Topology Homework #2 - Solution Key

Pb. 1: (a) $X \setminus \text{int}(A)$ is a closed set (since $\text{int}(A)$ is open)

and $X \setminus \text{int}(A) \supseteq X \setminus A$ (since $\text{int}(A) \subseteq A$)

Thus $X \setminus \text{int}(A) \supseteq \overline{X \setminus A}$ as $\overline{X \setminus A}$ is the smallest closed set containing $X \setminus A$.

$X \setminus (\overline{X \setminus A})$ is an open set (since $\overline{X \setminus A}$ is closed)

$X \setminus (\overline{X \setminus A}) \subseteq A$, since $X \setminus A \subseteq \overline{X \setminus A}$ so $X \setminus (\overline{X \setminus A}) \subseteq X \setminus (X \setminus A) = A$

Thus $X \setminus (\overline{X \setminus A}) \subseteq \text{int}(A)$, as $\text{int}(A)$ is the largest open set contained in A.

so $X \setminus \text{int}(A) \subseteq \overline{X \setminus A}$.

We proved $\overline{X - A} = X \setminus \text{int}(A)$.

(b) Replace A by $X \setminus A$ in (a). We get

$\overline{X \setminus (X \setminus A)} = X \setminus \text{int}(X \setminus A)$, so $\overline{A} = X \setminus \text{int}(X \setminus A)$

so $X \setminus \overline{A} = \text{int}(X \setminus A)$

Certainly, other perfectly good solutions are possible for this problem.

Pb. 2. By contradiction, suppose there exist

$U \neq \emptyset$ open, $U \subseteq \overline{A}$.

Then $U \cap A \neq \emptyset$ (since U is a nghd. of points from $\overline{A}$).

Fix a point $a \in U \cap A$.

Since A is relatively discrete, $\exists\, V$ open s.t. $A \cap V = \{a\}$.

Without loss of generality, we can assume $V \subseteq U \subseteq \overline{A}$, as otherwise we replace V by $U \cap V$ (note that $U \cap V \neq \emptyset$, as $a \in U \cap V$)

Since X contains no isolated points $V \neq \{a\}$, so fix another point $b \in V \subseteq \overline{A}$, $b \neq a$.

Since X is Hausdorff, $\exists$ an open set W ~~which we can assume~~ ~~$W \subseteq V$~~ so that $b \in W \subseteq V \subseteq \overline{A}$ and $a \notin W$.

Since $b \in W$ open and $b \in \overline{A}$ $\Rightarrow$ there is a point $\tilde{a} \in W \cap A \neq \emptyset$

Note that $\tilde{a} \neq a$ since $a \notin W$

But $W \subseteq V$ so it follows that $\tilde{a} \in A \cap V$

This contradicts $A \cap V = \{a\}$

Note 1: You don't really use that (X,d) is a metric space but just that X is Hausdorff (which every metric space is)
In fact, I think you only need X to be T_1-space

Note 2: Some of you had a good argument, but assuming that A is closed. It is not true that a relatively discrete set is closed. Example: Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}^*\} \subseteq \mathbb{R}$

Then A is relatively discrete in $\mathbb{R}$, but is not closed as $0 \in \overline{A}$ but $0 \notin A$

Pb. 3. Parts (a) and (b) are just applications of the definitions and DeMorgan's laws.

For (a), if $A = \bigcap_{n=1}^{\infty} U_n$, with U_n open $\forall n \in \mathbb{N}^*$,

then $X \setminus A = X \setminus \bigcap_{n=1}^{\infty} U_n \overset{\text{De Morgan}}{=} \bigcup_{n=1}^{\infty} (X \setminus U_n)$

But $X \setminus U_n$ is closed for any $n \in \mathbb{N}^*$, thus

$X \setminus A$ is a countable union of closed sets, so $X \setminus A$ is an F_σ set.

Part (b) is completely similar.

(c) Assume A is an F_σ-set. Then A can be written as

$A = \bigcup_{n=1}^{\infty} F_n$ with F_n closed, $\forall n \in \mathbb{N}^*$.

Let $C_1 = F_1$, $C_2 = F_1 \cup F_2$, ..., $C_k = \bigcup_{n=1}^{k} F_n$, ...

All sets C_k are closed, as finite unions of closed sets, and clearly $C_1 \subseteq C_2 \subseteq \dots \subseteq C_k \subseteq C_{k+1} \subseteq \dots$

It is also obvious that $\bigcup_{k=1}^{\infty} C_k = \bigcup_{n=1}^{\infty} F_n = A$

(d) Given a set A, we proved that $f : X \to \mathbb{R}$
$f(x) = d(x, A)$ is a continuous function.

