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Symplectic Forms and Cohomology Decomposition of almost Complex Four-Manifolds

Tedi Draghici[1], Tian-Jun Li[2], and Weiyi Zhang[2]

[1]Department of Mathematics, Florida International University, Miami, FL 33199, USA and [2]School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

Correspondence to be sent to: draghici@fiu.edu

For any compact almost complex manifold (M, J), the last two authors [8] defined two subgroups $H_J^+(M)$, $H_J^-(M)$ of the degree 2 real de Rham cohomology group $H^2(M, \mathbb{R})$. These are the sets of cohomology classes which can be represented by J-invariant, respectively, J-antiinvariant real 2-forms. In this paper, it is shown that in dimension 4 these subgroups induce a cohomology decomposition of $H^2(M, \mathbb{R})$. This is a specifically four-dimensional result, as it follows from a recent work of Fino and Tomassini [6]. Some estimates for the dimensions of these groups are also established when the almost complex structure is tamed by a symplectic form and an equivalent formulation for a question of Donaldson is given.

1 Introduction

In this paper, we continue to study differential forms on an almost complex four-manifold (M, J) following [8]. We are particularly interested in the subgroups $H_J^+(M)$ and $H_J^-(M)$ of the degree 2 real de Rham cohomology group $H^2(M; \mathbb{R})$. These are the sets of cohomology classes which can be represented by J-invariant, respectively, J-antiinvariant real 2-forms. The goal pursued by defining these subgroups is simple: understand the effects of the action of the almost complex structure on forms at the level of cohomology and

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introduce the idea of (real) cohomology type, via the almost complex structure. Certainly, the subgroups $H_J^{\pm}(M)$ and their dimensions $h_J^{\pm}$ are diffeomorphism invariants of the almost complex manifold (M, J). We would like to show that these invariants appear to be interesting, particularly so in dimension 4. Here is the outline of our paper.

Our first main result, Theorem 2.3 in Section 2, shows that on any compact almost complex four-manifold the subgroups $H_J^+(M)$ and $H_J^-(M)$ will induce a direct sum decomposition of $H^2(M, \mathbb{R})$. With the terminology introduced in [8], Theorem 2.3 says that any almost complex structure on a compact four-dimensional manifold is C^∞-pure and full. See Section 2 for precise definitions. Theorem 2.3 turns out to be specifically a four-dimensional result. Indeed, Example 3.3 of Fino and Tomassini [6] shows the existence of a compact six-dimensional nilmanifold with an almost complex structure which is not C^∞-pure (the intersection of $H_J^+(M)$ and $H_J^-(M)$ is nonempty). (We learned of the preprint [6] while putting together the final form of our paper. There are further interesting links between [6] and our paper; see further comments in Section 2. The overlap is minimal though.) Taking products of this example with arbitrary almost complex manifolds, one obtains examples in all dimensions ≥ 6 of almost complex structures which are not C^∞-pure.

Also in Section 2, for a compact four-manifold with an *integrable* J, we show that subgroups $H_J^+(M)$ and $H_J^-(M)$ relate naturally with the (complex) Dolbeault cohomology groups. We also show that a complex type decomposition for cohomology does not hold for nonintegrable almost complex structures (see Lemma 2.12 and Corollary 2.14).

In Section 3, we focus on almost complex structures J which admit compatible or tame symplectic forms and we give estimates for the dimensions $h_J^{\pm}$ in this case. If there are J-compatible symplectic forms, then the collection of cohomology classes of all such forms, the so-called J-compatible cone, $\mathcal{K}_J^c(M)$, is a subcone of $H^2(M; \mathbb{R})$. In fact,

$$\mathcal{K}_J^c(M) \subset H_J^+(M)$$

as a (nonempty) open convex cone. Thus, it is important to determine h_J^+ as well as $H_J^+(M)$. Moreover, it is shown in [8] that if J is also C^∞-full (the sum of $H_J^+(M)$ and $H_J^-(M)$ is $H^2(M; \mathbb{R})$), then

$$\mathcal{K}_J^t(M) = H_J^-(M) + \mathcal{K}_J^c(M),$$

where $\mathcal{K}_J^t(M)$ is the collection of cohomology classes of J-tamed symplectic forms. Thus, it is also important to understand the group $H_J^-(M)$.

Our investigation of almost complex structures which are tamed by symplectic forms is also motivated by the following question of Donaldson [4].

Question 1.1. If J is an almost complex structure on a compact four-manifold M which is tamed by a symplectic form ω, is there a symplectic form compatible with J?

In [8], it was shown that the question has an affirmative answer when J is integrable. For progress on a related problem proposed by Donaldson, the symplectic Calabi–Yau equation and its relation to Question 1.1, the reader is referred to [4, 11–13].

We observe in Theorem 3.3 that an estimate on h_J^+ which is immediate for compatible J's can be carried over to the case of tamed J's as well. Section 3 ends with an equivalent formulation of Donaldson's Question 1.1.

In a later paper [5], we will further study the group H_J^-.

Convention: The groups indexed by (p, q) arise from *complex* differential forms. The groups indexed by $\pm$ arise from *real* differential forms.

