Name: ____________________ **PantherID:** ____________________

Additional review problems for Midterm – Topology – Fall 2015

1. (a) Suppose $f : X \to Y$ is a homeomorphism of topological spaces and suppose x_0 is an isolated point of X. Show that $f(x_0)$ is an isolated point of Y.

(b) Is the statement in (a) still true if "homeomorphism" is replaced by just "continuous map"?

2. (a) Let X, Y topological spaces and consider the product topology on $X \times Y$. Prove that the natural projections $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are open maps.

Note: We proved in class that π_X, π_Y are continuous and, moreover, that the product topology is the coarsest topology on $X \times Y$ so that the projections are continuous. The above exercise gives a different property of the projections. Be sure to give a complete proof.

(b) Does the statement of part (a) remain true in the case of an infinite product
(b1) with the box topology ?
(b2) with the product topology?

3. For any function $f : X \to Y$, define the graph of f, $\Gamma_f \subseteq X \times Y$, by

$$\Gamma_f := \{(x, y) \in X \times Y \mid y = f(x)\} .$$

(a) Prove that the projection π_X restricted to Γ_f is a bijection between Γ_f and X.

(b) Prove that f is continuous if and only if $\pi_X|_{\Gamma_f} : \Gamma_f \to X$ is homeomorphism.

(c) Prove that if Y is Hausdorff and f is continuous then Γ_f is a closed subset of $X \times Y$.

4. Let $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ be maps between topological spaces and let $f = f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ be the cartesian product of these two maps, defined by

$$f((x_1, x_2)) := (f_1(x_1), f_2(x_2)),\ \forall x_1 \in X_1, x_2 \in X_2.$$

Show that f is continuous if and only if f_1 and f_2 are continuous.

5. Show that $X \times Y$ and $Y \times X$ are homeomorphic.

6. Show that a complete metric space with no isolated points must be uncountable. *Hint:* Use Baire's Theorem.

7. (a) Show that the addition map $+ = add : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $add(x, y) := x + y$, is continuous.

(b) Show that the division map $\div = div : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$, $div(x, y) \to \frac{x}{y}$, is continuous.

8. Let A be the subset of $\mathbb{R}^\omega$ consisting of sequences that are "eventually zero", that is all $\{x_n\}_{n=1}^{\infty}$, so that $x_n \neq 0$ for only finitely many values of n. Find the closure of A in $\mathbb{R}^\omega$ if:
(a) $\mathbb{R}^\omega$ has the box topology;
(b) $\mathbb{R}^\omega$ has the product topology.

9. Let $A = (0, 1]$ be the subset of $\mathbb{R}$ consisting of all real numbers which are strictly greater than 0 and less than or equal to 1.

(a) Find the closure of A, interior of A and boundary of A in $\mathbb{R}$ with the usual topology.

(b) Find the closure of A, interior of A and boundary of A in $\mathbb{R}$ with the lower limit topology.

10. (a) Prove that the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases},$$

is discontinuous at every point.

(b) Prove that $g(x) = xf(x)$, where $f(x)$ is the function in part (a), is continuous only at $x = 0$, but is discontinuous at every other point.

(c) Give an example of a function $h : \mathbb{R} \to \mathbb{R}$ which is continuous at $x = 0$ and at $x = 1$, but is discontinuous at all other points.

(d) Is there an example of a real function (from $\mathbb{R}$ to $\mathbb{R}$) which is continuous at all integers, but is discontinuous everywhere else?