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Local models and integrability of certain almost Kähler 4-manifolds

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We classify, up to a local isometry, all non-Kähler almost Kähler 4-manifolds for which the fundamental 2-form is an eigenform of the Weyl tensor, and whose Ricci tensor is invariant with respect to the almost complex structure. Equivalently, such almost Kähler 4-manifolds satisfy the third curvature condition of A. Gray. We use our local classification to show that, in the compact case, the third curvature condition of Gray is equivalent to the integrability of the corresponding almost complex structure.

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1. Introduction

Let (M, Ω) be a symplectic manifold of dimension $2n$. An almost complex structure J is called *compatible* with the symplectic form Ω, if there exists a Riemannian metric g such that

$$\Omega(\cdot, \cdot) = g(J\cdot, \cdot).$$

In this case, the metric g is also called compatible with Ω, while the triple (g, J, Ω) is referred to as an *almost Kähler structure* on M. If, additionally, the compatible almost complex structure J is integrable, then (g, J, Ω) is a Kähler structure on M. Almost Kähler manifolds for which the almost complex structure J is not integrable will be called *strictly* almost Kähler.

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Gromov's theory of pseudo-holomorphic curves [27], Taubes' characterization of Seiberg-Witten invariants of symplectic 4-manifolds [45,46] and the recent works of Donaldson [17,18] conclusively situated the study of (strictly) almost Kähler manifolds into the mathematical main-stream. A very recent work of LeBrun [38], for instance, inspired by some aspects of Taubes' construction of solutions of the Seiberg-Witten equations on symplectic 4-manifolds, relates the existence of strictly almost Kähler 4-manifolds having particular properties of the curvature with some fundamental problems in Riemannian Geometry, such as the existence and the uniqueness of Einstein metrics, or of metrics minimizing volume among all Riemannian metrics satisfying point-wise lower bounds on sectional curvatures.

Despite the now compelling appeal of the "Riemannian" aspect of almost Kähler geometry, the subject suffered for a long time from a genuine lack of interesting examples. For instance, an old still open conjecture of Goldberg [23] affirms that there are no *Einstein*, strictly almost Kähler metrics on a compact symplectic manifold. K. Sekigawa proved [44] that the conjecture is true if the scalar curvature is non-negative. The case of negative scalar curvature is still wide open, despite of the recent discovery of *complete* Einstein strictly almost Kähler manifolds of any dimension $2n \geq 6$ [6], and of (local) Ricci-flat strictly almost Kähler metrics of dimension four [4,9,41]. Indeed, in addition to some subtle topological obstructions in dimension four [32,36–38], it turns out that there is a number of (rather not well understood yet) local obstructions to the existence of Einstein strictly almost Kähler metrics compatible with a given symplectic form [2]. Further progress on the Goldberg conjecture seems to hinge on a better understanding of these obstructions.

Classically, on a compact manifold M, the Einstein condition appears as the Euler-Lagrange equation of the *Hilbert functional* **S**, the total scalar curvature, acting on the space of all Riemannian metrics on M of a given volume. A "symplectic" setting of this variational problem was first considered by Blair and Ianus [13]: let (M, Ω) be a compact symplectic manifold and restrict the functional **S** to the space of Ω-compatible Riemannian metrics, then the critical points are the almost Kähler metrics (g, J, Ω) whose Ricci tensor Ric is J-invariant, i.e. satisfies:

$$\mathrm{Ric}(J\cdot, J\cdot) = \mathrm{Ric}(\cdot, \cdot). \tag{1}$$

The Euler-Lagrange equation (1) is a weakening of both the Einstein and the Kähler conditions. Furthermore, Blair [12] observed that for any almost Kähler metric (g, J) the following relation holds:

$$\frac{1}{4}\int_M |\nabla J|^2 d\mu + \mathbf{S}(g) = \frac{4\pi}{(n-1)!}(c_1 \cdot [\Omega]^{\wedge(n-1)})(M),$$

