Final Exam - MAD 2104 - Summer A 2014

1. (10 pts) Are $\neg(p \leftrightarrow q)$ and $p \leftrightarrow (\neg q)$ logically equivalent? Justify. Any style OK, but use at least one sentence in your justification.

2. (16 pts) On the set of all integers, **Z**, consider the congruency mod 5 relation, that is

$$m \equiv n \pmod 5 \text{ if and only if } 5 | (m - n).$$

(a) Show that the congruency mod 5 relation is an equivalence relation on **Z**.

(b) Six integers are picked randomly. Show that among these six there are at least two integers whose difference is divisible by 5. For full credit here mention who are the pigeons, who are the pigeonholes (boxes), etc.

3. (10 pts) Write the sum below using summation notation and find the exact value of the sum.

$$4 + 9 + 14 + 19 + 24 + 29 + \ldots + 194 + 199$$

4. (14 pts) Recursively define $a_0 = 1$, $a_1 = 3$, $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$.

(a) (2 pts) Calculate a_2, a_3.

(b) (2 pts) Guess a formula for a_n.

(c) (10 pts) Prove your guess using mathematical induction. Specify which type of induction you are using.

5. (24 pts) Answer and very briefly justify (4 pts each).

(a) How many edges does K_n have? Recall that K_n is the complete (simple, undirected) graph with n vertices.

(b) What is the coefficient of $x^3 y^5$ in $(3x - 2y)^8$?

(c) What is the exact value of $P(10, 3)$?

(d) What is the value of the sum $C(n, 0) + C(n, 1) + C(n, 2) + ... + C(n, n-1) + C(n, n)$?

(e) A bowl contains 6 red balls, 9 yellow balls. What is the minimum number of balls which must be drawn (blindly) to be sure at least three balls of the same color are drawn?

(f) Is there an undirected graph with three vertices of degree 3 and two vertices of degree 4? If yes, draw one such graph, if no, explain why it there is no such graph.

6. (12 pts) Give your answer and a brief explanation for each of the following:

(a) How many bit strings of length nine contain exactly four 1s?

(b) How many bit strings of length nine start with 11 or end with 00?

(c) How many bit strings of length nine are palindromes?

7. (14 pts) Use mathematical induction to prove ONE of these.

(A) $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$, for any $n \geq 1$.

(B) Prove that for any positive integer n, a rectangular checkerboard of dimensions $(3 \cdot 2^n)$ by 2^n can be covered with right triominoes.

8. (6 pts) Recall that CAE is a permutation of ACE. How many permutations of the letters $ABCDEFGH$ contain the string DAC? Explain your work.

9. (10 pts) A club has 10 men and 15 women.

(a) In how many ways a committee of 2 men and 3 women can be selected?

(b) How many ways are there to select a committee of 5 people and a president for this committee, if the president must be a woman.

10. (10 pts) The English alphabet contains 21 consonants and 5 vowels. How many strings of six lowercase letters contain:

(a) at least one vowel?

(b) exactly two vowels?

11. Give an algebraic proof (10pts) or a combinatorial proof (14pts) for the identity

$$kC(n, k) = nC(n - 1, k - 1), \text{ where } 1 \leq k \leq n .$$

You may give two proofs, but you will get credit for only one of the two (the higher score).

12. (12 pts) Describe all values of m, n for which the complete bipartite graph $K_{m,n}$ has
(a) an Euler circuit.
(b) an Euler path.
No need to find explicitly an Euler circuit or an Euler path, but explain clearly the result you are using.

13. (12 pts) What is the chromatic number of W_7? Justify your answer.