Quiz 5 – MAD 2104 - Summer A 2014 **NAME:** Solution Key

This is a take-home quiz. Due date is Monday, June 16. Even for the short problems, please add a brief explanation to the answer.

1. (8 pts) (a) How many functions are there from a set with five elements to a set with three elements?

$f:\{1,2,3,4,5\}\to\{a,b,c\}$ – f is determined by $f(1), f(2), f(3), f(4), f(5)$. There are 3 choices for each. Thus there are

3^5 functions

(b) How many **one-to-one** functions are there from a set with five elements to a set with three elements?

None: By pigeonhole principle, there will be two inputs which will have same output, so there is no one-to-one function from a set with 5 elements to a set with 3 elements.

(c) How many **one-to-one** functions are there from a set with five elements to a set with eight elements?

$P(8,5) = 8\cdot 7\cdot 6\cdot 5\cdot 4$

8 choices for $f(1)$, 7 choices for $f(2)$, ... , 4 choices for $f(5)$.

(d) How many **onto** functions are there from a set with eight elements to a set with five elements?

The answer is $8\times 7\times 6\times 5\times 75$

I may explain this in class.

2. (8 pts) How many strings of five lowercase letters in the English alphabet contain:

(a) the letter a? Let A = set of 5 letter words containing "a", U = set of all 5 letter words:

$|A| = |U| - |\overline{A}| = \boxed{26^5 - 25^5}$

(b) the letters a and b? With the notations from (a), also let B = set of 5 letter words containing "b"

$|A\cap B| = |U| - |\overline{A\cap B}| = |U| - |\overline{A}\cup\overline{B}| = |U| - (|\overline{A}| + |\overline{B}| - \underbrace{|\overline{A}\cap\overline{B}|}_{\text{5 letter words with no "a" nor "b"}}) = \boxed{26^5 - 25^5 - 25^5 + 24^5}$

(c) the letters a and b in consecutive positions with a preceding b, with all letters distinct?

Think of "ab" as a single entity. It can occur in 4 positions in the string of 5 letters. The other 3 letters can be chosen in $24\times 23\times 22$ ways. Thus $\underline{4\times 24\times 23\times 22}$

(d) the letters a and b, where a is somewhere to the left of b in the string, with all letters distinct?

There are $C(5,2)$ ways of choosing two spots out of 5 where to have the letters (a,b). Once the two spots are chosen "a" will have the left position and "b" the right position. The other 3 letters can be chosen in $P(24,3) = 24\times 23\times 22$ ways (as order matters)

Thus the answer for (d) is

$C(5,2)\times P(24,3) = 10\times 24\times 23\times 22$

3. (8 pts) (a) Use the binomial theorem to show that

$$C(n,0) - C(n,1) + C(n,2) - C(n,3) + \ldots + (-1)^n C(n,n) = 0.$$

(b) How many subsets with an odd number of elements does a set with 10 elements have?

(a) Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} C(n,k)\, x^{n-k} y^k$

Take $x=1$, $y=-1$

$$0 = (1+(-1))^n = \sum_{k=0}^{n} C(n,k)\cdot(-1)^k = C(n,0) - C(n,1) + C(n,2) - \ldots + (-1)^n C(n,n)$$

(b) From the binomial theorem with $x=y=1$, we also know that
$C(n,0) + C(n,1) + C(n,2) + C(n,3) + \ldots + C(n,n) = 2^n$

But from part (a), we have $C(n,0) + C(n,2) + C(n,4) + \ldots = C(n,1) + C(n,3) + \ldots$

Thus, we get $C(n,1) + C(n,3) + \ldots = C(n,0) + C(n,2) + \ldots = \frac{1}{2}\cdot 2^n = 2^{n-1}$

But in a ~~sub~~set with n elements there are $C(n,1)$ subsets with one element
$C(n,3)$ subsets with 3 elements and so on.

Thus a set with n elements has 2^{n-1} subsets with an odd number of elements.
For $n=10$, $2^{10-1} = 2^9 = 512$ subsets with odd number of elements.

4. (4 pts) How many numbers must be selected from the set $\{1,3,5,7,9,11,13,15,17\}$ to guarantee that at least one pair of these numbers have an average of 9?

Consider the partition $\{1,17\} \cup \{3,15\} \cup \{5,13\} \cup \{7,11\} \cup \{9\}$
~~for the given set~~ where each set we think as a "box".
The last box containing just the '9' is not a "good" box
as it does not contain two elements with an average of 9.
Thus, to guarantee that we'll get two elements with
average 9, we need to select <u>six</u> elements from the
original set.