Name: Solution Key PantherID:

Exam 1 MAC-2313 Spring 2012

To receive credit you MUST SHOW ALL YOUR WORK.

1. (15 pts) Given the vectors $\mathbf{u} = -\mathbf{i}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, find each of the following:

(a) the angle between $\mathbf{u}$ and $\mathbf{v}$ (answer as inverse trig function ok);

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$ $\quad \|\vec{u}\| = 1$, $\|\vec{v}\| = \sqrt{4+1+4} = 3$

$-2 = 1 \cdot 3 \cdot \cos\theta \Rightarrow \cos\theta = -\frac{2}{3} \Rightarrow \theta = \arccos\left(-\frac{2}{3}\right)$

(b) two vectors $\mathbf{u}^{\parallel}$, $\mathbf{u}^{\perp}$, so that $\mathbf{u} = \mathbf{u}^{\parallel} + \mathbf{u}^{\perp}$, $\mathbf{u}^{\parallel} \parallel \mathbf{v}$, $\mathbf{u}^{\perp} \perp \mathbf{v}$.

$\vec{u}^{\parallel} = \lambda \vec{v}$

$\vec{u} = \lambda\vec{v} + \vec{u}^{\perp}$ Take ~~inner~~ dot product with $\vec{v}$

$\vec{u} \cdot \vec{v} = \lambda \vec{v} \cdot \vec{v} + 0$

$\Rightarrow \lambda = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$ so $\vec{u}^{\parallel} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$

$\vec{u}^{\parallel} = -\frac{2}{9}\vec{v} = -\frac{4}{9}\vec{i} - \frac{2}{9}\vec{j} - \frac{4}{9}\vec{k}$

$\vec{u}^{\perp} = \vec{u} - \vec{u}^{\parallel} = -\frac{5}{9}\vec{i} + \frac{2}{9}\vec{j} + \frac{4}{9}\vec{k}$

(c) the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

$\text{Area} = \|\vec{u} \times \vec{v}\|$ $\quad \vec{u} \times \vec{v} = 2\vec{j} - \vec{k}$

$\text{Area} = \sqrt{4+1} = \sqrt{5}$

$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$

2. (10 pts) Match the following equations with the appropriate surface:

(i) $x^2 - 2y^2 - 3z^2 = 1$ $\longleftrightarrow$ (c)

(ii) $2y^2 + 3z^2 = 1$ $\longleftrightarrow$ (d)

(iii) $(x+1)^2 + 2(y-1)^2 + 3(z-2)^2 = 10$ $\longleftrightarrow$ (a)

(iv) $x - 2y^2 - 3z^2 = 1$ $\longleftrightarrow$ (e)

(v) $(x+1)^2 + 2(y-1)^2 - 3(z-2)^2 = 10.$ $\longleftrightarrow$ (b)

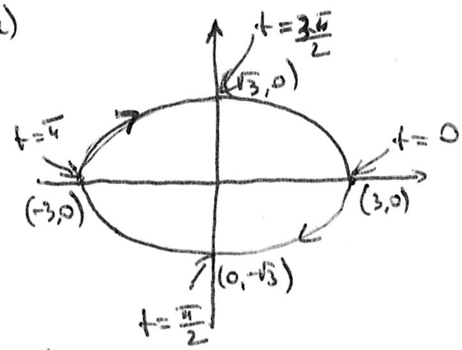
(a) ellipsoid (b) hyperboloid with one sheet (c) hyperboloid with two sheets

(d) elliptic cylinder (e) elliptic paraboloid

3. (15 pts) (a) (5 pts) Sketch the graph of $\mathbf{r}(t) = 3\cos t\,\mathbf{i} - \sqrt{3}\sin t\,\mathbf{j}$ in 2-space, indicating the direction of increasing t.

(b) (10 pts) Find the unit tangent vector to the curve when $t = \pi/3$ and write the parametric equations of the tangent line to the curve when $t = \pi/3$.

