Name: ______________________________ **PantherID:** ______________________________

Homework 4 - Topology – Fall 2015 **Due Tuesday, Oct. 27, 2015**

1. (10 pts) Exercise 11, page 107 textbook.

2. (10 pts) Exercise 16, page 107 textbook.

3. (10 pts) The subset of $\mathbb{R}^\omega$ defined by

$$H = \prod_{n\geq 1} [0, 1/n]$$

is called the Hilbert cube.
(a) Show that H is closed in $\mathbb{R}^\omega$ with respect to both the box and the product topologies.

(b) Find $int(H)$ if $\mathbb{R}^\omega$ has the box topology and find $int(H)$ if $\mathbb{R}^\omega$ has the product topology.

4. (10 points bonus) Let ℓ^2 be the subset of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x} = \{x_n\}_n$ such that $\sum x_n^2$ converges. Then the formula

$$d(\mathbf{x}, \mathbf{y}) = \Big(\sum_{n=1}^{\infty} (x_n - y_n)^2 \Big)^{1/2}$$

defines a metric on ℓ^2, called the ℓ^2-metric. The space ℓ^2 can be endowed with four different topologies: three subspace topologies inherited from $\mathbb{R}^\omega$ – the box, the uniform and the product topologies – and one given by the ℓ^2-metric.

(a) Show that on ℓ^2 the following strict inclusions between the four topologies hold:

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology} \supset \text{product topology} \quad .$$

(b) Note that the Hilbert cube H from Problem 3 is a subspace of ℓ^2. Show that on H the following relations between the four induced subspace topologies hold:

$$\text{box topology} \supset \ell^2\text{-topology} = \text{uniform topology} = \text{product topology} \quad .$$

Comment: The last problem contained a serious mistake in the first version. Since it is a bit tedious to give all details for part (a), I will consider Problem 4 as optional, but you'll receive bonus points if you can do it.