Name: Solution Key **Panther ID:** ____________

Exam 1 Calculus II Spring 2015

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (10 pts) (a) Is the sequence $\{n^2 - 10n\}_{n=1}^{+\infty}$ eventually monotone? Justify your answer. Yes

Let $a_n = n^2 - 10n$.

$a_{n+1} - a_n = (n+1)^2 - 10(n+1) - (n^2 - 10n) = \cancel{n^2} + 2n + 1 \cancel{-10n} - 10 \cancel{-n^2} \cancel{+10n}$

So $a_{n+1} - a_n = 2n - 9$

Clearly $a_{n+1} - a_n > 0$ for $n \geq 5$

so $\{a_n\}_n$ is eventually strictly increasing

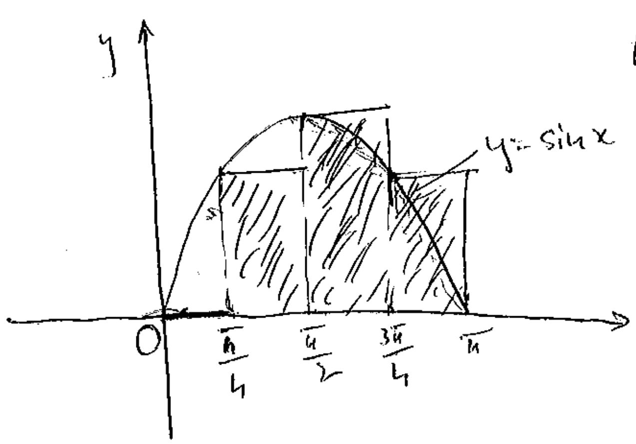
(b) Is the sequence $\{n^2 - 10n\}_{n=1}^{+\infty}$ convergent? Justify your answer. No!

$\{a_n\}_n$ is not convergent since

$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} (n^2 - 10n) = \lim_{n\to+\infty} n(n-10) = +\infty$

In other words, $\{a_n\}_n$ is not bounded from above.

2. (10 pts) On a graph of $y = \sin(x)$ shade in the area corresponding to L_4, the left-endpoint approximation with 4 subdivisions of $\int_0^\pi \sin x\, dx$. Then find the exact value of L_4. (OK if your answer contains π, or square-roots).



L_4 = shaded area

$L_4 = (\sin 0)\cdot\frac{\pi}{4} + \sin\left(\frac{\pi}{4}\right)\cdot\frac{\pi}{4} + \sin\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{4} + \sin\left(\frac{3\pi}{4}\right)\cdot\frac{\pi}{4}$

$L_4 = 0 + \frac{\sqrt{2}}{2}\cdot\frac{\pi}{4} + 1\cdot\frac{\pi}{4} + \frac{\sqrt{2}}{2}\cdot\frac{\pi}{4}$

$L_4 = \frac{\pi}{4}\left(2\cdot\frac{\sqrt{2}}{2} + 1\right) = \boxed{\frac{\pi}{4}\left(\sqrt{2} + 1\right)}$

3. (10 pts) In each case answer True or False. No justification necessary. (2 pts each)

(a) Sequences are functions. **True** **False**

(b) A bounded sequence must be convergent. **True** **False**

(c) For any $n \geq 1$, $(1+2+3+...+n)^2 = 1^3+2^3+3^3+...+n^3$. **True** **False**

(d) The average value of a **linear** function $f(x)$ on an interval $[a,b]$ is the same as $f(\frac{a+b}{2})$. **True** **False**

(e) The function $f(x) = \frac{1}{\sqrt{x}}$ is integrable on the interval $[0,1]$. **True** **False**

4. (16 pts) Evaluate each of the following series or show it diverges:

(a) $\ln\left(\frac{1}{3}\right) + \ln\left(\frac{3}{5}\right) + \ln\left(\frac{5}{7}\right) + \ln\left(\frac{7}{9}\right) + ...$ $= \sum_{k=1}^{\infty} \ln\left(\frac{2k-1}{2k+1}\right)$

Let $S_n = \sum_{k=1}^{n} \ln\left(\frac{2k-1}{2k+1}\right) \underset{\substack{\uparrow \\ \text{properties} \\ \text{of logs}}}{=} \sum_{k=1}^{n} \left[\ln(2k-1) - \ln(2k+1)\right] =$

$= (\ln 1 - \cancel{\ln 3}) + (\cancel{\ln 3} - \cancel{\ln 5}) + (\cancel{\ln 5} - \cancel{\ln 7}) + \dots + (\cancel{\ln(2n-1)} - \ln(2n+1))$

$S_n = \ln 1 - \ln(2n+1) = -\ln(2n+1)$. As $\lim_{n\to+\infty} S_n = -\infty$ it follows that this series diverges (to $-\infty$)

(b) $\sum_{k=2}^{+\infty} \frac{(-2)^k}{3^{k-1}} = \frac{(-2)^2}{3} + \frac{(-2)^3}{3^2} + \frac{(-2)^4}{3^3} + \dots =$

you should realize it's a geometric series

$= \frac{(-2)^2}{3} \cdot \left[1 + \frac{(-2)}{3} + \frac{(-2)^2}{3^2} + \dots\right] =$

