction expansion of $\sqrt{2}$ has period one, while the expansion of $\sqrt{3}$ has period two. Experimenting with continued fraction expansions of square roots, answer the following. Give reasons for your answers if you can.

(a) For which n does $\sqrt{n}$ have a continued fraction expansion of period one?

(b) For which n does $\sqrt{n}$ have a continued fraction expansion of period two?

4.4 Searching for Transcendental Numbers

We have studied irrational numbers from three different perspectives: algebra, geometry, and analysis. We have distinguished different types of irrationals: quadratic numbers, constructible numbers, polygon numbers, arithmetic numbers, and algebraic numbers. We have met just about all types of known irrational numbers. But we haven't yet found a home for the two most famous irrational numbers of all, π and e. We are in good company. They were not proved to be irrational until the mid-1700s; this was accomplished by John Lambert (1728–1777). And they were not proved to be nonalgebraic until the late 1800s; Charles Hermite (1822–1901) proved in 1872 that e was not algebraic, Ferdinand Lindemann (1852–1939) proved in 1882 that π was not algebraic. In fact, it was not until 1844 that a nonalgebraic irrational number was known at all. It was constructed by the French mathematician J. Liouville (1809–1882). The proof that a new kind of number existed is fascinating; we shall return to it later. In the twentieth century the search for these new numbers has not produced a great many of them. In 1900, the German mathematician David Hilbert (1862–1943) presented his 23 famous problems at the International Congress in Paris, and problem 7 posed the question of whether the number $2^{\sqrt{2}}$ was algebraic or not. It was shown to be not algebraic in 1934 by the Russian mathematician Aleksander Gelfond (1906–1968). But today, more than 60 years later, many numbers are only suspected of being nonalgebraic. Here we shall carry out our own search for these rare, yet plentiful numbers. They are called transcendental numbers.

Definition 4.4.1 *A number is a* **transcendental number** *if it is not algebraic.*

Notation: We denote the set of real transcendental numbers by $\mathbb{T}$.

That such numbers exist, and exist in great numbers, was proved by the German mathematician Georg Cantor (1845–1918). His proof involves counting members of infinite sets. This involves taking a brief detour into set theory. We shall examine the sizes of different infinite sets. We know, for example, that the set of natural numbers is infinite, basically because there is no largest natural number. We shall say that any set is countable if we can attach a unique natural number to every element; that is, if we can count every member. This makes sense. What may be difficult, however, is the reasoning behind some of the proofs. Since we do not intend to give a background in set theory, reading of the proofs will be slow going for those who have not had experience with functions and one-to-one correspondences, but the concepts are not that difficult. Let us recall a few basic definitions. A function is a **one-to-one correspondence** if it sends distinct elements to distinct images; that is, if $f(a) = f(b)$ then $a = b$. A function maps A **into** B if the set $\{f(x) : x \in A\} \subseteq B$. A function maps A **onto** B if the set $\{f(x) : x \in A\} = B$.

Definition 4.4.2 *A set S is* **countable** *if there is a one-to-one function from S into $\mathbb{N}$, the set of natural numbers.*

Example 4.4.3

(a) The even positive numbers are countable. Let f be the counting function that assigns even positive numbers to natural numbers as follows:

$$2 \to 1, \quad 4 \to 2, \quad 6 \to 3, \ldots, 2n \to n.$$

So $f(2n) = n$.

(b) The integers are countable. Let I be the counting function that assigns to each integer a natural number as follows:

$$0 \to 1, \ 1 \to 2, \ -1 \to 3, \ 2 \to 4, \ -2 \to 5, \ldots.$$

So $I(0) = 1$ and for $n > 0, I(n) = 2n$ and $I(-n) = 2n + 1$. We will refer to this particular counting function I again.

(c) Let $S = \{x : 0 < x < 1 \text{ and } x \text{ is rational}\}$. This set of rationals between 0 and 1 is countable. Let R be the counting function that assigns to each rational in S a natural number as follows:

$$1/2 \to 1,\ 1/3 \to 2,\ 2/3 \to 3,\ 1/4 \to 4,\ 3/4 \to 6, \ldots.$$

In general, $R(a/b) = a + (b-1)(b-2)/2$ where a/b is reduced to lowest terms. We shall refer to this function R again. Notice that R maps the set of rationals into, rather than onto, the set of natural numbers. That is because R does not assign a value to a/b if the fraction is not in lowest terms. So, for example, the number 5 is not the image of any fraction. Under the formula its pre-image would be 2/4, but this fraction is not in lowest terms. ■

The following theorem is useful to us. It employs some ingenious counting methods; in particular, it relies on the fundamental theorem of arithmetic and the fact that there are infinitely many primes.

