NAME: Solution Key. Panther ID: ____________

Worksheet week 10 - MAC 2312, Spring 2015

1. You are the treasurer of the new island kingdom Polar Koordinatea where no calculators are allowed. The Queen summons you and gives you 24h to design a coin for the kingdom. After a sleepless night, you come up with two proposals:

(i) The coin is the circle $r = 1$, having inside the rose $r = \cos(3\theta)$ whose petals are plated in gold;

(ii) The coin is circle $r = 1$, having inside the rose $r = \sin(5\theta)$ whose petals are plated in gold.

(a) Draw the two designs you submit to the Queen.

(b) Seeing the designs, the Queen decides: "Make the one that has more gold. Bring it tomorrow!" You go and spend one more sleepless night lost in polar computations. What do you tell the Queen?

(c) After your answer next day, the Queen decides again: "You modify the design (i) as follows. Inside the coin $r = 1$, draw also the circle $r = 1/2$. The part of the rose $r = \cos(3\theta)$ which is inside $r = 1/2$ shall be covered with platinum, the rest of the rose shall be covered with gold. And this shall be the coin of Polar Koordinatea!"

Just when you are about to leave happy, the Queen says: "I would like to know by tomorrow if more platinum or more gold is needed for the new coin. Tell me please the exact difference between the two areas." Can you answer this?

(a) The drawing should be supported by a table (θ, r), or at least by an explanation of the values of θ corresponding to maximum r.

(b) For a rose with an odd number of petals, when $\theta \in [0, 2\pi]$ each petal is covered twice, thus integrating from θ to will give us twice the area inside the rose

If $r = \cos((2k+1)\theta)$ or $r = \sin((2k+1)\theta)$ we get

$$2 \cdot A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_0^{2\pi} \cos^2((2k+1)\theta)\, d\theta = \frac{1}{2} \int_0^{2\pi} \frac{1 + \cos(2(2k+1)\theta)}{2}\, d\theta$$

$$\text{so } 2 \cdot A = \frac{1}{4} \left(\int_0^{2\pi} 1\, d\theta + \int_0^{2\pi} \cos((4k+2)\theta)\, d\theta \right)$$

$$2 \cdot A = \frac{1}{4} \left(2\pi + \frac{1}{(4k+2)} \sin((4k+2)\theta) \Big|_{\theta=0}^{2\pi} \right) = \frac{1}{4}(2\pi + 0) = \frac{\pi}{2}$$

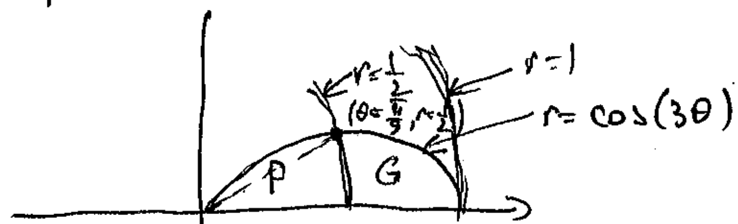
Thus $A = \frac{\pi}{4}$. The same answer is obtained for $r = \sin((2k+1)\theta)$.

Thus all roses with an odd number of petals have the same area, which is $\frac{1}{4}$ of the area of the disk they are inscribed in.

You tell the queen the two designs use the same amount of gold

Part c

I show the picture for platinum (P) & gold (G) in one half of a petal.



First find the intersection of the curves $r=\frac{1}{2}$ and $r=\cos(3\theta)$.

It occurs when $\cos(3\theta)=\frac{1}{2}$, so (first) when $3\theta=\frac{\pi}{3}$, so $\theta=\frac{\pi}{9}$

The area of gold in ~~fo~~ one half of a petal is computed by

$$A_{\text{Gold for } \frac{1}{2}\text{ petal}} = \int_0^{\frac{\pi}{9}} \frac{1}{2}\cos^2(3\theta)\,d\theta - \int_0^{\frac{\pi}{9}} \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{9}} \frac{1+\cos(6\theta)}{2}\,d\theta - \frac{1}{8}\cdot\frac{\pi}{9} =$$

$$= \frac{1}{4}\left(\theta + \frac{\sin(6\theta)}{6}\right)\Bigg|_0^{\frac{\pi}{9}} - \frac{\pi}{72} =$$

$$= \frac{1}{4}\cdot\left(\frac{\pi}{9} + \frac{\sin\left(\frac{2\pi}{3}\right)}{6}\right) - \frac{\pi}{72} = \frac{\pi}{36} - \frac{\pi}{72} + \frac{1}{24}\cdot\frac{\sqrt{3}}{2} = \frac{\pi}{72} + \frac{\sqrt{3}}{48}$$

Thus the total area of the gold in the rose is

$$A_{\text{Gold}} = 6\cdot\left(\frac{\pi}{72} + \frac{\sqrt{3}}{48}\right) = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

Area of platinum is

$$A_{\text{Platinum}} = A_{\text{Rose}} - A_{\text{Gold}} = \frac{\pi}{4} - \frac{\pi}{12} - \frac{\sqrt{3}}{8} = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

The difference

$$A_{\text{Gold}} - A_{\text{Platinum}} = \frac{\pi}{12} + \frac{\sqrt{3}}{8} - \frac{\pi}{6} + \frac{\sqrt{3}}{8} = \frac{\sqrt{3}}{4} - \frac{\pi}{12} = \frac{3\sqrt{3}-\pi}{12}$$

Since $3\sqrt{3} = \sqrt{27} > 5 > \pi$ it follows that the design contains more gold than platinum.