2 Cohomology Decomposition of almost Complex Four-Manifolds

2.1 The groups $H_J^\pm$

Let M be a compact $2n$-dimensional manifold and suppose J is an almost complex structure on M. J acts on the bundle of real 2-forms Λ^2 as an involution, by $\alpha(\cdot,\cdot) \to \alpha(J\cdot, J\cdot)$, thus we have the splitting:

$$\Lambda^2 = \Lambda_J^+ \oplus \Lambda_J^-. \tag{1}$$

We will denote by Ω^2 the space of 2-forms on M (C^∞-sections of the bundle Λ^2), Ω_J^+ the space of J-invariant 2-forms, etc. For any $\alpha \in \Omega^2$, the J-invariant (resp. J-antiinvariant) component of α with respect to the decomposition (1) will be denoted by α' (resp. α'').

Definition 2.1 [8]**.** Let $\mathcal{Z}^2$ denote the space of closed 2-forms on M and let $\mathcal{Z}_J^\pm = \mathcal{Z}^2 \cap \Omega_J^\pm$. Define

$$H_J^\pm(M) = \{\mathfrak{a} \in H^2(M;\mathbb{R}) | \exists\ \alpha \in \mathcal{Z}_J^\pm \text{ such that } [\alpha] = \mathfrak{a}\}. \tag{2}$$

2.2 The type decomposition of $H^2(M;\mathbb{R})$

Obviously,

$$H_J^+(M) + H_J^-(M) \subseteq H^2(M;\mathbb{R}),$$

but if J is not integrable, it is not clear whether equality holds and whether the intersection of the two subspaces is trivial. Thus, the following definitions were also introduced in [8].

Definition 2.2. (i) J is said to be C^∞-pure if $H_J^+ \cap H_J^- = 0$; and
(ii) J is said to be C^∞-full if $H^2(M;\mathbb{R}) = H_J^+(M) + H_J^-(M)$.

Note: The terms *pure* and *full* almost complex structures were also defined in [8] in terms of currents. We will not use these in this paper, so we refer the reader to [8] and [6] for more details on this. Note also that the paper of Fino and Tomassini provides a number of interesting cases when the notions of pure and full almost complex structures are equivalent to the C^∞ counterparts (Theorems 3.7 and 4.1 in [6]). See also Remark 2.7.

Our first result is the following theorem.

Theorem 2.3. If M is a compact four-dimensional manifold, then any almost complex structure J on M is C^∞-pure and full. Thus, there is a direct sum cohomology decomposition

$$H^2(M;\mathbb{R}) = H_J^+(M) \oplus H_J^-(M). \tag{3}$$

Before the proof, we should set some more preliminaries and notations. The particularity of dimension 4 is that the Hodge operator $*_g$ of a Riemannian metric g on M also acts as an involution on Λ^2. Thus, we have the well-known selfdual, anti-selfdual splitting of the bundle of 2-forms:

$$\Lambda^2 = \Lambda_g^+ \oplus \Lambda_g^-. \tag{4}$$

We will denote by $\Omega_g^\pm$ the space of sections of $\Lambda_g^\pm$ and by α^+, α^- the selfdual, anti-selfdual components of a 2-form α. Since the Hodge de Rham Laplacian commutes with $*_g$, the decomposition (4) holds for the space of harmonic 2-forms $\mathcal{H}_g$ as well. By Riemannian Hodge theory, we get the metric induced cohomology decomposition

$$H^2(M;\mathbb{R}) = \mathcal{H}_g = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-. \tag{5}$$

As in Definition 2.1, one can define

$$H_g^\pm(M) = \{\mathfrak{a} \in H^2(M;\mathbb{R}) | \exists\, \alpha \in \mathcal{Z}_g^\pm \text{ such that } [\alpha] = \mathfrak{a}\}.$$

Of course, $\mathcal{Z}_g^{\pm} := \mathcal{Z}^2 \cap \Omega_g^{\pm} = \mathcal{H}_g^{\pm}$, so clearly $H_g^{\pm}(M) = \mathcal{H}_g^{\pm}$, and (5) can be written as

$$H^2(M;\mathbb{R}) = H_g^+ \oplus H_g^-.$$

We will need the following special feature of the Hodge decomposition in dimension 4.

Lemma 2.4. If $\alpha \in \Omega_g^+$ and $\alpha = \alpha_h + d\theta + \delta\Psi$ is its Hodge decomposition, then $(d\theta)_g^+ = (\delta\Psi)_g^+$ and $(d\theta)_g^- = -(\delta\Psi)_g^-$. In particular, the 2-form

$$\alpha - 2(d\theta)_g^+ = \alpha_h$$

is harmonic and the 2-form

$$\alpha + 2(d\theta)_g^- = \alpha_h + 2d\theta$$

is closed.