where ∇ is the Riemannian connection of g, $|\cdot|$ is the point-wise norm induced by g, and c_1 and $d\mu = \frac{1}{n!}\Omega^{\wedge n}$ are respectively the first Chern class and the the volume form of (M, Ω). Note that the right hand-side of the above equality is a symplectic invariant (hence is independent of the choice of a particular Ω-compatible metric); thus the Ω-compatible almost Kähler structures satisfying (1) are also the critical points of the *Energy functional*, which acts on the space of Ω-compatible almost complex structures by

$$\mathbf{E}(J) = \int_M |\nabla J|^2 d\mu.$$

From this point of view, critical almost Kähler metrics have been recently studied in [34]. Clearly, the functional **E** (resp. **S**) is bounded from below (resp. from above), the Kähler metrics being minima of **E** (resp. maxima of **S**). However, a direct variational approach of finding extremal metrics for these functionals seems not to be easily applicable as it may happen that the infimum of **E** be zero, although M does not carry Kähler structures at all (see [34]). In fact, apart from the explicit compact examples of [1,16] (which, multiplied by Kähler manifolds, provide examples of any dimension $2n \geq 6$), almost nothing seems to be known in general about existence of compact strictly almost Kähler metrics satisfying (1). In particular, no such example is yet known on a compact symplectic 4-manifold.

One reason for which some technical difficulties appear in applying global methods is perhaps hidden in the fact that even locally the equation (1) is difficult to be solved. Indeed, it consists of a system of PDE's for a compatible metric with Ω, which does not satisfy the Cartan test [2]. One finds instead a number of nontrivial relations between higher jets of J and the curvature of g. On a compact manifold precisely these relations are at the origin of the integrability results obtained in [2,5,9,20,44]. The few known (local) examples appear, in fact, as a byproduct of additional geometric structure which makes the study of the Euler-Lagrange equation (1) more tractable.

The additional structure relevant to this paper comes as an extra-assumption imposed on the curvature tensor of the almost Kähler 4-manifold, namely

$$W(\Omega) = \nu\Omega, \tag{2}$$

where W is the Weyl tensor of g, viewed as a symmetric traceless endomorphism acting on the space of 2-forms; thus, equation (2) is equivalent to Ω being an eigenform of the Weyl tensor, in which case the eigenfunction ν is smooth and equal, up to a factor $1/6$, to the so-called conformal scalar curvature κ of (g, J) (see Section 2.3 for a precise definition).

The condition (2) in its own right appears to be natural in the context of four dimensional almost Kähler geometry. Indeed, motivated by harmonic maps theory, C. Wood [48] showed that an almost Kähler 4-manifold (M, g, J, Ω) satisfies (2) if and only if J is a critical point of the energy functional **E**, but under variations through all almost complex structures compatible with the given metric g; for this reason such almost complex structures were called *harmonic* in [48]. Furthermore, almost Kähler metrics saturating the new curvature estimates of [38] must all satisfy (2).

Let us finally mention that each of the conditions (1) and (2) corresponds to the vanishing of a certain irreducible component of the curvature R under the action of $U(2)$ on the space of algebraic curvature tensors [47]. Taken together, (1) & (2) are equivalent to the more familiar *third curvature condition of Gray* (see [26]):

$$R_{XYZU} = R_{JXJYJZJU}. \tag{3}$$

Evidently, the curvature of a Kähler manifold satisfies (3), hence any of the conditions (1) and (2).

