Name: **Solution Key** Panther ID: _____

Exam 3   Calculus II   Spring 2015

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) In each case answer True or False. No justification needed. (2 pts each)

(a) In xy coordinates, the curve $r = 3\sin\theta$ is a circle that passes through the origin.   **True**

(b) If the series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to+\infty} a_k = 0$ **True**
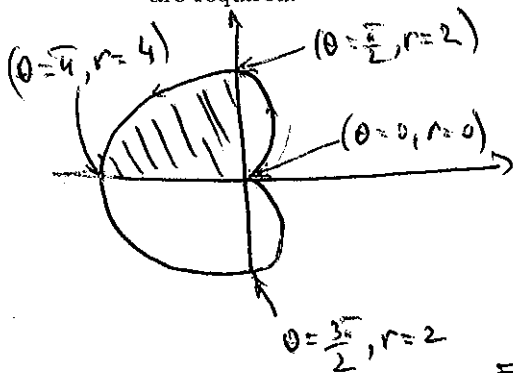
(c) If $0 < a_k < \frac{1}{k}$ for all $k \geq 1$, then $\sum_{k=1}^{\infty} a_k$ is convergent.   **False**

(d) If $\sum_{k=1}^{\infty} u_k$ is convergent, then $\sum_{k=1}^{\infty} |u_k|$ is convergent.   **False**

(e) $1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \ldots = 1$ **True**

(f) If $a_k > 0$, for all k, and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} (a_k)^2$ also converges.   **True**

2. (12 pts) Find the area enclosed by the cardioid $r = 2 - 2\cos\theta$ in the second quadrant. Sketch and computation are required.

$$A = \int_{\theta=\frac{\pi}{2}}^{\theta=\pi} \frac{1}{2} r^2 \, d\theta = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}\left(2(1-\cos\theta)\right)^2 d\theta =$$

$$= 2\int_{\frac{\pi}{2}}^{\pi} (1-\cos\theta)^2 \, d\theta = 2\int_{\frac{\pi}{2}}^{\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= 2\int_{\frac{\pi}{2}}^{\pi} \left(1 - 2\cos\theta + \frac{1+\cos(2\theta)}{2}\right) d\theta = 2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin(2\theta)}{4}\right]\Big|_{\frac{\pi}{2}}^{\pi}$$

$$\boxed{A = \frac{3\pi}{2} + 4}$$

3. (20 pts) Determine whether each of the following series converges or diverges. Full justification is required.

(a) $\sum\limits_{k=1}^{\infty} \frac{k^2}{2^k}$ Ratio Test

$P = \lim\limits_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim\limits_{k \to \infty} \frac{(k+1)^2}{2^{k+1}} \cdot \frac{2^k}{k^2}$

$P = \frac{1}{2} \lim\limits_{k \to \infty} \frac{(k+1)^2}{k^2} = \frac{1}{2} < 1$

The series converges by the ratio test $\S$

(b) $\sum\limits_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$ Integral test

$\int_{2}^{+\infty} \frac{1}{x\sqrt{\ln x}} dx = \int_{\ln 2}^{+\infty} \frac{1}{\sqrt{w}} dw$

$w = \ln x$
$dw = \frac{1}{x} dx$

$= 2\sqrt{w} \Big|_{\ln 2}^{+\infty} = +\infty$ integral diverges.

Since $f(x) = \frac{1}{x\sqrt{\ln x}}$ is decreasing,

the series $\sum\limits_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$ is divergent by the integral test

4. (20 pts) For each of the following series, determine if the series is divergent (D), conditionally convergent (CC), or absolutely convergent (AC). Answer **and carefully** justify your answer. Very little credit will be given just for a guess. Most credit is given for the quality of the justification. (10 pts each)

(a) $\frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + ... = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot k}{2k+1}$

Note that $\lim\limits_{k \to \infty} \frac{k}{2k+1} = \frac{1}{2} \neq 0$

So $\lim\limits_{k \to \infty} \frac{(-1)^{k-1} \cdot k}{2k+1}$ D.N.E.

Thus the series **diverges** by the kth-term test

(b) $\sum\limits_{k=0}^{\infty} (-1)^k \frac{k}{k^2+1}$

Test A.C. $\sum\limits_{k=0}^{\infty} \left| (-1)^k \frac{k}{k^2+1} \right| = \sum\limits_{k=0}^{\infty} \frac{k}{k^2+1}$

This is comparable with $\sum\limits_{k=1}^{\infty} \frac{1}{k}$

Use limit comparison test

$L = \lim\limits_{k \to \infty} \frac{\frac{k}{k^2+1}}{\frac{1}{k}} = \lim\limits_{k \to \infty} \frac{k^2}{k^2+1} = 1$

Since $0 < L < \infty$ and the harmonic series diverges, it follows that $\sum \frac{k}{k^2+1} = +\infty$

Test C.C. A.ST.