Theorem 4.4.4 *If $S_1, S_2, \ldots S_n, \ldots$ is a countable collection of countable sets, then $\bigcup S_i$ is countable.*

Proof Here is how we shall assign the counting numbers. We assume that each of the sets, S_i, has a counting function (call it f_i) that counts the specific set, S_i. Let g count the members of $\bigcup S_i$ as follows: If $x \in \bigcup S_i$, then let $j = \min\{i : x \in S_i\}$. Let $g(x) = p_j^{f_j(x)}$. Here p_j represents the jth prime. Now this function g is one-to-one because, by definition, x is picked from a particular set location so j is unique and f_j is one-to-one. □

Theorem 4.4.5 *The set of rational numbers is countable.*

Proof This follows from Theorem 4.4.4 because the rational numbers are a countable set of countable sets. Adopting the temporary notation $(n, n+1)$ to indicate the set of rational numbers between n and $n+1$ and recalling that $\mathbb{Z}$ represents the set of integers, we may write

$$\mathbb{Q} = \mathbb{Z} \cup (0,1) \cup (-1,0) \cup (1,2) \cup (-2,-1) \cup \ldots.$$

Example 4.4.3 (b) shows this is a countable collection of sets, Example 4.4.3 (c) shows each of the sets is countable. □

Example 4.4.6

(a) Let us see what a counting function will look like that can count the rationals. It will be made up of the functions I and R from Example 4.4.3. Let $x = -3/5$. And let the following sets be

$$\mathbb{Z} = S_1, \quad (0,1) = S_2, \quad (-1,0) = S_3, \ldots.$$

We find that $x \in (-1,0) = S_3$. In fact, $x = -1 + 2/5$. Now the counting function R assigns 8 to 2/5 because $8 = 2 + (3)(4)/2$. So $g(x) = 5^8$. We use 5 because it is the third prime.

(b) Let us find what number is assigned to $x = 64/17$. Now $x = 3 + 13/17$ so $x \in (3,4) = S_8$. Also, $R(13/17) = 13 + (15)(16)/2 = 133$. And 19 is the eighth prime number. So $g(x) = 19^{133}$. ■

We now set about counting the algebraic numbers. Recall that an algebraic number is a solution to a polynomial equation with integer coefficients.

Definition 4.4.7 *The number x is an algebraic number of* **degree** *n if x is a solution to a polynomial of degree n with integer coefficients and n is the smallest degree polynomial for which this is true.*

Theorem 4.4.8 *The set of algebraic numbers of degree n is countable.*

Proof Let S be the set of algebraic numbers of degree n. We describe a function, g, that counts the members of S. Let s be an algebraic number of degree n. So s is a solution to the equation $a_0 + a_1x + \cdots + a_nx^n = 0$. Recalling the function I: $I(0) = 1$ and for $n > 0, I(n) = 2n$ and $I(-n) = 2n + 1$, let

$$g(s) = 2^{I(a_0)} \times 3^{I(a_1)} \times 5^{I(a_2)} \times \cdots \times {p_{n+1}}^{I(a_n)} \times p_{n+i+1}$$

where s is the ith of the n possible real solutions (recall the fundamental theorem of algebra) numbered from smallest to largest and p_n is the nth prime number. If s is a multiple root we can count it multiple times, as we did when counting fractions in Example 4.4.3 (c). Also recall that I is the function that counts the integers as defined in 4.4.3 (b). □

Example 4.4.9

Consider the middle, or second, real solution, call it s, to the polynomial equation $x^5+0x^4+0x^3+0x^2-5x+1=0$. There are three real solutions to this quintic. Incidentally, they are all nonarithmetic numbers, as Theorem 4.1.16 tells us. Using the scheme used in the proof of Theorem 4.4.8, let's find out what number will be assigned to s. Note that $I(1)=2$, $I(0)=1$, and $I(-5)=11$. Notice also that 2 is the first prime, 3 is the second, and 13 is the sixth. We shall call our solution s the second solution. Thus

$$g(s)=2^2\times 3^{11}\times 5^1\times 7^1\times 11^1\times 13^2\times 19.$$

If we chose the smallest, or first, real root, the counting number would be

$$2^2\times 3^{11}\times 5^1\times 7^1\times 11^1\times 13^2\times 17.$$ ■

Theorem 4.4.10 *There are a countable number of algebraic numbers.*

Proof Since there are a countable number of algebraic numbers of degree n and a countable number of degrees, this follows from Theorem 4.4.4. □

Now comes the surprise: We cannot count the real numbers; there are too many. Since we can count the algebraic numbers, the rest of the real numbers, the transcendentals, must not be countable. That means there are lots more transcendental numbers than algebraic numbers.

Theorem 4.4.11 (Cantor) *The real numbers cannot be counted.*

Proof Consider S, the set $\{x\colon 0<x\leq 1$, where x is a real number$\}$. Suppose we have found a counting function f from S to N. We shall use the following notation: If $x\in S$, then $f(x)=n$, for some natural number n, and we shall represent x by the decimal expansion

$$x=0.a_{n,1}a_{n,2}\dots a_{n,k}\dots.$$

We shall assume that a number ending in all 9s will be designated by its equivalent that ends in all 0s. We say this for uniqueness of representation. Now we construct a real number as follows

$$r=0.b_1b_2\dots b_k\dots,$$

where $b_k=0$, if $a_{k,k}\neq 0$, $b_k=1$, if $a_{k,k}=0$.

