Correct solution for Pb. 5 with the units as given. There is still an ambiguity about the units for the medium resistance. Let's agree that sentence as "the medium offers resistance equal to $4x'$ Newtons, where x' is the velocity in cm/s."

5. (18 pts) A spring is such that a force of 10 newtons would stretch it 5cm. The spring hangs vertically and a 2-kg mass is attached to it. After this 2-kg mass comes to rest in its equilibrium position, it is pulled down 3cm below this position and released at $t = 0$ (with zero initial velocity). The medium offers resistance equal to $4x'$, ⇐ Newton where x' is the velocity in centimeters per second.

(a) (6 pts) Set up as an IVP problem.

(b) (10 pts) Solve the IVP to find the displacement function $x(t)$.

(c) (2 pts) Is the motion underdamped (or oscillatory damped), critically damped, or overdamped?

(a) The general equation for damped (and free) motion is

$$m x'' + a x' + k x = 0, \quad (*)$$

where $x(t)$ is the deformation beyond the equilibrium position at time t

Here is the first issue with units. Everywhere in the problem cm are used for x and cm/s for x'. It would seem to be ok to use $k = \frac{10}{5} = 2 \frac{N}{cm}$ and plug in constants to get the DE.

$$2x'' + 4x' + 2x = 0 \quad \text{when we have } x(t) \text{ in centimeters}$$

But this is incorrect! From $F = m \cdot a$, $\underline{1N = 1\,kg \cdot \frac{m}{s^2}}$.

Thus, for the term mx'' to be expressed in N (as kx and ax' are), as the mass is in kg, the acceleration x'' has to be expressed in $\frac{m}{s^2}$. Thus, for consistency, we should work with the deformation expressed in meters.

Now there is one more issue with the notations used in the pb. It says x' is the velocity at time t in cm/s, thus it is understood that $x(t)$ is the deformation in cm.

So let us denote $\tilde{x}(t)$ the deformation at time t in meters.

Thus the equation $(*)$ ~~should~~ for $\tilde{x}$ is

$$m\tilde{x}'' + \tilde{a}\tilde{x}' + \tilde{k}\tilde{x} = 0 \quad (\tilde{*})$$

These are the constants: $\underline{m \text{ is } 2\,kg}$

→ Continued on next page

$\tilde{k} = \frac{10\,N}{5\cdot 10^{-2}\,m} = 200 \left(\frac{N}{m}\right)$

$\tilde{a}$ is <u>tricky</u> (this is the final issue with the units)

$\tilde{a}$ is <u>not 4</u> and <u>is not</u> $\frac{4}{100} = 0.04$

The resistance force from the medium (in Newtons) is numerically equal to $4x'$ with x' velocity in cm/s. So the constant 4 is not just a scalar, but it has <u>units</u>, which are $\frac{N}{\frac{cm}{s}} = \frac{N\cdot s}{cm}$.

When we go to <u>meters</u> the $4\frac{N\cdot s}{cm}$ becomes $4\frac{N\cdot s}{10^{-2}m} = 4\cdot 10^{2}\frac{N\cdot s}{m} = \underline{400}\frac{N\cdot s}{m}$

Here is a better way to see this:

$\tilde{x}(t) = 10^{-2}\cdot x(t)$ (as $1cm = 10^{-2}\,m$)

$\tilde{x}' = 10^{-2}x'$ so $x' = 10^{2}\tilde{x}' = 100\,\tilde{x}'$

where x' is expressed in $\frac{cm}{s}$ and $\tilde{x}'$ in $\frac{m}{s}$.

Thus $|F_{resistance}| = 4x' = 400\tilde{x}'$.

Thus $\underline{\tilde{a} = 400}$.

The DE $(\tilde{*})$ is then $\underline{2\tilde{x}'' + 400\tilde{x}' + 200\tilde{x} = 0}$ or

$\tilde{x}'' + 200\tilde{x}' + 100\tilde{x} = 0$

with initial conditions $\tilde{x}(0) = 0.03 = 3\cdot 10^{-2}\,m$

$\tilde{x}'(0) = 0$

(b) Characteristic equation is

$\lambda^2 + 200\lambda + 100 = 0$ (but the left side <u>is not</u> $(\lambda+10)^2$!)

The roots are both real and negative

$\lambda_1 = -100 - 10\sqrt{99}$, $\lambda_2 = -100 + 10\sqrt{99}$

so the motion is overdamped

→ continuation

The general solution is

$$\tilde{x}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \text{ where } \lambda_1 = -100 - 10\sqrt{99},\ \lambda_2 = -100 + 10\sqrt{99}.$$

The constants c_1, c_2 are obtained from the initial conditions

$\tilde{x}(0) = 0.03$ and $\tilde{x}'(0) = 0$.

Get the system $\begin{cases} c_1 + c_2 = 0.03 \\ c_1\lambda_1 + c_2\lambda_2 = 0 \end{cases}$ whose solutions are

$$c_1 = \frac{0.03\lambda_2}{\lambda_2 - \lambda_1} = \frac{0.03(-10+\sqrt{99})}{2\sqrt{99}} \qquad c_2 = -\frac{0.03\lambda_1}{\lambda_2 - \lambda_1} = \frac{0.03(10+\sqrt{99})}{2\sqrt{99}}$$

so we get the expression for $\tilde{x}(t)$ (deformation in meters.)

Since the problem started with $x(t)$ in centimeters the expression for it is $x(t) = 10^2 \tilde{x}(t)$ or

$$x(t) = \frac{3(-10+\sqrt{99})}{2\sqrt{99}} e^{(-100-10\sqrt{99})t} + \frac{3(10+\sqrt{99})}{2\sqrt{99}} e^{(-100+10\sqrt{99})t}$$

(c) Motion is overdamped as explained above (both λ_1, λ_2 are real & negative)

Final Note:

The similar Pb. 5 page 209 text (section 5.3) looks to have a wrong answer in the back of the book because of a mistake in transforming the $280x'$ term!