Proof. Since $*\alpha = \alpha$, by the uniqueness of the Hodge decomposition, we have $*(d\theta) = \delta\Psi$, $*(\delta\Psi) = d\theta$. The lemma follows. ■

Remark 2.5. The decomposition $\alpha = \alpha_h + 2(d\theta)_g^+$ for a selfdual form α can also be seen as the Hodge decomposition for Ω_g^+ associated to the elliptic differential complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega_g^+ \longrightarrow 0.$$

Suppose now that J is an almost complex structure and g is a J-compatible Riemannian metric on the four-manifold M in the sense that g is J-invariant, i.e. $g(Ju, Jv) = g(u, v)$. The pair (g, J) defines a J-invariant 2-form ω by

$$\omega(u, v) = g(Ju, v). \tag{6}$$

Such a triple (J, g, ω) is called an almost Hermitian structure. An almost Hermitian structure (J, g, ω) is called almost Kähler if ω is closed.

Given J, we can always choose a compatible g. The relations between the decompositions (1) and (4) on a four-dimensional almost Hermitian manifold are

$$\Lambda_J^+ = \underline{\mathbb{R}}(\omega) \oplus \Lambda_g^-, \tag{7}$$

$$\Lambda_g^+ = \underline{\mathbb{R}}(\omega) \oplus \Lambda_J^-, \tag{8}$$

$$\Lambda_J^+ \cap \Lambda_g^+ = \underline{\mathbb{R}}(\omega), \quad \Lambda_J^- \cap \Lambda_g^- = 0. \tag{9}$$

The following lemma is an immediate consequence of (8).

Lemma 2.6. Let (M^4, g, J, ω) be a four-dimensional almost Hermitian manifold. Then $\mathcal{Z}_J^- \subset \mathcal{H}_g^+$ and the natural map $\mathcal{Z}_J^- \to H_J^-$ is bijective. More precisely, if $\mathcal{H}_g^{+,\omega^\perp}$ denotes the subspace of harmonic selfdual forms pointwise orthogonal to ω, we have

$$H_J^- = \mathcal{Z}_J^- = \mathcal{H}_g^{+,\omega^\perp}. \tag{10}$$

In particular, any closed J-antiinvariant form α ($\alpha \not\equiv 0$) is nondegenerate on an open dense subset $M' \subseteq M$.

Proof. Since $\Lambda_J^- \subset \Lambda_g^+$, a closed J-antiinvariant 2-form is a selfdual harmonic form. In particular, there exists no nontrivial exact J-antiinvariant 2-form. Thus, the natural map $\mathcal{Z}_J^- \to H_J^-$ is bijective. The equality (identification) (10) is obvious. For the last statement, note that any selfdual form is nondegenerate on the complement of its nodal set $M' = M \setminus \alpha^{-1}(0)$. On the other hand, any harmonic form satisfies the unique continuation property, so if $\alpha \not\equiv 0$, its nodal set $\alpha^{-1}(0)$ has empty interior. In fact, from [2] it is known more: $\alpha^{-1}(0)$ has Hausdorff dimension ≤ 2. ∎

We are now ready to give the proof of Theorem 2.3

Proof of Theorem 2.3. Let g be a J-compatible Riemannian metric, and let ω be the 2-form defined by (g, J). We start by proving that J is C^∞-pure. If $\mathfrak{a} \in H_J^+ \cap H_J^-$, let $\alpha' \in \mathcal{Z}_J^+$, $\alpha'' \in \mathcal{Z}_J^-$, be representatives for $\mathfrak{a}$. Then

$$\mathfrak{a} \cup \mathfrak{a} = \int_M \alpha' \wedge \alpha'' = 0,$$

but by Lemma 2.6, we also have

$$\mathfrak{a} \cup \mathfrak{a} = \int_M \alpha'' \wedge \alpha'' = \int_M |\alpha''|_g^2 \, d\mu_g.$$

Thus, $\alpha'' = 0$, so $\mathfrak{a} = 0$.

Next, we prove that J is C^∞-full. Suppose the contrary. Then there exists a class $\mathfrak{a} \in H^2(M; \mathbb{R})$ which is (cup product) orthogonal to $H_J^+ \oplus H_J^-$. Since $H_g^- \subset H_J^+$, we can assume $\mathfrak{a} \in H_g^+$. Let α be the harmonic, selfdual representative of $\mathfrak{a}$ and denote $f =< \alpha, \omega >$. The function f is not identically zero, as otherwise it follows from Lemma 2.6 that $\mathfrak{a} \in H_J^-$. Now we apply Lemma 2.4 to the selfdual form $f\omega$. The closed form $(f\omega)_h + 2(f\omega)^{\mathrm{exact}}$ is also J-invariant; indeed, it is equal to $f\omega + 2((f\omega)^{\mathrm{exact}})_g^-$. (Here and later, we shall denote α^{exact} the exact part from the Hodge decomposition of a form α.) Thus,

$(f\omega)_h + 2(f\omega)^{\text{exact}}$ is a representative for a class $\mathfrak{b} \in H_J^+$. But

$$\begin{aligned}\mathfrak{a} \cup \mathfrak{b} &= \int_M < \alpha, (f\omega)_h + 2(f\omega)^{\text{exact}} > \; d\mu_g \\ &= \int_M < \alpha, f\omega + 2((f\omega)^{\text{exact}})_g^- > \; d\mu_g = \int_M f^2 \, d\mu_g \neq 0.\end{aligned}$$

This contradicts the assumption that $\mathfrak{a}$ is orthogonal to $H_J^+ \oplus H_J^-$. ■

Remark 2.7. (i) Combining Theorem 2.3 with Theorem 3.7 from [6], it follows that any almost complex structure on a compact four-dimensional manifold is not just pure and full for forms, but for currents as well.