Notice that r cannot be identical with any of the real numbers we have counted because it differs at the kth place. So our assumption that the counting function counted all real numbers was wrong. Thus there is no such function; the reals are uncountable. □

Corollary 4.4.12 *The transcendental numbers cannot be counted.*

Now that we know that there are oodles of transcendental numbers, the search is on to find them. We have examined solutions of polynomial equations, numbers that can be constructed from straightedge and compass, polygon numbers, and limits of continued fraction expansions. We have not found a single transcendental number among them and, had we not been clued in that π and e are transcendental, we could not point to a single example. Of course, we can generate a few transcendentals from the ones we already know as this theorem shows.

Theorem 4.4.13 *If x is a transcendental number and y is algebraic, then $x+y$, $x-y$, xy, y/x, x/y, x^n, and $\sqrt[n]{x}$ are all transcendental numbers.*

Proof We proceed by assuming the contrary and deriving a contradiction. Here is how it works for the sum of two numbers. Let x be a transcendental number and y an algebraic number. Then suppose that z is algebraic, where $z=x+y$. It follows that $x=z-y$ and this implies x is algebraic; a contradiction. So z must be transcendental. (Now this is an easy proof compared to what has come before.) □

So we know that arithmetic combinations π with algebraic numbers and e with algebraic numbers will yield transcendental numbers. We should note that Theorem 4.4.13 implies that $\sqrt{\pi}$ is transcendental. This fact shows that it is impossible, with straightedge and compass, to square the circle. That is, it is impossible to build a square with the same area as the area of a circle. This was one of the famous unsolved problems that the Greek mathematicians posed.

Before we unleash a whole host of transcendental numbers on you, numbers that have been proved transcendental only recently (in this century), let us study the first transcendental that was constructed. A couple of the proofs are tough going and are included for the sake of completeness, but the ideas behind them are in the spirit of our examination of continued fractions, best possible approximants, and the rates of convergence of rationals to reals. The number we study was found by the French mathematician Joseph Liouville (1809–1882); it is

$$1/10^{1!} + 1/10^{2!} + 1/10^{3!} + \cdots + 1/10^{n!} + \cdots$$
$$= 0.110001000000000000000000100\ldots0001000\ldots.$$

The 5th 1 is positioned in the 120th place.

Corollary 3.3.15 tells us that the nth convergent p_n/q_n of the real number R is closer to R than $1/q_n^2$. Theorem 4.3.24 tells us that, for square roots, $\sqrt{k}$, we can improve on this and find convergents that are closer to $\sqrt{k}$ than $1/2q_n^2$. In fact, while the proof is a bit beyond us, the truth of the matter is that given any real number R, there is a rational number p/q closer to R than $1/\sqrt{5}q^2$. Furthermore, the constant $\sqrt{5}$ cannot be improved upon because of our old friend, the golden mean, ϕ. The following theorem says what we mean.

Theorem 4.4.14

1. *Given any real number R, there is a rational number p/q such that $|R - p/q| < 1/(\sqrt{5}q^2)$.*
2. *Let $k > \sqrt{5}$. Given any positive integer q, $|\phi - p/q| > 1/kq^2$ for all rational numbers p/q.*

Part (2) of the theorem tells us that the speed of convergence to ϕ by fractions is necessarily restrained. This type of restraint holds for all algebraic numbers. As an example, we show that for $\sqrt{2}$ we may state the constraint like this.

Theorem 4.4.15 *Given any positive integer q, $|\sqrt{2} - p/q| > 1/3q^2$, for all rational numbers p/q.*

Proof For $q = 1$ the theorem holds right away because p may be 1 or 2 and, in either case, $|\sqrt{2} - p/q| > 1/3$. Now if $q > 1$, then let us assume, for the sake of being perverse, that $|p/q - \sqrt{2}| \le 1/3q^2$. Thus $p/q < \sqrt{2} + (1/3q^2)$ for some q. Because we know that $\sqrt{2} < 10/7$ and since $q > 1$, we have $p/q < 10/7 + 1/12$. So we have $p/q + \sqrt{2} < 10/7 + 10/7 + 1/12 < 3$. Now

$$|p^2/q^2 - 2| = (p/q - \sqrt{2})(p/q + \sqrt{2})| < 1/3q^2 \times 3 = 1/q^2.$$

Therefore, $|p^2 - 2q^2| < 1$. Since p and q are natural numbers, it follows that $p^2 - 2q^2 = 0$ and so $p/q = \sqrt{2}$. But this cannot be because $\sqrt{2}$ is irrational. So $|\sqrt{2} - p/q| > 1/3q^2$. □

Here is how the rate of convergence may be governed for general algebraic numbers.