$= \frac{(-2)^2}{3} \cdot \underbrace{\sum_{k=0}^{\infty} \left(\frac{-2}{3}\right)^k}_{\substack{\text{standard geometric series} \\ \text{with } r = -\frac{2}{3}}} = \frac{4}{3} \cdot \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{4}{\cancel{3}} \cdot \frac{\cancel{3}}{5}$

$\boxed{= \frac{4}{5}}$

as $|r| = \frac{2}{3} < 1$ series converges

$\boxed{\text{So } \sum_{k=2}^{\infty} \frac{(-2)^k}{3^{k-1}} \text{ converges to } \frac{4}{5}.}$

5. (28 pts) Compute each integral and simplify your answer when possible (7 pts each):

(a) $\int_1^2 \left(1+\frac{1}{x}\right) dx = \left(x + \ln x\right)\Big|_{x=1}^{x=2}$

$= (2+\ln 2) - (1+\ln 1)$

$= 1 + \ln 2$

(b) $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx = \left(\arcsin x\right)\Big|_{x=0}^{x=\frac{1}{2}}$

$= \arcsin\left(\frac{1}{2}\right) - \arcsin 0$

$= \frac{\pi}{6} - 0 = \boxed{\frac{\pi}{6}}$

(c) $\int_0^{\pi/4} \tan x \sec^2 x\, dx =$

sub. $w = \tan x$

$dw = \sec^2 x\, dx$

$= \int_{w=0}^{w=1} w\, dw = \frac{w^2}{2}\Big|_{w=0}^{w=1} = \boxed{\frac{1}{2}}$

(d) $\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\, dx = \int_{w=1}^{w=9} \frac{\frac{1}{3}dw}{\sqrt{w}} =$

Sub. $w = x^3+1$

$dw = 3x^2 dx$

$\frac{1}{3}dw = x^2 dx$

$= \frac{1}{3}\int_1^9 w^{-\frac{1}{2}} dw = \frac{1}{3}\cdot 2w^{\frac{1}{2}}\Big|_{w=1}^{w=9}$

$= \frac{2}{3}\left(\sqrt{9} - \sqrt{1}\right) = \boxed{\frac{4}{3}}$

6. (12 pts) A snail is moving on the x-axis so that its position (in meters) with respect to the origin is given by $s(t) = (t-1)(t-3)$, where t is the time in hours, $0 \le t \le 3$. $s(t) = t^2 - 4t + 3$

(a) Does the snail have a constant acceleration during the motion? Justify your answer.

Yes. $v(t) = s'(t) = 2t - 4$

$a(t) = s''(t) = 2\ \frac{m}{h^2}$ ← so acceleration is indeed constant for this motion

(b) Find the total distance traveled by the snail in the time interval $0 \le t \le 3$ hours.

With integral: Tot. dist. traveled $= \int_0^3 |2t-4|\, dt = \int_0^2 2|t-2|dt + \int_2^3 2|t\ldots$

$= -\int_0^2 2(t-2)\, dt + \int_2^3 2(t-2)\, dt =$ etc.

Without integral.

Since $v(t) = 2(t-2)$, we know that $v(t) < 0$ when $t \in [0,2)$ and $v(t) > 0$ when $t \in (2,3]$

So in the first two seconds the snail moves to the left, while from $t \in [2,3]$ it moves to the right

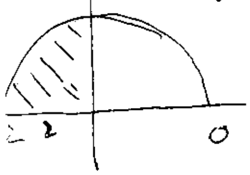
Total distance $= |s(2) - s(0)| + |s(3) - s(2)|$

$= |-1-3| + |0-(-1)| = 4 + 1 = 5\,m$

7. (12 pts) Given $F(x) = \int_{-2}^{x} \sqrt{4-t^2}\, dt$, compute each of the following and give a brief explanation:

(a) $F(0) = \int_{-2}^{0} \sqrt{4-t^2}\, dt$

Geometry $= \frac{1}{4}$ Area of disk $= \frac{1}{4} \cdot \pi \cdot 2^2 = \boxed{\pi}$



(b) $F'(0) = \boxed{\sqrt{4-0^2} = 2}$

$F'(x) = \frac{d}{dx}\left(\int_{-2}^{x} \sqrt{4-t^2}\, dt\right)$

$F'(x) = \sqrt{4-x^2}$

FTC part (ii)

(c) $F''(0) = \boxed{0}$

$F''(x) = \left(\sqrt{4-x^2}\right)'$

$= \left(\left(4-x^2\right)^{\frac{1}{2}}\right)' =$

$= \frac{1}{\cancel{2}}\left(4-x^2\right)^{-\frac{1}{2}} \cdot \left(-\cancel{2}x\right)$

$F''(x) = \frac{-x}{\sqrt{4-x^2}}$

8. (12 pts) Choose ONE to prove. If possible, use sentences or formulas with complete justifications. The grading will be based on the clarity of your logic and explanations, as much as on any calculations involved.

(a) Show that the harmonic series diverges.

(b) State FTC, both parts. Prove the part of FTC about $\int_a^b f(x)\, dx = F(b) - F(a)$. You may use without proof the other part of FTC.

See the text or your notes