Solution Homework 5: MAA 4211 Spring 2002

5, page 64. (i) We'll show that $\lim_{x\to a} h(x) = L$ using the sequential characterization of limits (Theorem 3.6).

Let $x_n \in I \setminus \{a\}$ be an arbitrary sequence such that $x_n \to a$. Because $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$, from Theorem 3.6 we conclude that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = L$. From the hypothesis, we also deduce that $f(x_n) \leq h(x_n) \leq g(x_n)$. Thus, applying the squeeze theorem for sequences (Theorem 2.9, part (i)), it follows that $h(x_n) \to L$ as $n \to \infty$. Applying again Theorem 3.6, it follows that $\lim_{x\to a} h(x) = L$. □

(ii) We'll again use the sequential characterization of limits. Let $x_n \in I \setminus \{a\}$ be an arbitrary sequence such that $x_n \to a$. Because $\lim_{x\to a} f(x) = 0$, it follows that $f(x_n) \to 0$ as $n \to \infty$. Since $|g(x)| \leq M$ for all $x \in I \setminus \{a\}$, we get in particular that $|g(x_n)| \leq M$ for all $n \in \mathbf{N}$, thus the sequence $g(x_n)$ is bounded. From the squeeze theorem for sequences (Theorem 2.9, part (ii)), it follows that $f(x_n)g(x_n) \to 0$. Hence from Theorem 3.6, we get $\lim_{x\to a} f(x)g(x) = 0$. □

8, page 70.

($\Rightarrow$) First assume that $\lim_{x\to\infty} f(x) = L$ and let $x_n \in (a, \infty)$ such that $x_n \to \infty$ as $n \to \infty$. We want to show that $f(x_n) \to L$.

Let $\epsilon > 0$. Since $\lim_{x\to\infty} f(x) = L$, there exists $M \in \mathbf{R}$ such that for any $x > M$, $|f(x) - L| < \epsilon$. Because $x_n \to \infty$, there exists a rank $N_0 \in \mathbf{N}$ such that $\forall n \geq N_0$, $x_n > M$. Combining these, it follows that $\forall n \geq N_0$, $|f(x_n) - L| < \epsilon$. Thus $f(x_n) \to L$.

($\Leftarrow$) Conversely, now assume that $f(x_n) \to L$ for any sequence $x_n \in (a, \infty)$ such that $x_n \to \infty$, and we'd like to prove that $\lim_{x\to\infty} f(x) = L$. We do this using a proof by contradiction. Suppose that $f(x)$ does not converge to L as x approaches a. Then there exists an $\epsilon_0 > 0$ such that for any $M \in \mathbf{R}$, there exists a real number x_M (which depends on M) such that $x_M > M$ and $|f(x_M) - L| \geq \epsilon_0$. In particular, for each natural number $n > a$ (taken as M in the above), we'll find a real number x_n, such that $x_n > n$ and $|f(x_n) - L| \geq \epsilon_0$. The inequality $x_n > n$ for any $n > a$ implies that the sequence $x_n \to \infty$, thus by assumption $f(x_n) \to L$ as $n \to \infty$. Thus $|f(x_n) - L| < \epsilon_0$, for n sufficiently large. But this is in contradiction with $|f(x_n) - L| \geq \epsilon_0$, for any $n \in \mathbf{N}$, $n > a$. □