Theorem 4.4.16 (Liouville) *Let z be an algebraic number of degree $n > 1$ and let $r_m = p_m/q_m$ be a sequence of rational numbers converging to z. Then, for a sufficiently large M, $|z - p_m/q_m| > 1/q_m^{n+1}$ for all $q_m > M$.*

Proof Suppose that z is a solution to the polynomial equation $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0$. Then

$$f(r_m)/(r_m - z) = (f(r_m) - f(z))/(r_m - z) =$$
$$a_n(r_m^{n-1} + r_m^{n-2}z + \cdots + r_m z^{n-2} + z^{n-1}) +$$
$$a_{n-1}(r_m^{n-2} + r_m^{n-3}z + \cdots + r_m z^{n-3} + z^{n-2}) + \cdots +$$
$$a_3(r_m^2 + r_m z + z^2) + a_2(r_m + z) + a_1.$$

Letting m be such that $|z - r_m| < 1$, we may say that, for sufficiently large m,

$$f(r_m)/(r_m - z) < n|a_n|(|z| + 1|)^{n-1} + (n-1)|a_{n-1}|(|z| + 1|)^{n-2} + \cdots$$
$$+3|a_3|(|z| + 1|)^2 + 2|a_2|(|z| + 1|) + |a_1| = M.$$

Let $q_m > M$. Then $|z - r_m| > |f(r_m)|/M > |f(r_m)|/q_m$.

Now

$$|f(r_m)| = |(a_n p_m^n + a_{n-1}p_m^{n-1}q_m + \cdots + a_1 p_m q_m^{n-1} + a_0 q_m^n)/q_m^n|.$$

Note that r_m cannot be a solution to $f(x) = 0$ because if it were we could factor out $(x - r_m)$ and so z would necessarily be of lesser degree. Hence $f(r_m) \neq 0$. Furthermore, the numerator of this fraction is an integer so it must be at least 1. We conclude that $|z - r_m| > (1/q_m)(1/q_m^n) = 1/q_m^{n+1}$. □

Using his theorem, Loiuville constructed a transcendental number. Notice that its decimal expansion is characterized by rapidly increasing stretches of zeros of length $m!$.

Theorem 4.4.17 *The number $z = 1/10^{1!} + 1/10^{2!} + \cdots + 1/10^{n!} + \cdots$ is transcendental.*

Proof Let $r_m = p_m/q_m = 1/10^{1!} + 1/10^{2!} + \cdots + 1/10^{m!} = p_m/10^{m!}$. Then $|z - r_m| < (10)(1/10^{(m+1)!})$. Now if z is an algebraic number of degree n, then Liouville's theorem says that $|z - r_m| > 1/10^{(n+1)m!}$ for sufficiently large m. So

$$1/10^{(n+1)m!} < (10)(1/10^{(m+1)!}) = 1/10^{(m+1)!-1}.$$

But this is false for $m > n$, so z is transcendental. □

Now that we have seen that the rate of convergence of the continued fraction to an algebraic number is restrained, it makes sense to explore the entries in the continued fraction expansion. For quadratic numbers, the entries are periodic. It has been theorized, but not proved, that the entries are bounded for all algebraic numbers. Certainly bounded entries are consistent with a restrained rate of convergence. We shall note that the continued fraction expansion for some known transcendental numbers, in particular those based on the number e, are unbounded. Further we recall from Section 3.3 the dramatic changes in the entries of the expansion of π; the number 292 is the 5th entry in the continued fraction expansion. We shall leave this notion of restrained versus erratic behavior of continued fraction expansions for a project (Project 5.21).

Let us now open the flood gates of the transcendental dam. As mentioned in the introduction to Section 4.4, Hilbert's seventh problem was solved by Gelfond. But not only did Gelfond show that the number $2^{\sqrt{2}}$ is transcendental, he proved that a whole class of numbers like $2^{\sqrt{2}}$ is transcendental.

Theorem 4.4.18 (Gelfond) *The number z^y is transcendental if z is algebraic (not 0 or 1) and y is irrational and algebraic (it may be complex).*

Example 4.4.19

(a) Consider $10^{1/2}$. This number is irrational because if $10^{1/2} = p/q$, then $q^2 = 10p^2$. This is an impossibility because $10p^2$ has, in its prime factorization, an odd number of 2s and 5s while q^2 cannot. Of course, we know this number to be root constructible; after all, it is $\sqrt{10}$.

(b) Consider $10^{\sqrt{2}}$. This number is transcendental by Theorem 4.4.18.