(a)



parametrically

$$\begin{cases} x = 3\cos t \\ y = -\sqrt{3}\sin t \end{cases} \Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1 \Leftrightarrow \frac{x^2}{9} + \frac{y^2}{3} = 1 \text{ ellipse}$$

(b) $\vec{r}\,'(t) = -3\sin t\,\vec{i} - \sqrt{3}\cos t\,\vec{j}$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{\sqrt{9\sin^2 t + 3\cos^2 t}}\left(-3\sin t\,\vec{i} - \sqrt{3}\cos t\,\vec{j}\right)$$

$$\vec{T}\left(\frac{\pi}{3}\right) = \frac{1}{\frac{\sqrt{30}}{2}}\left(-\frac{3\sqrt{3}}{2}\vec{i} - \frac{\sqrt{3}}{2}\vec{j}\right) = -\frac{1}{\sqrt{10}}\left(3\vec{i} + \vec{j}\right)$$

Param. equations of tangent line

Point: $\vec{r}\left(\frac{\pi}{3}\right) = \frac{3}{2}\vec{i} - \frac{3}{2}\vec{j}$

Tang. line

$$\boxed{\begin{cases} x = \frac{3}{2} - 3t \\ y = -\frac{3}{2} - t \end{cases}}$$

4. (12 pts) Determine if the lines

$L_1: \; x = 1 + t,\; y = -1 + 2t,\; z = 2 + t \qquad L_2: \; x = 2 + s,\; y = 2 + 3s,\; z = 4 + 2s$

are parallel, intersecting, or skew.

We should check if the system below has a solution or not.

$$\begin{cases} 1 + t = 2 + s \\ -1 + 2t = 2 + 3s \\ 2 + t = 4 + 2s \end{cases} \Leftrightarrow \begin{cases} t - s = 1 \\ 2t - 3s = 3 \\ t - 2s = 2 \end{cases}$$

From $\begin{cases} t - s = 1 \\ t - 2s = 2 \end{cases}$ we get $\begin{cases} t = 0 \\ s = -1 \end{cases}$

The equation $2t - 3s = 3$ is also verified so the system does have a solution

The lines are <u>intersecting</u> at the point <u>$P(1, -1, 2)$</u>.

5. (18 pts) (a) (6 pts) Show that the line L given by $x = 1 + 2t$, $y = -1 + t$, $z = 4$ is parallel to the plane π whose equation is $x - 2y + z = 0$.

(b) (12 pts) Find the equation of the plane that contains the line L and is perpendicular to the plane π (where L and π are those in part (a)).

Solution 1:

(a) The normal of the plane π is given by $\vec{n}_1 = \langle 1, -2, 1 \rangle$,

while the directional vector of the line is $\vec{u} = \langle 2, 1, 0 \rangle$

Since $\vec{u} \cdot \vec{n}_1 = 2 - 2 = 0$, it follows that $\vec{u}$ is parallel to the plane.

Thus the line is either parallel with π or contained in π.

But the point $(1, -1, 4)$ is on the line, but is not in π.

So the line must be parallel to π.

Solution 2: We find out if the line intersects the plane.

For that, we should solve

$$(1 + 2t) - 2(-1 + t) + 4 = 0$$

we get $7 = 0$ so there is no solution

The line is parallel to the plane.

(b) The normal $\vec{n}_2$ of the plane we are looking for is given by

$$\vec{n}_2 = \vec{u} \times \vec{n}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix} = \vec{i} - 2\vec{j} - 5\vec{k}$$

A point from the line, which will also be a point in our plane is $(1, -1, 4)$

The equation of the required plane is:

$$\boxed{1 \cdot (x-1) - 2(y+1) - 5(z-4) = 0}$$

6. (18 pts) (a) (10 pts) Compute the curvature, $\kappa(t)$, of the curve

$$\mathbf{r}(t) = \sqrt{2}\cos t\,\mathbf{i} + \sqrt{2}\cos t\,\mathbf{j} + 2\sin t\,\mathbf{k}$$

(b) (8 pts) Show that the curve in part (a) is a circle, confirming your result from (a). *Hint:* Check that the curve lies on the sphere $x^2 + y^2 + z^2 = 4$ and on a certain plane.