In [3], we have constructed an explicit family of strictly almost Kähler metrics that all satisfy the third Gray condition (3). The main purpose of this paper is to close the circle of ideas from our previous work [3], by showing that, conversely, any strictly almost Kähler 4-manifold which satisfies (3) is locally modeled by a metric in this family:

Theorem 1. *Let $(\Sigma, g_\Sigma, \Omega_\Sigma)$ be an oriented Riemann surface with metric g_Σ and volume form Ω_Σ, and let $h = w + iv$ be a non-constant holomorphic function on Σ, whose real part w is everywhere positive. On the product of Σ with $\mathbb{R}^2 = \{(z, t)\}$ consider the symplectic form*

$$\Omega = \Omega_\Sigma - dz \wedge dt \tag{4}$$

and the compatible Riemannian metric

$$g = g_\Sigma + w dz^{\otimes 2} + \frac{1}{w}(dt + v dz)^{\otimes 2}; \tag{5}$$

Then,

(i) *(g, Ω) defines a strictly almost Kähler structure whose curvature satisfies the third Gray condition* (3)*.*
(ii) *For any connected strictly almost Kähler 4-manifold (M, g, Ω) whose curvature satisfies* (3) *there exists an open dense subset U with the following property: in a neighborhood of any point of U, (g, Ω) is homothetic to an almost Kähler structure given by* (4-5)*.*

This local classification of strictly almost Kähler 4-manifolds satisfying (3) includes as particular cases some previously known results from [3] and [9]. Indeed, the *Einstein* metrics in our family correspond to the case when (Σ, g_Σ) is a flat surface so that g is a Gibbons-Hawking metric [22] with respect to a translation-invariant harmonic function; we get precisely the selfdual Ricci-flat strictly almost Kähler examples of [9,41,43]. Furthermore, if we consider the half-plane realization of the hyperbolic space $(\mathbb{H}^2, g = \frac{dx^2+dy^2}{x^2})$ and take the holomorphic function $h = x + iy$, then (4-5) gives the four dimensional proper 3-symmetric space $(\mathrm{Isom}(\mathbb{E}^2) \cdot Sol_2)/SO(2)$ (see [25,33]), which can be characterized as the only strictly almost Kähler 4-manifold whose curvature tensor preserves the type decomposition of complex 2-forms [3] (a condition known as the *second curvature condition of Gray* [26]).

To prove Theorem 1 we use local methods which have originated from [9], and have been further developed in [3]. However, the more general result of the current paper requires two main new ingredients. The first comes in the form of a Unique Continuation Principle for the Nijenhuis tensor, N, of an almost Kähler 4-manifold satisfying (3) (Proposition 1), which allows us to realize the dense set U in Theorem 1 as the subset of points where N does not vanish. We then consider on U the orthogonal splitting of the tangent bundle TM into the sum of two linear complex sub-bundles, D and $D^{\perp}$, where $D = \{TM \ni X : \nabla_X J = 0\}$ is the so-called *Kähler nullity* of J, while its orthogonal complement, $D^{\perp}$, is identified with the canonical bundle K_J of J (see Sect. 2.2). This allows us to define another almost complex structure, I, which is equal to J when acting on D, but to $-J$ on $D^{\perp}$. The almost complex structure I is compatible with g, and induces the opposite orientation on M. Motivated by this fact, we shall often refer to (g, I) as the *opposite* almost Hermitian structure associated to (g, J). The second new ingredient, which is the heart of the proof of the above Theorem 1, consists of showing that for any strictly almost Kähler metric (g, J) satisfying (3), the opposite almost Hermitian structure (g, I) is, in fact, a Kähler structure (Proposition 2). Once this is achieved, Theorem 1 follows by [3, Theorem 2] (see also Theorem 3 below).

Note that the metric (5) of Theorem 1 is endowed with an $\mathbb{R}^2$-isometric action which is *surface-orthogonal* and preserves the symplectic form (4); such an action cannot be extended to a Hamiltonian toric action on a *compact* symplectic 4-manifold. One is then lead to suspect that there will be no compact examples of strictly almost Kähler 4-manifolds satisfying (3) (although there are complete ones [3]). We confirm this with the following integrability theorem, which is a direct generalization of results of [5] and [9]:

Theorem 2. *A compact almost Kähler 4-manifold whose curvature satisfies the third Gray condition* (3) *is necessarily Kähler* .