(c) Consider $10^{\log 2}$. This number is 2, by definition. ■

Theorem 4.4.20 *If x is a natural number that is not a power of 10, then $\log_{10} x$ is transcendental.*

Proof Let $y = \log_{10} x$. Suppose that y is not transcendental. Since x is not a power of 10, y cannot be rational. We leave the proof of this as an exercise. Thus y is irrational and algebraic. It follows from Theorem 4.4.18 that 10^y is transcendental. But $x = 10^y$ and x is a natural number. This contradiction tells us that $y = \log_{10} x$ is transcendental. □

So we have uncovered a whole new line of transcendental numbers; $\log_{10} x$ for natural numbers x that are not powers of 10. So, for example, $\log 2 = 0.301029995664\ldots$, and $\log 3 = 0.47712125472\ldots$ are transcendental.

But the theorem that opens up the treasure chest of transcendentals is as follows.

Theorem 4.4.21 *If $z \neq 0$ is an algebraic (it may be complex) number, then e^z is transcendental.*

As in the case of Gelfond's theorem, the proof is well beyond this book; it can be found in advanced books on number theory.

Let us recall some facts about the functions e^z, $\cos x$ and $\sin x$ that we have picked up in a calculus course. We let z represent a complex number and x represent a real number.

(i) $e^z = 1 + z + z^2/2! + z^3/3! + \cdots + z^n/n! + \cdots$

(ii) $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots + (-1)^n x^{2n}/(2n)! + \cdots$

(iii) $\sin x = x - x^3/3! + x^5/5! - x7/7! + \cdots + (-1)^n x^{2n+1}/(2n+1)! + \cdots$

(iv) $e^{ix} = \cos x + i \sin x$

Theorem 4.4.22

1. *Let x be an algebraic number. If $x \neq 0$, then $\cos x$ is a transcendental number. If $x \neq 1$, then $\cos^{-1}(x)$ is transcendental.*

2. *If $x \neq 0$, then e^x transcendental. If $x \neq 1$, then $\ln(x)$ is transcendental.*

Proof If $\cos x$ were algebraic so, too, would $i \sin x$ be algebraic. Then their sum would be algebraic. But we know that $e^{ix} = \cos x + i \sin x$ and since x is an algebraic number and $x \neq 0$, Theorem 4.4.21 tells us that e^{ix} is transcendental. This contradiction establishes that $\cos x$ is transcendental. If $\cos^{-1}(x) = y$ were algebraic and $x \neq 1$ then $y \neq 0$ and $\cos y = x$ is transcendental. But x was assumed to be algebraic, so this contradiction proves that $\cos^{-1}(x)$ is transcendental if $x \neq 1$. We leave the rest of the proof as an exercise. □

Theorem 4.4.23 *e^{π} is transcendental.*

Proof We know that $e^{i\pi} = \cos \pi + i \sin \pi = -1$, so $e^{-\pi} = (-1)^i = i^{2i}$ and $e^{\pi} = i^{-2i}$. Since i is algebraic it follows from Theorem 4.4.18 that i^{-2i} is transcendental. □

Example 4.4.24

Here are examples of transcendental numbers. We can display most of these with a hand calculator.

(a) $\cos(\sqrt{2}) = 0.155943694765\ldots$

(b) $\cos^{-1}(4/5) = 0.643501108793\ldots$

(c) $e^{\pi} = 23.1406926328\ldots$

(d) $\ln(\sqrt[3]{2}) = 0.231049060187\ldots$

(e) $i^i = 0.207879576350761908546955\ldots$. ■

We have been flirting with complex numbers in the past two theorems and part (e) of the last example. In elementary algebra we have learned about adding, subtracting, multiplying, and dividing complex numbers, but raising complex numbers to complex powers belongs in an advanced course. This would be a good time to make plans to take such a course. Making sense of numbers such as i^i is actually a complex task—no pun intended.

It looks as though we can prove that most anything is transcendental. But there are many elementary numbers that, unbelievably, are not understood at all. Not only have the following numbers not been proved to be transcendental, it is not even known whether they are irrational. If you can believe this, they might be rational numbers. Here are some examples.

Example 4.4.25

It is not known whether the following numbers are rational, arithmetic, algebraic, or transcendental.

$$\pi + e, \quad \pi \times e, \quad \pi^e, \quad 2^{\pi}, \quad 2^e, \quad \pi^{\pi}, \quad e^e$$ ■

Since e^z and the trigonometric functions of sine and cosine offer a power series representation, we can home in on a transcendental number with as great an accuracy as we wish. We need not be limited to the decimal expansion on our caculator.