(ii) Theorem 2.3 does not generalize to higher dimensions. A six-dimensional almost complex manifold which is not C^∞-pure is given in Example 3.3 of [6]. Higher dimensional examples can be obtained from the following simple observation: if (M_1, J_1), (M_2, J_2) are almost complex manifolds and one of them is not C^∞-pure, then $(M_1 \times M_2, J_1 \oplus J_2)$ is not C^∞-pure either.

By contrast, note the following result (also proved in [6], Proposition 3.2).

Proposition 2.8. If J is an almost complex structure on a compact manifold M^{2n} and J admits a compatible symplectic structure, then J is C^∞-pure.

Proof. On any almost Hermitian manifold (M^{2n}, g, J, ω), if $\alpha \in \Omega_J^-$, then

$$*_g(\alpha) = \alpha \wedge \omega^{n-2}. \tag{11}$$

Thus, if ω is symplectic and α is closed, (11) implies that $*_g(\alpha)$ is also closed. Hence, for any almost Kähler structure (g, J, ω), $\mathcal{Z}_J^- \subset \mathcal{H}_g^2$. It is straightforward now to generalize the first part of the proof of Theorem 2.3. Let $\mathfrak{a} \in H_J^+ \cap H_J^-$, and let $\alpha' \in \mathcal{Z}_J^+$, $\alpha'' \in \mathcal{Z}_J^-$ be representatives for $\mathfrak{a}$. Then

$$\mathfrak{a} \cup \mathfrak{a} \cup [\omega]^{n-2} = \int_M \alpha' \wedge \alpha'' \wedge \omega^{n-2} = 0,$$

but by (11), we also have

$$\mathfrak{a} \cup \mathfrak{a} \cup [\omega]^{n-2} = \int_M \alpha'' \wedge \alpha'' \wedge \omega^{n-2} = \int_M |\alpha''|_g^2 \, d\mu_g.$$

Thus, $\alpha'' = 0$, so $\mathfrak{a} = 0$. ■

2.3 The complexified H^2

2.3.1 *The groups* $H_J^{p,q}$

In all of the above, we referred to decompositions of *real* 2-forms. We now present the relation with the more familiar splitting of bigraded *complex* 2-forms:

$$\Lambda^2_{\mathbb{C}} = \Lambda_J^{2,0} \oplus \Lambda_J^{1,1} \oplus \Lambda_J^{0,2}. \tag{12}$$

The relation between the decompositions (1) and (12) is well known:

$$\begin{aligned} \Lambda_J^+ &= \left(\Lambda_J^{1,1}\right)_{\mathbb{R}}, \\ \Lambda_J^- &= \left(\Lambda_J^{0,2} \oplus \Lambda_J^{2,0}\right)_{\mathbb{R}}. \end{aligned} \tag{13}$$

Note that the bundle Λ_J^- inherits an almost complex structure, still denoted J, by

$$\beta \in \Lambda_J^- \;\rightarrow\; J\beta \in \Lambda_J^-, \text{ where } J\beta(X, Y) = -\beta(JX, Y).$$

Definition 2.9. Let $H_J^{p,q}$ be the subspace of the complexified de Rham cohomology $H^2(M;\mathbb{C})$, consisting of classes which can be represented by a complex closed form of type (p, q).

Lemma 2.10. The groups $H_J^{p,q}$ have the following properties:

$$H_J^{p,q} = \overline{H_J^{q,p}}, \tag{14}$$

$$\begin{aligned} H_J^{p,p} &= \left(H_J^{p,p} \cap H^{2p}(M;\mathbb{R})\right) \otimes \mathbb{C}, \\ \left(H_J^{p,q} + H_J^{q,p}\right) &= \left(\left(H_J^{p,q} + H_J^{q,p}\right) \cap H^{p+q}(M;\mathbb{R})\right) \otimes \mathbb{C}. \end{aligned} \tag{15}$$

Proof. Relation (14) follows from the fact that a complex form Ψ is closed if and only if its conjugate $\overline{\Psi}$ is closed. The equalities in (15) follow from (14) and the following fact: let V be a real vector space and W a complex subspace of $V \otimes_{\mathbb{R}} \mathbb{C}$, which as a subspace is invariant under conjugation. Then W is the complexification of $W \cap V$ (see Remark 2.5 on p.139 in [3]). ■

We now investigate the relation between the groups $H_J^{\pm}$ and $H_J^{p,q}$. As we shall see in Lemma 2.12, when J is not integrable, there is an important difference compared to what (13) would have predicted.

Lemma 2.11. For a compact almost complex manifold (M, J) of any dimension,

$$H_J^+ = H_J^{1,1} \cap H^2(M; \mathbb{R}), \tag{16}$$

and

$$H_J^{1,1} = H_J^+ \otimes_{\mathbb{R}} \mathbb{C}. \tag{17}$$

Proof. The relation (17) is a consequence of (16) and (15) with $(p, p) = (1, 1)$. So we just need to prove (16).