Define, for $n \in \mathbb{N}^*$, $U_n = \{ \underset{\in X}{x} \mid d(x,A) < \frac{1}{n} \}$

As $U_n = f^{-1}\left((-\infty, \tfrac{1}{n})\right)$, U_n is open, since f is continuous

Moreover $\bigcap_{n=1}^{\infty} U_n = \{ x \mid \forall n \ d(x,A) < \frac{1}{n} \} = \{ x \mid d(x,A) = 0 \}$

But we proved in class that $\{ x \mid d(x,A) = 0 \} = \overline{A}$

So $A = \overline{A}$ by assumption, we proved $A = \bigcap U_n$ so A is an F_σ set.

Pb 4. a) Fix $x \in X$. We'll show that $X \setminus \{x\}$ is open.

Let $y \in X \setminus \{x\}$. By the Hausdorff assumption,
$\exists$ U, V open so that $y \in V$, $x \in U$ and $U \cap V = \emptyset$.
In particular $y \in V$ and $x \notin V$, thus $y \in \underset{\text{open}}{V} \subseteq \mathbb{R} \setminus \{x\}$.

This proves that $\mathbb{R} \setminus \{x\}$ is open, so $\{x\} = \mathbb{R} \setminus (\mathbb{R} \setminus \{x\})$ is closed.

Note: You don't really need the full "strength" of the Hausdorff assumption, but just the fact that for every two points x, y there exist open sets U, V so that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.
This is the (weaker than Hausdorff) T_1-separation axiom. You can show that the T_1-separation axiom is actually equivalent to the requirement that every singleton $\{x\}$ is a closed set.

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b) If U, V are open sets in the finite complement topology on $\mathbb{R}$, using De Morgan,
$$\mathbb{R} \setminus (U \cap V) = (\mathbb{R} \setminus U) \cup (\mathbb{R} \setminus V) \text{ is a finite set,}$$
as both $(\mathbb{R} \setminus U)$ and $(\mathbb{R} \setminus V)$ are finite.
As $\mathbb{R}$ is infinite, it follows that $U \cap V$ must be infinite, in particular, $U \cap V$ must be non-empty.
Thus, we proved that the intersection of any two open sets in the finite complement topology on $\mathbb{R}$ is non-empty.
This space is therefore not Hausdorff

Pb. 4 (b) - continuation

As closed sets in the finite complement topology are all finite sets, it is obvious that $\{x\}$ is closed.

Continuing the previous note,

$(\mathbb{R}, \mathcal{T}_{\text{finite compl.}})$ is an example of a T_1-space which is not a T_2-space (i.e. not Hausdorff)

Pb. 4 (c). Let $\{x_n\}_n$ so that $x_n \neq x_m$ $\forall n \neq m$ and let $y \in \mathbb{R}$.

Let U open set in $\mathcal{T}_{\text{fin.compl.}}$ with $y \in U$.

Then $\mathbb{R} \setminus U$ is a finite set.

Since $f : \mathbb{N} \to \mathbb{R}$ $f(n) = x_n$ is a one-to-one function (by the assumption $x_n \neq x_m$ if $n \neq m$)

and since $\mathbb{R} - U$ is finite, it follows that

$f^{-1}(\mathbb{R} \setminus U) = \{n \mid x_n = f(n) \in \mathbb{R} \setminus U\}$ is also a finite set

Let $N = \max \{n \mid x_n \in \mathbb{R} \setminus U\}$

Then $\forall n \geq N+1$ we have $x_n \in U$. Since U was arbitrary open nghd. of y, this shows that $\{x_n\}_n \to y$. Since y was arbitrary it follows that $\{x_n\}_n \to y$, $\forall y \in \mathbb{R}$.

Without the assumption $x_n \neq x_m$, $\forall n \neq m$, the statement is no longer true.

Let $\{x_n\}_n = \{c\}_n$ be a constant sequence.

Then, we claim $\{x_n\}_n \to c$, but $\{x_n\}_n \not\to y$ if $y \neq c$

Pb 4(c) continuation

if U open, $c \in U$, obviously $\underset{c}{\overset{\shortparallel}{x_n}} \in U$, $\forall n$

so $\{\underset{c}{\overset{\shortparallel}{x_n}}\}_n \to c$

but if $y \neq c$, let $U = \mathbb{R} \setminus \{c\}$. Then $y \in U$

but $x_n \notin U$ $\forall n \in \mathbb{N}$

so $\{\underset{c}{\overset{\shortparallel}{x_n}}\}_n \not\to y \neq c$.

4(d) If $(X, \mathcal{T}_{trivial} = \{\emptyset, X\})$

then a sequence $\{x_n\}_n$ in X converges to any limit $y \in X$.

Any neighborhood of y is X itself, so certainly $x_n \in X$ $\forall n \in \mathbb{N}$.