Example 4.4.26

We know that $\cos(1)$ is transcendental. Now

$$\cos x = 1 - x^2/2! + x^4/4! + \cdots + (-1)^n x^{2n}/(2n)! + \cdots.$$

The calculator tells us that $\cos(1) = 0.540302305868\ldots$. The series tells us that

$$\cos(1) = 1 - 1/2! + 1/4! + \cdots + (-1)^n/(2n)! + \cdots.$$

If we want the series for $\cos(1)$ to 20-place accuracy, we need only go out 12 places because $1/22! = 8.9 \times 10^{-22}$. This is feasible. ■

While we have listed lots of exotic transcendental numbers, none of them has a decimal pattern that can be remembered. Here is one that does. The decimal built from the counting numbers is transcendental.

$$0.12345678910111213141516171819202122232425\ldots$$

We finish up our brief look at transcendentals by revisiting π and e one last time. The number e is the less well known of the two. It is not an everyday number like π is. It is known as the base of the natural logarithms, and it was born less than 300 years ago. As we have seen in Theorem 4.4.21, the series e^z is invaluable to us in our search for transcendentals. And the exponential function e^x is known to all calculus students and all students of science who study exponential growth. We have mentioned that e has an unbounded continued fraction expansion. Incredibly, its expansion follows a pattern.

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$$

So we may approximate e with fractions to as close as we like. Also, there are patterned continued fraction expansions based on terms using e. Here are two examples; we leave it as an exercise to find more.

$$(e-1)/(e+1) = [0; 2, 6, 10, 14, 18, \ldots]$$

$$(e^2-1)/(e^2+1) = [0; 1, 3, 5, 7, 9, \ldots]$$

We may also approximate e to as great an accuracy as we like with its series expansion:

$$e = 1 + 1 + 1/2! + 1/3! + \cdots + 1/n! + \cdots.$$

Here is e to 21 places:

$$e = 2.718281828459045235360\ldots.$$

Unquestionably, π is the most famous number in all of mathematics. It is a most natural of numbers to consider—the ratio of the circumference to the diameter of a circle. Not surprisingly, it occurs in formulas for circular objects in geometry. We all know them.

$C = 2\pi r$; C stands for the circumference of a circle with radius r.
$A = \pi r^2$; A stands for the area of a circle with radius r.
$V = (4/3)\pi r^3$; V stands for the volume of a sphere with radius r.
$S = 4\pi r^2$; S stands for the surface area of a sphere with radius r.

But π occurs in all fields of mathematics—and in the most unexpected places. Here are some examples:

$$e^{i\pi} = -1.$$

This truly remarkable fact follows from $e^{iz} = \cos z + i \sin z$.

$$n! \approx (\sqrt{2\pi n})n^n e^{-n}$$

This is Stirling's formula, which was mentioned in the exercises of Section 1.3. It is a good approximation of $n!$ as n gets large. Notice that this formula relates three interesting numbers: $n!, e$, and π.

$$f(x) = e^{-x^2}/\sqrt{2\pi}$$

This is the definition for the normal distribution, that bell-shaped curve we see in statistical data.

$$F(n)/n \approx 6/\pi^2$$

Here $F(n)$ stands for the number of square-free numbers $\leq n$. This approximation becomes very good as n gets large. A square-free number is a number made up of primes raised to the first power. For example, if $n = 10$, then the square-free numbers ≤ 10 are 2, 3, 5, 6, 7, and 10. There are six of them; and 6/10 is close to $6/\pi^2$.

As pervasive and fundamental a number as π is, it is nearly intractable from a numerical standpoint. As we have stated, it was not proved to be irrational until the mid-1700s, and it was not found to be transcendental until the late 1800s. But it has been recognized and studied for as long as mathematicians have lived. In the Old Testament, I Kings 7:23 implies that $\pi = 3$. The Babylonians around 2000 B.C. thought π to be either 3 or $3\frac{1}{8}$. Around 1500 B.C. in the *Rhind Papyrus*, $\pi = 256/81 \approx 3.16049$. Archimedes, around 200 B.C., approximated π using a 96-sided regular polygon. He found $3\frac{10}{71} < \pi < 3\frac{1}{7}$. Ptolemy, the great astronomer, about 400 years later approximated π with 377/120. This is correct to four places. In the third century A.D. the Chinese geometer Liu Hui approximated π using a 192-sided regular polygon. His estimate was 3.1416. This estimate was also recorded in the sixth century a.d.by the Hindu astonomer Aryabhata, in the *Aryabhitiya* Verse II 28: "Add 4 to 100, multiply by 8, and add 62000. The result is approximately the circumference of a circle of which the diameter is 20000." In the fifth century A.D. the Chinese mathematician Zu Chongzhi found the approximation of 355/113, which is correct to six places.

Let us begin with a method for approximating π that captures the spirit of both Archimedes and Liu Hui. This approximation involves inscribing regular $(k \times 2^n)$-gons in a unit circle. You can approximate either the area or the circumference of the circle using larger and larger n. We shall approximate the circumference of by finding the perimeter of a regular 2^n-gon. Figure 4.10 depicts a unit circle with center at O. The side of a polygon is depicted by PQ. The unit segment OR bisects the $\angle QOP$ and is perpendicular to PQ. The point S is the intersection of OR and PQ. Let x denote the length of PQ. Let h denote the length of OS. Let y denote the length of PR.