The inclusion $H_J^+ \subseteq H_J^{1,1} \cap H^2(M; \mathbb{R})$ is clear, so we now prove the converse inclusion. An element in $H_J^{1,1} \cap H^2(M; \mathbb{R})$ can be represented by a complex d closed (1,1) form $\rho = \sigma + d\tau$, with σ a d closed real form. So it is also represented by the real d closed (1,1) form $\frac{1}{2}(\rho + \bar{\rho}) = \sigma + d(\tau + \bar{\tau})$.

When J is integrable, the same argument appears in the proof of Theorem 2.13 in [3]. ■

The next lemma is a well-known result (see e.g. [9]), recast in our terminology. It can also be seen as a consequence and as a slight extension of Hitchin's lemma [7].

Lemma 2.12. Let J be an almost complex structure on a compact four-manifold.

$$\left(H_J^{2,0} + H_J^{0,2}\right) = \begin{cases} H_J^- \otimes_{\mathbb{R}} \mathbb{C}, & \text{if } J \text{ is integrable,} \\ 0, & \text{if } J \text{ is not integrable.} \end{cases} \tag{18}$$

In particular, if J is integrable, then

$$H_J^- = \left(H_J^{2,0} + H_J^{0,2}\right) \cap H^2(M; \mathbb{R}). \tag{19}$$

Proof. A (complex) form $\Phi \in \Omega_J^{2,0}$ is of the form

$$\Phi = \beta + iJ\beta, \text{ where } \beta \in \Omega_J^-.$$

Assume $\beta \not\equiv 0$. The point of the lemma is that $d\beta = 0$ and $d(J\beta) = 0$ occur simultaneously if and only if J is integrable. To see this, let $Z_j = X_j - iJX_j$, $j = 1, 2, 3$ be arbitrary (1,0) vector fields. Then

$$d\Phi(Z_1, \overline{Z_2}, \overline{Z_3}) = -\Phi([\overline{Z_2}, \overline{Z_3}]^{1,0}, Z_1).$$

Assuming $d\beta = d(J\beta) = 0$, i.e. $d\Phi = 0$, the above relation implies $[\overline{Z_2}, \overline{Z_3}]^{1,0} = 0$. This follows first on the set $M' = M \setminus \beta^{-1}(0)$, but then everywhere on M by continuity, since M' is dense in M (see Lemma 2.6). This implies the integrability of J.

Conversely, assume that J is integrable and we want to show that $d\beta = 0$ iff $d(J\beta) = 0$. Using $d = \partial + \bar{\partial}$ and $2\beta = \Phi + \bar{\Phi}$, we have

$$2d\beta = (\partial + \bar{\partial})(\Phi + \bar{\Phi}) = \bar{\partial}\Phi + \partial\bar{\Phi}\ .$$

(We used that $\partial\Phi = 0$ since it is a (3,0) form on a complex surface.) Thus, $d\beta = 0$ iff $\bar{\partial}\Phi = 0$. Similarly, $d(J\beta) = 0$ iff $\bar{\partial}(i\Phi) = 0$. But it is obvious that $\bar{\partial}\Phi = 0$ iff $\bar{\partial}(i\Phi) = 0$. ■

Remark 2.13. There are examples of nonintegrable almost complex structures for which the real group H_J^- is nonzero, although, as shown above, the complex group $H_J^{2,0} + H_J^{0,2}$ is always zero in this case. See Example 2.18 and the remark that follows.

By the above two lemmas, we get the following corollary.

Corollary 2.14. Suppose J is an almost complex structure on a compact four-manifold. Then J is always complex C^∞-pure in the sense $H_J^{1,1} \cap H_J^{2,0} \cap H_J^{0,2} = \{0\}$. Moreover, J is also complex C^∞-full, i.e.

$$H^2(M; \mathbb{C}) = H_J^{1,1} \oplus H_J^{2,0} \oplus H_J^{0,2},$$

if and only if J is integrable or $h_J^- = 0$.

2.3.2 *Dolbeault decomposition when J is integrable*

When J is integrable, there is the Dolbeault decomposition which has long been discovered. We briefly recall this decomposition and relate it to the groups $H_J^{p,q}$ introduced in Section 2.3.1.

The Fröhlicher spectral sequence of the double complex

$$(\Omega^*(M) \otimes \mathbb{C} = \oplus\Omega^{p,q}, \partial, \bar{\partial})$$

reads (see p.41–5, p.140–1 in [3]):

$$E_1^{p,q} = H_{\bar{\partial}}^{p,q}(M) \Rightarrow H^{p+q}(M; \mathbb{C}).$$

The resulting Hodge filtration on $H^2(M; \mathbb{C})$ reads:

$$H^2(M; \mathbb{C}) = F^0(H^2) \supset F^1(H^2) \supset F^2(H^2) \supset 0,$$

where

$$F^p(H^2) = \{[\alpha], \alpha \in \oplus_{p'+q'=2, p'\geq p}\Omega^{p',q'} | d\alpha = 0\}. \tag{20}$$