The proof of Theorem 2 involves the local result of Theorem 1 and the unique continuation property established in Proposition 1, together with some global arguments relying on the Enriques-Kodaira classification of compact complex surfaces and structure results for non-singular holomorphic foliations [15,24] and authomorphism groups [39] of ruled surfaces.

This article can be seen as a natural continuation (and conclusion) of our previous work [3]. The authors have nonetheless endeavored to make the current paper as self-contained as possible. The necessary background of almost Kähler geometry and a quick review of previous results is displayed in Sections 2 and 3 below. The proofs of Theorems 1 and 2 are then presented in Sections 4 and 5.

2. Elements of almost Kähler geometry

2.1. Type-decompositions of forms and vectors

Throughout the paper, (M, g, J, Ω) will denote an almost Kähler manifold of (real) dimension 4, where: g is a Riemannian metric, J is a g-orthogonal almost-complex structure – i.e. (M, g, J) is an almost Hermitian manifold – and $\Omega(\cdot,\cdot) = g(J\cdot,\cdot)$ is the fundamental 2-form of (g, J), which is closed, and therefore (M, Ω) is a symplectic manifold.

We denote by: TM the (real) tangent bundle of M; T^*M the (real) cotangent bundle; $\Lambda^r M$, $r = 1, \ldots, 4$ the bundle of real r-forms; $\langle\cdot,\cdot\rangle$ the inner product induced by g on these bundles (or on their tensor products), with the following conventions for the wedge product and for the inner product on forms:

$$(\alpha_1 \wedge \ldots \wedge \alpha_r)_{X_1,\ldots,X_r} = \det((\alpha_i(X_j))),$$

$$\langle \alpha_1 \wedge \ldots \wedge \alpha_r, \beta_1 \wedge \ldots \wedge \beta_r \rangle = \det(\langle \alpha_i, \beta_j \rangle),$$

where det denotes the determinant.

Using the metric, we shall implicitly identify vectors and covectors and, accordingly, a 2-form ϕ with the corresponding skew-symmetric endomorphism of the tangent bundle TM, by putting: $\langle \phi(X), Y \rangle = \phi(X, Y)$ for any vector fields X, Y. Also, if $\phi, \psi \in T^*M^{\otimes 2}$, by $\phi \circ \psi$ we understand the endomorphism of TM obtained by the composition of the endomorphisms corresponding to the two tensors.

The almost-complex structure J gives rise to a type decomposition of complex vectors and forms. By convention, J acts on the cotangent bundle T^*M by $(J\alpha)_X = -\alpha_{JX}$, so that J commutes with the Riemannian duality between TM and T^*M. We shall use the standard decomposition of the complexified cotangent bundle

$$T^*M \otimes \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M,$$

given by the $(\pm i)$-eigenspaces of J, the type decomposition of complex 2-forms

$$\Lambda^2 M \otimes \mathbb{C} = \Lambda^{1,1}M \oplus \Lambda^{2,0}M \oplus \Lambda^{0,2}M,$$

and the type decomposition of symmetric (complex) bilinear forms

$$S^2 M \otimes \mathbb{C} = S^{1,1}M \oplus S^{2,0}M \oplus S^{0,2}M.$$

Besides the type decomposition of complex forms, we shall also consider the splitting of real 2-forms into J-invariant and J-anti-invariant ones; for any real section of $\Lambda^2 M$ (resp. of $S^2 M$), we shall use the super-script $'$ to denote the projection to the real sub-bundle $\Lambda_{\mathbb{R}}^{1,1} M$ (resp. $S_{\mathbb{R}}^{1,1} M$) of J-invariant 2-forms (resp. symmetric 2-tensors), while the super-script $''$ stands for the projection to the bundle $[\![\Lambda^{0,2} M]\!]$ (resp. $[\![S^{0,2} M]\!]$) of J-anti-invariant ones; here and henceforth $[\![\cdot]\!]$ denotes the real vector bundle underlying a given complex bundle. Thus, for any section ψ of $\Lambda^2 M$ (resp. of $S^2 M$), we have the orthogonal splitting $\psi = \psi' + \psi''$, where