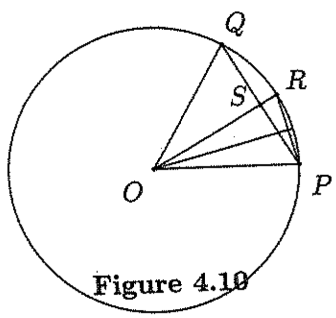

Figure 4.10

Lemma 4.4.27 *The length* $y = \sqrt{2-\sqrt{4-x^2}}$.

Proof The Pythagorean theorem gives us

$$h^2 + (x/2)^2 = 1; \quad y^2 = (1-(h)^2 + (x/2)^2.$$

It follows that

$$y^2 = (1-(h)^2 + 1 - h^2 = 2 - 2h = 2 - 2\sqrt{1-(x/2)^2} = 2 - \sqrt{4-x^2}.$$

So $y = \sqrt{2-\sqrt{4-x^2}}$. □

Theorem 4.4.28 *The perimeter of a regular* 2^n*-gon inscribed in a unit circle is*

$$2^n \times \sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$$

where there are $n-1$ *twos under the square root signs.*

Proof We proceed by induction on n for the following statement:

$\mathcal{P}(n)$: The length of the side of a 2^n-gon inscribed in a unit circle is

$$\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}.$$

$\mathcal{P}(2)$ is true because the side of a square inscribed in a unit circle is of length $\sqrt{2}$.

Suppose $\mathcal{P}(n)$ is true. So $\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$ is the length of the side of a regular 2^n-gon inscribed in a unit circle where there are $n-1$ twos in the expression. Now consider $\mathcal{P}(n+1)$: Lemma 4.4.29

says that the length of the side of a regular polygon with twice the number of sides is $\sqrt{2-\sqrt{4-x^2}}$, where

$$x = \sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \text{ with } n-1 \text{ twos.}$$

But this expression is $\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$ with n twos. In order to get the perimeter we simply multiply the length of the side by 2^n. □

This theorem shows what we suspected about the sequence from Example 4.3.2 (d): that $2^n \times \sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$ converges to π where this expression has n twos under the square roots.

Corollary 4.4.29 *The sequence*

$\{s_n\} = 2^n \times \sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$ *converges to* π *where the expression has* n *twos under the square roots.*

Proof This follows from Theorem 4.4.30, noting that the expression

$$2^{n+1} \times \sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$$

when divided by 2 gives the measure for the circumference of a semi-circle of radius 1, which is π. □

Today we can approximate π with an inexpensive calculator to several places. Many calculators show it as 3.14159265359. This is accurate to 11 places. In Section 3.3 we looked at its continued fraction expansion. It begins $[3; 7, 15, 1, 292, \ldots]$.

			3	7	15	1	292...
p_k	0	1	3	22	333	355	103993...
q_k	1	0	1	7	106	113	33102...

We see that

$$|355/113 - \pi| < 1/(113)(33102) \approx .000000267,$$

so this convergent is a very good approximation. Here is the continued fraction expansion a bit further:

$$[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, \ldots].$$

Unfortunately, the continued fraction expansion does not show a pattern. As we have mentioned, it does show a dramatic jump in size of entries with 292 in the fourth place. This is an early symptom of erratic behavior in the rate of convergence of the continued fraction to π. This, in turn, is an indication of what we already know; π is not algebraic.

With the aid of formulas, it is possible to calculate π to many places. Around 1600, it was calculated to 35 decimals and around 1700 it was up to 100 decimal places. When π was shown to be irrational in 1761, the search for more digits could no longer be driven by the search for a cycling of the digits. Its irrationality meant that this could not happen. But the lure of π to some mathematicians is inescapable, and more accuracy was calculated. In 1853, William Shanks calculated π to 707 places. It was pretty rough going past 1000 digits until the age of computers. For example, in 1949, π was known to 2037 places and it took 70 hours of calculation to arrive at this. In 1961, 100,000 places were found by computer in 9 hours. And computers have gotten much faster. In 1975, the millionth place was found. In 1989, the billionth digit was found.

Now that it is possible to find π to such enormous accuracy, it is possible to analyze trends in the occurrence of digits. Yet they appear to be perfectly random. In 1988 a statistical analysis of the first 29,360,000 digits of π was conducted. The most frequent digit was 4, which appeared 2,938,787 times, while the least frequent digits was 7, which occurred 2,934,083 times. While, in a random sequence, we would expect that each digit would occur about 2,936,000 times, this variation is not at all unreasonable. With 29,360,000 random digits the chances that there would be a string of nine straight instances of the same number is 29.36%. Indeed there is one such string: Nine consecutive 7s occur. As we continue to probe deeper into the infinity of the expansion, we could argue that there will be strings of hundreds of the same digit. Indeed, we could argue that any sequence you would ever want would eventually show up—just as we might argue that, given a typing monkey and a word processor and an infinite amount of time, the monkey would eventually type the

Bible word for word (thus indicating that $\pi = 3$). It might take a while, though.