Since

$$H^{p,q}_{\bar\partial}(M) = E^{p,q}_1 \to E^{p,q}_\infty = \frac{F^p(H^{p+1}(M;\mathbb{C}))}{F^{p+1}(H^{p+1}(M;\mathbb{C}))},$$

if the Fröhlicher spectral sequence degenerates at E_1, then

$$H^{p,q}_{\bar\partial}(M) \cong \frac{F^p(H^{p+1}(M;\mathbb{C}))}{F^{p+1}(H^{p+1}(M;\mathbb{C}))}. \tag{21}$$

For $p+q=2$, let

$$'H^{p,q}(M) = F^p(H^2) \cap \overline{F^q(H^2)}. \tag{22}$$

Lemma 2.15. $'H^{p,q}$ consists of de Rham classes which can be represented by a form of type (p,q), i.e

$$'H^{p,q} = H^{p,q}_J. \tag{23}$$

This should be known to experts; we record the argument here since it is useful to elucidate the relation between H^+_J and $H^{1,1}_{\bar\partial}$.

Proof. $F^2(H^2)$ consists of de Rham classes which can be represented by a form of type $(2,0)$. Consequently, $\overline{F^2(H^2)}$ consists of classes of $(0,2)$ forms.

It remains to show that $F^1(H^2) \cap \overline{F^1(H^2)}$ consists of de Rham classes which can be represented by a closed form of type $(1,1)$. First of all, every such de Rham class lies in $F^1(H^2)$ and $\overline{F^1(H^2)}$. On the other hand, by definition, a class is in $F^1(H^2) \cap \overline{F^1(H^2)}$ if and only if it is represented by closed forms $\alpha_1 = \alpha_1^{1,1} + \alpha_1^{2,0}$ and $\alpha_2 = \alpha_2^{1,1} + \alpha_2^{0,2}$. Now $\alpha_1 - \alpha_2 = d\beta$, and it is easy to see that $\alpha_1 - d\beta^{1,0} = \alpha_2 + d\beta^{0,1}$ is a d closed (1,1) form representing the same class. ∎

A weight 2 *formal* Hodge decomposition is a decomposition of the form

$$H^2(M;\mathbb{C}) = \oplus_{p+q=2}{}'H^{p,q}. \tag{24}$$

Theorem 2.16 [3]**.** If (M,J) is a Kähler manifold or a complex surface, then the Fröhlicher spectral sequence degenerates at E_1, and there is a weight 2 formal Hodge decomposition.

Consequently,

$$
\begin{aligned}
H_{\bar\partial}^{2,0} &= E_\infty^{2,0} \cong F^2(H^2) \cong {}'H^{2,0},\\
H_{\bar\partial}^{1,1} &= E_\infty^{1,1} \cong \frac{F^1(H^2)}{F^2(H^2)} \cong F^1(H^2) \cap \overline{F^1(H^2)} \cong {}'H^{1,1},\\
H_{\bar\partial}^{0,2} &= E_\infty^{0,2} \cong \frac{H^2(M;\mathbb{C})}{F^1(H^2)} \cong \overline{F^2(H^2)} \cong {}'H^{0,2}.
\end{aligned} \tag{25}
$$

Together with (23), (16), and (19), we conclude the following theorem.

Proposition 2.17. If J is integrable on a compact four-manifold, then

$$H_J^{p,q} = H_{\bar\partial}^{p,q}, \tag{26}$$

and

$$H_J^+ = H_{\bar\partial}^{1,1} \cap H^2(M;\mathbb{R}), \quad H_J^- = \left(H_{\bar\partial}^{2,0} \oplus H_{\bar\partial}^{0,2}\right) \cap H^2(M;\mathbb{R}). \tag{27}$$

Let us denote the dimension of $H_J^\pm$ by $h_J^\pm$. When J is integrable, it follows from Proposition 2.17 that

$$h_J^+ = h_{\bar\partial}^{1,1}, \quad h_J^- = 2h_{\bar\partial}^{2,0}. \tag{28}$$

Together with the signature theorem (Theorem 2.7 in [3]), we get

$$h_J^+ = \begin{cases} b^- + 1 & \text{if } b_1 \text{ even} \\ b^- & \text{if } b_1 \text{ odd}, \end{cases} \quad h_J^- = \begin{cases} b^+ - 1 & \text{if } b_1 \text{ even} \\ b^+ & \text{if } b_1 \text{ odd}. \end{cases} \tag{29}$$

It is a deep, but now well-known fact that the cases b_1 even/odd correspond to whether the complex surface (M, J) admits or not a compatible Kähler structure.

Notice that when J is integrable the dimensions $h_J^\pm$ are topological invariants. Such properties will not hold for general almost complex structures. In fact, we conjecture that for generic almost complex structures $h_J^- = 0$. However, there are examples of nonintegrable almost complex structures with $h_J^- \neq 0$. Here is one simple construction of such examples.