$\lim\limits_{k \to \infty} \frac{k}{k^2+1} = 0$ and $\frac{k}{k^2+1}$ is decrea

(show this)

$\sum\limits (-1)^k \frac{k}{} $ is C.C.

5. (14 pts) (a) (8 pts) Use the definition to find the Taylor series for $f(x) = \sin x$ at $x_0 = \pi/4$. It's OK not to use summation notation for your result, but write six, seven terms of the series to clearly show the pattern

(b) (6 pts) Use the Taylor polynomial of degree 2 of $f(x) = \sin x$ at $x_0 = \pi/4$ to get an approximation of $\sin(43°)$.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

$f(x) = \sin x$

$f'(x) = \cos x$

$f^{(2)}(x) = -\sin x$ ⎫ pattern will repeat

$f^{(3)}(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f^{(2)}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(3)}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

So the Taylor series for $f(x) = \sin x$ at $x_0 = \frac{\pi}{4}$ is

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{1}{1!}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2}\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2}\frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4 +$$

$$+ \frac{\sqrt{2}}{2}\frac{1}{5!}\left(x-\frac{\pi}{4}\right)^5 - \frac{\sqrt{2}}{2}\frac{1}{6!}\left(x-\frac{\pi}{4}\right)^6 - \dots$$

(b) $\sin x \approx \frac{\sqrt{2}}{2}\left(1 + \frac{1}{1!}\left(x-\frac{\pi}{4}\right) - \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2\right)$

$\sin(43°) = \sin\left(43\cdot\frac{\pi}{180}\right) \approx \frac{\sqrt{2}}{2}\left[1 + \left(\frac{43\pi}{180} - \frac{\pi}{4}\right) - \frac{1}{2}\left(\frac{43\pi}{180} - \frac{\pi}{4}\right)^2\right]$

6. (10 pts) Start from a familiar MacLaurin series and do some operations on it to obtain the MacLaurin series for $f(x) = \arctan x$. Hint: First think of $(\arctan x)'$.

$(\arctan x)' = \frac{1}{1+x^2}$

$\displaystyle\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ (*) (geometric series)

Substitute x by $(-x^2)$ in (*) to get

$\displaystyle\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k}$

Now integrate both sides

$\displaystyle\arctan x = \int\frac{1}{1+x^2}dx = \int\sum_{k=0}^{\infty}(-1)^k x^{2k}dx = \sum_{k=0}^{\infty}(-1)^k\int x^{2k}dx$

$\displaystyle\arctan x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1} + C$

since $\arctan(0) = 0$ it follows that $C = 0$

so $\displaystyle\arctan x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{\dots}$

7. (12 pts) Find the radius of convergence and the interval of convergence (with endpoints) of the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) \cdot 2^{2k+1}} (x-1)^{2k+1}$$

$$\rho = \lim_{k \to +\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k \to +\infty} \frac{|x-1|^{2(k+1)+1}}{(2(k+1)+1) \cdot 2^{2(k+1)+1}} \cdot \frac{(2k+1) \cdot 2^{2k+1}}{|x-1|^{2k+1}}$$

$$\rho = \lim_{k \to +\infty} \frac{2k+1}{2k+3} \cdot \frac{|x-1|^2}{4} = \frac{|x-1|^2}{4}$$

radius of convergence ← so $R = 2$

$$\rho = \frac{|x-1|^2}{4} < 1 \iff |x-1|^2 < 4 \iff |x-1| < 2$$

$$|x-1| < 2 \iff -2 < x-1 < 2 \iff -1 < x < 3$$

Test end-points

$x = -1$ $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) \cdot 2^{2k+1}} (-2)^{2k+1} = -\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$ converges by A.S.T.

$x = 3$ $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) 2^{2k+1}} \cdot 2^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$ also converges by A.S.T.

so $\boxed{I = [-1, 3]}$

8. (12 pts) Choose ONE to prove:

(a) State and prove the integral test.

(b) State and prove the k-th term divergence test.

(c) Prove or disprove the statement in Pb. 1 (f):

For (a) or (b) see notes or textbook

If $a_k > 0$, for all k and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} (a_k)^2$ also converges .

(c) is <u>True</u> : <u>Proof</u> : $\sum_{k=1}^{\infty} a_k$ converges $\Rightarrow \lim_{k \to \infty} a_k = 0$ (k^{th} term test)

Then $\lim_{k \to \infty} a_k = 0 \Rightarrow 0 < a_k \leq 1$ for $k \geq k_0$.

Thus $0 < a_k^2 \leq a_k$ for $k \geq k_0$ so

$$\sum a_k^2 \leq \sum a_k .$$

Thus $\sum^{\infty} a_k^2$ converges by simple comparison test.