Here is a listing of the digits up to the first 0, which, surprisingly, does not occur until the thirty-second decimal place:

$$\pi = 3.14159265358979323846264338327950\ldots.$$

It's only right that this famous number, π, that has no pattern to its decimal expansion and no pattern to its continued fraction expansion can be built up through infinite additions and infinite multiplications with some of the most beautiful and intriguing patterns in all of mathematics. We conclude the book with some of these magnificent formulas.

The first formula for π was found by Francois Viete (1540–1603) the father of modern algebra. It is one of many strange and fascinating equalities.

1. $2/\pi = \sqrt{1/2} \times \sqrt{1/2 + 1/2\sqrt{1/2}} \times \sqrt{1/2 + 1/2\sqrt{1/2 + 1/2\sqrt{1/2}}} \times \cdots$

In 1699, π was calculated to 71 decimal places using the formula

2. $\pi = 2\sqrt{3}(1 - 1/(3\times 3) + 1/(3^2\times 5) - 1/(3^3\times 7) + 1/(3^4\times 9) - \cdots)$.

With the invention of calculus in the 1600s, several formulas were invented. They were all approximations of infinite processes, such as infinite series or infinite products. Here are some of the more beautiful formulas involving expressions with π.

3. $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \cdots$
4. $\pi\sqrt{2}/4 = 1 + 1/3 - 1/5 - 1/7 + 1/9 + 1/11 - 1/13 - 1/15 + \cdots$
5. $\pi^2/6 = 1 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots$
6. $\pi^2/8 = 1 + 1/3^2 + 1/5^2 + 1/7^2 + \cdots$
7. $\pi^2/12 = 1 - 1/2^2 + 1/3^2 - 1/4^2 + \cdots$
8. $(\pi - 3)/4 = 1/(2\times 3\times 4) - 1/(4\times 5\times 6) - 1/(6\times 7\times 8) - \cdots$
9. $\pi^2/6 = 2^2/(2^2-1)\times 3^2/(3^2-1)\times 5^2/(5^2-1)\times\cdots\times p^2/(p^2-1)\times\cdots$, where p is a prime.

EXERCISES

1. Show that the following sets, S, are countable by displaying a 1-1 function from S into $\mathbb{N}$.
   (a) S is the set of odd numbers (both negative and positive).
   (b) S is the set of integer lattice points.
   (c) S is the set of rational lattice points.

2. Using the functions R from Example 4.4.3 and g from Theorem 4.4.4, find $g(x)$ for the following fractions:
   (a) $x = 3/7$
   (b) $x = -11/16$
   (c) $x = 105/13$
   (d) Is there a natural number y for which there is no x such that $g(x) = y$? Explain.

3. Using functions I and g from Theorem 4.4.8, find $g(x)$ for the following algebraic numbers.
   (a) The two solutions of $x^2 + 7x + 4 = 0$
   (b) The smallest real solution for $x^5 + 8x^4 - 3x^2 + 13 = 0$
   (c) The largest solution to $x^3 - 3x^2 + x + 2 = 0$
   (d) The third smallest real solution for $x^7 - 3x^6 - 5x^5 + 3x^3 - 19x - 6 = 0$
   (e) Find natural numbers y such that there is no x for which $g(x) = y$.

4. Given that $0.1234567891011\ldots$ is transcendental, what can you say about
   (a) $17.181920212223\ldots$?
   (b) any number that begins with a natural number n and, following the decimal point, has a decimal expansion consisting of the string of successive digits of the natural numbers that follow n? Give a reason for your answer.

5. Let $y = \log_{10} x$. Show that if x is not a power of 10, then y cannot be rational.

6. Prove that if m and n are natural numbers, then $\sqrt{m}^{\sqrt{n}}$ is transcendental.

7. Show that if α is an acute angle of a Pythagorean triangle, then $\cos\alpha$ is transcendental.

8. Complete the proof of Theorem 4.4.22.
   (a) If $x \neq 0$ is an algebraic number, then e^x is transcendental.
   (b) If $x \neq 1$ is an algebraic number, then $\ln(x)$ is transcendental.

9. Using Theorem 4.4.23, prove that π is transcendental.

10. Show that the following numbers are transcendental:
    (a) $\sqrt{i}^{i}$
    (b) $i^{\sqrt{i}}$
    (c) $e^{\sqrt{\pi}}$

11. Find the ratio of the number of square-free numbers $\leq n$ to n, where n is
    (a) 100
    (b) 1000
    (c) How close to $6/\pi^2$ is the ratio becoming?

12. Look for other patterned continued fraction expansions for terms made from e; for example, $\sqrt[n]{e}$.

13. See how accurate the following terms are to π.
    (a) $99^2/(2206\sqrt{2})$
    (b) $(63/25)(17 + 15\sqrt{5})/(7 + 15\sqrt{5})$
    (c) $\sqrt[4]{9^2 + 19^2/22}$