Example 2.18. Let (M, g, J_0, ω_0) be a compact Kähler surface with $b^+ \geq 3$, and let Φ be a (not identically zero) holomorphic (2,0) form on M (existence of such Φ is guaranteed by the assumption $b^+ \geq 3$). Let $\beta = Re(\Phi)$, $J_0\beta = Im(\Phi)$ be the real and imaginary parts of Φ. Both β and $J_0\beta$ are closed J_0-antiinvariant forms. Let $f \in C^\infty(M)$ be an arbitrary, not identically zero, smooth function and consider the form $\omega_{f,\beta} = \omega_0 + f\beta$. Because β

is pointwise orthogonal to ω_0, the form $\omega_{f,\beta}$ is nondegenerate everywhere. Since $\omega_{f,\beta}$ is also g-selfdual, it induces a g-compatible almost complex structure J on M. J is not integrable except the case when $f = constant$ and (M, g, J_0) is hyper-Kähler (see, for instance [1]). On the other hand, $J_0\beta$ is a nontrivial closed J-antiinvariant form. The last statement is true because $J_0\beta$ is pointwise orthogonal to both ω_0 and β. Thus, H_J^- is nontrivial.

Remark 2.19. In the above example, J and J_0 are what we call metric related almost complex structures, as they share a common compatible metric. In [5], we compute the exact values of $h_J^\pm$ for all almost complex structures J which are metric related to integrable ones. Example 6.2 of [6] exhibits a compact four-manifold which admits no integrable complex structures, but which admits an almost complex structure with $h_J^- = 1$.

3 Estimates for $h_J^\pm$ when J is Tamed by a Symplectic Form

From Theorem 2.3, on any compact four-dimensional almost complex manifold (M, J), we have

$$h_J^+ + h_J^- = b_2. \tag{30}$$

The decomposition (8) also leads to the following immediate estimates

$$h_J^+ \geq b^-, \quad h_J^- \leq b^+. \tag{31}$$

One reason for our interest in $H_J^\pm$ stems from the following fact. If J admits compatible symplectic forms, then the set of all such forms, the J-compatible cone, $\mathcal{K}_J^c(M)$, is a (nonempty) open convex cone of $H_J^+(M)$ [8]. Thus, it is important to determine the dimension h_J^+ of $H_J^+(M)$.

In light of the question of Donaldson mentioned in the Introduction, it is also interesting to obtain information on the dimension h_J^+ in the case when J is just tamed by symplectic forms.

It was shown in [8] that an integrable J admits compatible Kähler structures if and only if it admits tamed symplectic forms. Thus, we can state (29) in this context as follows:

$$h_J^+ = \begin{cases} b^- + 1 & \text{if } J \text{ is tamed and integrable,} \\ b^- & \text{if } J \text{ is nontamed and integrable.} \end{cases} \tag{32}$$

3.1 A general estimate

When J admits a compatible symplectic form, we have the following easy improvement of (31).

Proposition 3.1. If J is almost Kähler, then

$$h_J^+ \geq b^- + 1, \quad h_J^- \leq b^+ - 1. \tag{33}$$

Actually, (33) can be obtained in a slightly more general setting from the following lemma.

Lemma 3.2. Suppose (M, g, J, ω) is a compact four-dimensional almost Hermitian manifold. Assume that the harmonic part ω_h of the Hodge decomposition of ω is not identically zero. Then (33) holds.

Proof. Let $\omega = \omega_h + d\theta + \delta\Psi$ be the Hodge decomposition of ω. From Lemma 2.4, $\omega + 2(d\theta)^- = \omega_h + 2d\theta$ is a closed J-invariant 2-form. By assumption, it represents a nontrivial cohomology class in $H_g^+ \cap H_J^+$ and the estimates follow. ■

Of course, if (M, g, J, ω) is almost Kähler, $\omega = \omega_h$, so Proposition 3.1 is obvious. More interestingly, Lemma 3.2 implies that the estimates (33) hold for tamed J's as well.

Theorem 3.3. Suppose J is tamed by a symplectic form ω. Then the estimates (33) still hold.

Proof. Write

$$\omega = \omega' + \omega'' \tag{34}$$

with $\omega' \in \Omega_J^+$ and $\omega'' \in \Omega_J^-$. Explicitly,

$$\omega'(v, w) = \frac{1}{2}\omega(v, w) + \frac{1}{2}\omega(Jv, Jw). \tag{35}$$

Then ω' is compatible with J and nondegenerate, thus it determines a Riemannian metric g. From the pair (ω, J), we actually get a conformal class of metrics, these for which $\Lambda_g^+ = \mathrm{Span}\{\omega, \Lambda_J^-\}$. The metric we fixed is singled out by imposing that $|\omega'|^2 = 2$.

We show that Lemma 3.2 can be applied to the almost Hermitian structure (g, J, ω'). It is enough to show that the harmonic part ω'_h is not identically zero. This is true because the following cup product is nonzero:

$$[\omega'_h] \cup [\omega] = \int_M \omega'_h \wedge \omega = \int_M (\omega'_h + 2d\theta) \wedge \omega =$$
$$= \int_M (\omega' + 2(d\theta)^-_g) \wedge (\omega' + \omega'') = \int_M \omega' \wedge \omega' \neq 0. \qquad \blacksquare$$

The following corollary is an immediate consequence.

