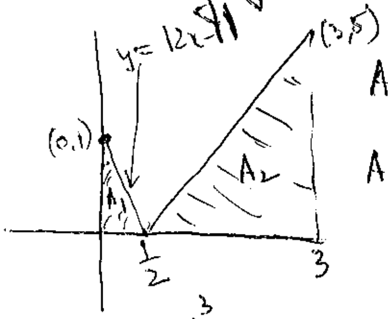
Worksheet week 3 - MAC 2312, Spring 2015 **NAME:** Solution Key

1. Use FTC or geometry to evaluate each integral:

(a) $\int_0^3 |2x-1|\,dx = A_1 + A_2$

Using geometry is probably easiest



$A_1 = \frac{1 \cdot \frac{1}{2}}{2} = \frac{1}{4}$

$A_2 = \frac{5 \cdot \frac{5}{2}}{2} = \frac{25}{4}$

Thus $\int_0^3 |2x-1|\,dx = \frac{1}{4} + \frac{25}{4} = \boxed{\frac{13}{2}}$

(b) $\int_1^2 \frac{x^2+1}{x}\,dx = \int_1^2 \left(\frac{x^2}{x} + \frac{1}{x}\right)dx$

$= \int_1^2 \left(x + \frac{1}{x}\right)dx =$

$= \left(\frac{x^2}{2} + \ln x\right)\Big|_{x=1}^{x=2}$

$= \boxed{\frac{3}{2} + \ln 2}$

(c) $\int_0^{\pi/3} \sec^2 x\,dx =$

$= (\tan x)\Big|_{x=0}^{x=\frac{\pi}{3}}$

$= \tan\frac{\pi}{3} - \tan 0$

$= \boxed{\sqrt{3}}$

2. Find the average value of $f(x) = \frac{1}{x^2+1}$ on the interval $[-1,1]$ and find all values of $x^* \in [-1,1]$ so that $f(x*)$ equals the average value of f on $[-1,1]$. Why are such values of x^* guaranteed to exist?

$$f_{ave} = \frac{\int_{-1}^{1} \frac{1}{1+x^2}\,dx}{1-(-1)} = \frac{(\arctan x)\Big|_{x=-1}^{x=1}}{2} = \frac{\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)}{2} = \boxed{\frac{\pi}{4}}$$

$x^* \in [-1,1]$ so that $f(x^*) = f_{ave}$ is guaranteed to exist by M.V.T. for integrals as $f(x) = \frac{1}{x^2+1}$ is continuous on $[-1,1]$.

To find x^*, solve $\frac{1}{x^2+1} = \frac{\pi}{4} \Rightarrow x^2+1 = \frac{4}{\pi} \Rightarrow x^2 = \frac{4}{\pi} - 1 \Rightarrow x^* = \pm\sqrt{\frac{4}{\pi} - 1}$

both values work as $\pm\sqrt{\frac{4}{\pi}-1} \in [-1,1]$

3. Use substitution to compute each integral:

(a) $\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{w=1}^{w=2} \frac{1}{\sqrt{w}}\,dw =$

subst. $w = \ln x$ (Note: $x=e \Rightarrow w = \ln e = 1$, $x = e^2 \Rightarrow w = \ln e^2 = 2$)

$dw = \frac{1}{x}dx$

$= \int_1^2 w^{-\frac{1}{2}}\,dw = 2w^{\frac{1}{2}}\Big|_1^2 = \boxed{2\sqrt{2} - 2}$

(b) $\int_0^1 \frac{x}{x^2+1}\,dx = \int_{w=1}^{w=2} \frac{\frac{1}{2}dw}{w} =$

sub. $w = x^2+1$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$= \frac{1}{2}\ln w\Big|_{w=1}^{w=2} = \boxed{\frac{1}{2}\ln 2}$

4. Given that $F(x) = \int_0^x \sqrt{8t - t^2}\, dt$, for $x \in [0, 8]$, do the following:

(a) Determine the values of $F(0)$, $F(4)$, $F(8)$. Hint: Complete the square and use geometry.
(b) Determine $F'(x)$ and $F''(x)$.
(c) Based on parts (a) and (b), sketch the graph of the function $y = F(x)$, for $x \in [0, 8]$. What kind of point is $x = 4$ for the graph of $y = F(x)$?

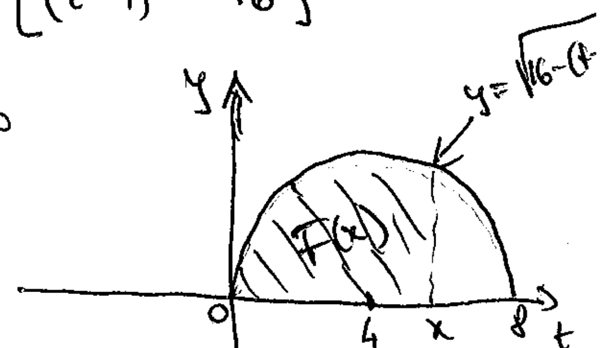
(a) Completion of the square: based on $(A \pm B)^2 = A^2 \pm 2AB + B^2$

$$8t - t^2 = -(t^2 - 8t) = -(t^2 - 2 \cdot t \cdot 4) =$$

$$= -(t^2 - 2 \cdot t \cdot 4 + 4^2 - 4^2) = -[(t-4)^2 - 16]$$

Thus $8t - t^2 = 16 - (t-4)^2$ so

$$F(x) = \int_0^x \sqrt{16 - (t-4)^2}\, dt$$



$F(0) = 0$, $F(4) = \int_0^4 \sqrt{16-(t-4)^2}\, dt = \frac{1}{4} \cdot \pi \cdot 4^2 = 4\pi$, $F(8) = \frac{1}{2}\pi \cdot 4^2 = 8\pi$

(b) By FTC (ii) $F'(x) = \frac{d}{dx}\left(\int_0^x \sqrt{8t - t^2}\, dt\right) = \sqrt{8x - x^2} = \sqrt{16 - (x-4)^2}$

$$F''(x) = \left(\left(16 - (x-4)^2\right)^{\frac{1}{2}}\right)' = \frac{1}{2}\left(16 - (x-4)^2\right)^{-\frac{1}{2}} \cdot (-2)(x-4) = -\frac{x-4}{\sqrt{16 - (x-4)^2}}$$

(c) Graph of $y = F(x)$

$F(x)$ is increasing, since $F'(x) > 0$ on $[0,8]$
$F(x)$ is concave up on $[0,4]$, as $F''(x) > 0$ on $[0,4)$
$F(x)$ is concave down on $[4,8]$, as $F''(x) < 0$ on $[4,8]$
$x = 4$ is an inflection point for $F(x)$.