$$\psi'(\cdot,\cdot) = \frac{1}{2}(\psi(\cdot,\cdot) + \psi(J\cdot, J\cdot)) \text{ and } \psi''(\cdot,\cdot) = \frac{1}{2}(\psi(\cdot,\cdot) - \psi(J\cdot, J\cdot)).$$

We finally define the $U(2)$-decomposition (with respect to J) of real 2-forms

$$\Lambda^2 M = \mathbb{R}\cdot\Omega \oplus \Lambda_0^{1,1} M \oplus [\![\Lambda^{0,2} M]\!], \tag{6}$$

where $\Lambda_0^{1,1} M$ is the sub-bundle of the *primitive* (1,1)-forms, i.e. the J-invariant 2-forms which are point-wise orthogonal to Ω. The above splitting fits in with the well known $SO(4)$-decomposition of $\Lambda^2 M$

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$$

into the sub-bundles $\Lambda^{\pm} M$ of selfdual (resp. anti-selfdual) forms. Indeed, we have

$$\Lambda^+ M = \mathbb{R}\cdot\Omega \oplus [\![\Lambda^{0,2} M]\!], \quad \Lambda^- M = \Lambda_0^{1,1} M. \tag{7}$$

Note that $S_{\mathbb{R}}^{1,1} M$ can be identified with $\Lambda_{\mathbb{R}}^{1,1} M$ via the complex structure J: for any $S \in S_{\mathbb{R}}^{1,1} M$,

$$(S \circ J)(\cdot,\cdot) := S(J\cdot,\cdot)$$

is the corresponding element of $\Lambda_{\mathbb{R}}^{1,1} M$. Also, the real bundle $[\![\Lambda^{0,2} M]\!]$ (resp. $[\![S^{0,2} M]\!]$) inherits a canonical complex structure, still denoted by J, which is given by

$$(J\psi)(X,Y) := -\psi(JX, Y), \ \forall \psi \in [\![\Lambda^{0,2} M]\!],$$

so that $([\![\Lambda^{0,2} M]\!], J)$ becomes isomorphic to the *anti-canonical* bundle $K_J^{-1}(M)$ of M. We adopt a similar definition for the action of J on $[\![S^{0,2} M]\!]$. Notice that, using the metric g, $[\![S^{0,2} M]\!]$ can be also viewed as the bundle of symmetric, J-anti-commuting endomorphisms of TM.

2.2. The $U(2)$-decomposition of the curvature

The Riemannian curvature R is defined by $R_{X,Y}Z = (\nabla_{[X,Y]} - [\nabla_X, \nabla_Y])Z$, where ∇ is the Levi-Civita connection of g. Using the metric, it will be considered as a section of the bundle $S^2(\Lambda^2 M)$ of symmetric endomorphisms of $\Lambda^2 M$, or of the tensor product $\Lambda^2 M \otimes \Lambda^2 M$, depending on the context. The conformal part of R, the Weyl tensor W, commutes with the Hodge operator $*$ acting on 2-forms and, accordingly, splits as $W = W^+ + W^-$, where $W^\pm = \frac{1}{2}(W \pm W \circ *)$. W^+ is called the *selfdual Weyl tensor*; it acts trivially on $\Lambda^- M$ and will be considered in the sequel as a field of (symmetric, trace-free) endomorphisms of $\Lambda^+ M$. Similarly, the *anti-selfdual Weyl tensor*, W^-, will be considered as a field of endomorphisms of $\Lambda^- M$.