(a) $$\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

$$\vec{r}\,'(t) = -\sqrt{2}\sin t\,\vec{i} - \sqrt{2}\sin t\,\vec{j} + 2\cos t\,\vec{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{2\sin^2 t + 2\sin^2 t + 4\cos^2 t} = \sqrt{4(\sin^2 t + \cos^2 t)} = \sqrt{4} = 2$$

$$\vec{r}\,''(t) = -\sqrt{2}\cos t\,\vec{i} - \sqrt{2}\cos t\,\vec{j} - 2\sin t\,\vec{k}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sqrt{2}\sin t & -\sqrt{2}\sin t & 2\cos t \\ -\sqrt{2}\cos t & -\sqrt{2}\cos t & -2\sin t \end{vmatrix} = (2\sqrt{2}\sin^2 t + 2\sqrt{2}\cos^2 t)\vec{i} - (2\sqrt{2}\sin^2 t + 2\sqrt{2}\cos^2 t)\vec{j} + 0\vec{k}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = 2\sqrt{2}\,\vec{i} - 2\sqrt{2}\,\vec{j}$$

$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = 2\sqrt{2}\cdot\sqrt{2} = 4$$

Thus $\kappa(t) = \dfrac{4}{2^3} = \boxed{\dfrac{1}{2}}$.

(b) Parametrically, the curve is given by $x = \sqrt{2}\cos t$, $y = \sqrt{2}\cos t$, $z = 2\sin t$.
But then $x^2 + y^2 + z^2 = 2\cos^2 t + 2\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$.
So the curve lies on the sphere with center at origin and radius 2.
Easily we see that the curve lies also on the plane $x = y$ or $x - y = 0$. This is a plane that goes through the origin.
The intersection of the sphere with this plane is our curve, thus it is a circle of radius 2. For a circle $\kappa = \frac{1}{r} = \frac{1}{2}$.

7. (12+3 pts) Find the arc length parametrization of the curve $r(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j} + 3\mathbf{k}$. (Bonus) Can you describe in words what does the curve represents in 3-space?

The curve is a spiral in the plane $z=3$, with center at $(0,0,3)$

$$\vec{r}\,'(t) = (e^t\cos t - e^t \sin t)\vec{i} + (e^t \sin t + e^t \cos t)\vec{j} + 0\vec{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2} = \sqrt{e^{2t}(\cos^2 t + \sin^2 t - \cancel{2\sin t\cos t} + \sin^2 t + \cos^2 t + \cancel{2\sin t \cos t})}$$

$$\|\vec{r}\,'(t)\| = \sqrt{e^{2t}\cdot 2} = \sqrt{2}\,e^t$$

$$s = \int_0^t \sqrt{2}\,e^t\,dt = \sqrt{2}(e^t - 1) \Rightarrow e^t - 1 = \frac{s}{\sqrt{2}} \Rightarrow e^t = 1 + \frac{s}{\sqrt{2}}$$

$$\Rightarrow t = \ln\left(1 + \frac{s}{\sqrt{2}}\right)$$

The arclength parametrization is

$$\boxed{\vec{r}(s) = \left(1+\frac{s}{\sqrt{2}}\right)\cos\left(\ln\left(1+\frac{s}{\sqrt{2}}\right)\right)\vec{i} + \left(1+\frac{s}{\sqrt{2}}\right)\sin\left(\ln\left(1+\frac{s}{\sqrt{2}}\right)\right)\vec{j} + 3\vec{k}}$$

8. (10 pts) Suppose that a particle is moving on a curve with constant speed. Show that at every moment the velocity vector is perpendicular to the acceleration vector.

$$\|\vec{r}\,'(t)\| = \text{const}. \Rightarrow \quad \|\vec{r}\,'(t)\|^2 = \text{const}, \text{ so } \vec{r}\,'(t)\bullet\vec{r}\,'(t) = \text{const}.$$

Take the derivative of both sides.

$$\frac{d}{dt}\left(\vec{r}\,'(t)\bullet\vec{r}\,'(t)\right) = \frac{d}{dt}(\text{const}) = 0.$$

We get $\vec{r}\,''(t)\bullet\vec{r}\,'(t) + \vec{r}\,'(t)\cdot\vec{r}\,''(t) = 0$

$$2\vec{r}\,'(t)\bullet\vec{r}\,''(t) = 0 \quad \text{so } \vec{r}\,'(t)\bullet\vec{r}\,''(t) = 0$$

But $\vec{v} = \vec{r}\,'$ and $\vec{a} = \vec{r}\,''$

So we proved that $\vec{v}$ and $\vec{a}$ are perpendicular at every point.