Corollary 3.4. If $b^+ = 1$ and J is tamed, then

$$h^+_J = 1 + b^- = b_2, \quad h^-_J = b^+ - 1 = 0. \tag{36}$$

Remark 3.5. Lemma 3.2 can also be applied to show that if $b^+ \geq 1$, then the estimates (33) even hold for generic nontamed almost complex structures J (but not all in view of (32)).

It has been shown in [10] that nontamed almost complex structures exist in any path-connected component of almost complex structures.

3.2 A formulation of Donaldson's question

We end this section by giving an equivalent formulation of Question 1.1. Suppose $\tilde{J}$ is an almost complex structure that is tamed by a symplectic form ω on a compact four-manifold M. As noted in the proof of Theorem 3.3, the pair $(\tilde{J}, \omega)$ gives rise to a conformal class of Riemannian metrics $[g]$, so that $\Lambda^+_{[g]} = \mathrm{Span}\{\omega, \Lambda^-_{\tilde{J}}\}$. In the proof of Theorem 3.3, we chose in this conformal class the metric that made ω', the $\tilde{J}$-invariant part of ω, have pointwise norm $\sqrt{2}$.

For the comments below, we prefer to use another natural metric in this conformal class: we choose the metric g so that $|\omega|^2_g = 2$ pointwise on M. Equivalently, g is chosen so that g and ω induce an almost Kähler structure (g, J, ω). Certainly, $\tilde{J}$ is also g-compatible, and let $\tilde{\omega}$ be the fundamental 2-form of $(g, \tilde{J})$. Then

$$\tilde{\omega} = f\omega + \gamma, \text{ with } \gamma \in \Omega^-_J, \ f \in C^\infty(M) \text{ so that } 2f^2 + |\gamma|^2 = 2. \tag{37}$$

Since $\tilde{J}$ is tamed by ω, the function f is strictly positive on M. Thus, we can think that $\tilde{J}$ is induced by the metric g and the 2-form $\omega + \frac{1}{f}\gamma$, up to conformal rescaling by f.

Conversely, let (M^4, g, J, ω) be an almost Kähler manifold and let $\alpha \in \Omega^-_J$. Denote $\tilde{\omega}_\alpha = \omega + \alpha$. This is a nondegenerate, g selfdual form, so (up to a conformal normalization) it induces another g-compatible almost complex structure which we denote $\tilde{J}_\alpha$. It is clear that $\tilde{J}_\alpha$ is tamed by ω.

Donaldson's Question 1.1 is equivalent to the following question.

Question 3.6. Is it true that for any almost Kähler manifold (M^4, g, J, ω) and any $\alpha \in \Omega_J^-$, the almost complex structure $\tilde{J}_\alpha$ is compatible with a symplectic form?

Using this setup and Lemma 2.4, we obtain the following partial result.

Proposition 3.7. With the notations above, if the 2-form α satisfies the pointwise condition

$$2 + |\alpha|^2 - 4|(\alpha^{\text{exact}})_g^-|^2 > 0, \tag{38}$$

then $\tilde{J}_\alpha$ is compatible with a symplectic form.

Proof. We just apply Lemma 2.4 to $\tilde{\omega}_\alpha = \omega + \alpha$. The form

$$\tilde{\omega}_\alpha + 2\left(\tilde{\omega}_\alpha^{\text{exact}}\right)_g^- = \omega + \alpha + 2(\alpha^{\text{exact}})_g^-$$

is closed and $\tilde{J}_\alpha$-invariant. Condition (38) is equivalent to this form being pointwise positive definite. ■

Remark 3.8. When α is closed (hence harmonic), condition (38) is trivially satisfied. In this case, $\tilde{\omega}_\alpha$ is itself a symplectic form. Proposition 3.7 basically says that if α is not too far from being closed, then $\tilde{J}_\alpha$ is compatible with a symplectic form. The result can be seen in relation with the openness result of Donaldson [4].

If α does not satisfy (38), Lemma 2.4 may still help in the search for a symplectic form compatible with $\tilde{J}_\alpha$. Let (M^4, g, J, ω) be the fixed almost Kähler structure. Note that by (7) any $\tilde{J}_\alpha$-invariant form Ω_α can be written as

$$\Omega_\alpha = f\tilde{\omega}_\alpha + \theta, \text{ with } f \in C^\infty(M) \text{ and } \theta \in \Omega_g^-.$$

Applying Lemma 2.4 to $f\tilde{\omega}_\alpha$, we get that Ω_α is also closed if and only if $\tilde{\theta} = \theta - 2((f\tilde{\omega}_\alpha)^{\text{exact}})_g^-$ is closed, hence harmonic. Thus, a potential symplectic form Ω_α which is $\tilde{J}_\alpha$-compatible must be of the type

$$\Omega_\alpha = f\tilde{\omega}_\alpha + 2((f\tilde{\omega}_\alpha)^{\text{exact}})_g^- + \tilde{\theta}, \text{ with } f \in C^\infty(M) \text{ and } \tilde{\theta} \in \mathcal{H}_g^-.$$

Now the question becomes how should one choose $f \in C^\infty(M)$ and $\tilde{\theta} \in \mathcal{H}_g^-$ to satisfy $\Omega_\alpha^2 > 0$ everywhere on M.

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