The Ricci tensor, Ric, is the symmetric bilinear form defined by $\mathrm{Ric}(X, Y) = \mathrm{tr}\,\{Z \to R_{X,Z}Y\}$; alternatively, $\mathrm{Ric}(X, Y) = \sum_{i=1}^4 \langle R_{X,e_i}Y, e_i\rangle$ for any g-orthonormal basis $\{e_i\}$. We then have $\mathrm{Ric} = \frac{s}{4}\,g + \mathrm{Ric}_0$, where s is the scalar curvature (that is, the trace of Ric with respect to g) and Ric_0 is the *trace-free Ricci tensor*. The latter can be made into a section of $S^2(\Lambda^2 M)$, then denoted by $\widetilde{\mathrm{Ric}_0}$, by putting $\widetilde{\mathrm{Ric}_0}(X \wedge Y) = \mathrm{Ric}_0(X) \wedge Y + X \wedge \mathrm{Ric}_0(Y)$; equivalently, for any section ϕ of $\Lambda^2 M$ we have

$$\widetilde{\mathrm{Ric}_0}(\phi) = \mathrm{Ric}_0 \circ \phi + \phi \circ \mathrm{Ric}_0.$$

It is readily checked that $\widetilde{\mathrm{Ric}_0}$ satisfies the first Bianchi identity, i.e. $\widetilde{\mathrm{Ric}_0}$ is a tensor of the same kind as R itself, as well as W^+ and W^-; moreover, $\widetilde{\mathrm{Ric}_0}$ anti-commutes with $*$, so that it can be viewed as a field of homomorphisms from $\Lambda^+ M$ into $\Lambda^- M$, or from $\Lambda^- M$ into $\Lambda^+ M$ (adjoint to each other); we eventually get the well-known Singer-Thorpe decomposition of R, see e.g. [11]:

$$R = \left(\begin{array}{c|c} W^+ + \frac{s}{12}\,\mathrm{Id}_{|\Lambda^+ M} & (\frac{1}{2}\widetilde{\mathrm{Ric}_0})_{|\Lambda^- M} \\ \hline (\frac{1}{2}\widetilde{\mathrm{Ric}_0})_{|\Lambda^+ M} & W^- + \frac{s}{12}\,\mathrm{Id}_{|\Lambda^- M} \end{array}\right).$$

In the presence of a g-orthogonal almost complex structure J, which induces the chosen orientation of M, the above $SO(4)$-decomposition can be further refined to get seven irreducible $U(2)$-invariant pieces [47]. To see this, we first decompose the traceless Ricci tensor into its J-invariant and J-anti-invariant parts, Ric_0' and $\mathrm{Ric}_0'' = \mathrm{Ric}''$, which gives rise to the decomposition

$$\widetilde{\mathrm{Ric}_0} = \left(\widetilde{\mathrm{Ric}_0'} \middle| \widetilde{\mathrm{Ric}_0''}\right).$$

We shall denote by ρ the *Ricci form* of (M, g, J), which is the $(1, 1)$-form corresponding to Ric' via J (we thus have $\rho = \mathrm{Ric}' \circ J$), and by $\rho_0 = \mathrm{Ric}_0' \circ J$ its primitive part.

Furthermore, with respect to (7), W^+ decomposes as

$$W^+ = \left(\begin{array}{c|c} \frac{\kappa}{6} & W_2^+ \\ \hline (W_2^+)^* & -\frac{\kappa}{12}\mathrm{Id}_{|[\![\Lambda^{0,2}M]\!]} + W_3^+ \end{array} \right),$$

where:

- the smooth function κ is the so called *conformal scalar curvature* of (g, J), defined by $\kappa = 3\langle W^+(\Omega), \Omega\rangle$; thus, κ is determined by the scale action of W^+ on any of the two factor of (7).
- the component W_2^+ is the piece of W^+ that interchanges the two factors of (7); it thus can be seen as a section of $K_J^{-1}(M)$ by the following equality

$$W_2^+ = \frac{1}{2}\Big(\rho_*'' \otimes \Omega + \Omega \otimes \rho_*''\Big), \tag{8}$$

  where ρ_*'' is the J-anti-invariant component of the so-called *star-Ricci form* $\rho_* = R(\Omega)$ of (g, J); recall that the *star-Ricci tensor* Ric^* is defined by $\mathrm{Ric}^*(\cdot,\cdot) = -\rho_*(J\cdot,\cdot)$; in particular, $\rho_*'' = 0$ if and only if $W^+(\Omega) = \frac{\kappa}{6}\Omega$;
- the component W_3^+ can be viewed as a section of $K_J^{-2}(M)$; with respect to any (local) section ϕ of $[\![\Lambda^{0,2}M]\!]$, it can be written as

$$W_3^+ = \frac{\lambda}{2}[\phi \otimes \phi - J\phi \otimes J\phi] + \frac{\mu}{2}[\phi \otimes J\phi + J\phi \otimes \phi], \tag{9}$$

  λ and μ being (locally defined) smooth functions; equivalently, W_3^+ is the component of W^+ that belongs to the bundle of symmetric endomorphisms of $[\![\Lambda^{0,2}M]\!]$ which anti-commute with J.

We thus get the $U(2)$-splitting of the Riemannian curvature of an almost Hermitian 4-manifold [47]:

$$R = \frac{s}{12}\mathrm{Id}_{|\Lambda^2 M} + W_1^+ + W_2^+ + W_3^+ + \widetilde{\mathrm{Ric}_0'} + \widetilde{\mathrm{Ric}_0''} + W^-, \tag{10}$$

where W_1^+ denotes the scalar component of W^+, specifically given by

$$W_1^+ = \frac{\kappa}{8}\Omega \otimes \Omega - \frac{\kappa}{12}\mathrm{Id}_{|\Lambda^+ M}. \tag{11}$$

The following immediate application of the decomposition (10) provides alternative definitions (in dimension 4) for the third curvature condition of Gray (3):

Lemma 1. *[47] For an almost Hermitian 4-manifold (M, g, J) the following conditions are equivalent:*

(i) $\mathrm{Ric}'' = 0$ *and* $\rho_*'' = 0$.

(ii) $\mathrm{Ric}'' = 0$ *and* $W(\Omega) = \frac{\kappa}{6}\Omega$.
(iii) $\mathrm{Ric}^* - \mathrm{Ric} = \frac{(\kappa - s)}{6} g$.
(iv) $R_{XYZU} = R_{JXJYJZJU}$.
(v) *R preserves the type decomposition* (*with respect to* J) *of real 2-forms.*

2.3. The Nijenhuis tensor and the curvature

A central object in our study is the *Nijenhuis tensor* (or *complex torsion*) of an almost complex structure J, defined by

$$N_{X,Y} = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

Thus, N is a 2-form with values in TM, which by Newlander-Nirenberg theorem [40] vanishes if and only if J is integrable.

Alternatively, the Nijenhuis tensor can be viewed as a map from $\Lambda^{1,0}M$ to $\Lambda^{0,2}M$ in the following manner: given a complex (1,0)-form ψ, we define by $\partial\psi$ and $\bar{\partial}\psi$ the projectors of $d\psi$ to $\Lambda^{2,0}M$ and $\Lambda^{1,1}M$, respectively. In general, $d \neq \partial + \bar{\partial}$, as $d\psi$ can also have a component of type (0,2), which we denote by $N(\psi)$; writing $\psi = \alpha + iJ\alpha$ where α is a real 1-form, one calculates

$$N(\psi)_{X,Y} = \frac{1}{2}\alpha(N_{X,Y}) = \frac{1}{4}\psi(N_{X,Y}), \quad \forall X, Y \in T^{0,1}M.$$

Equivalently, for every real 1-form α we have

$$(d^J\alpha)''_{X,Y} = (J(d\alpha)'')_{X,Y} + \frac{1}{2}J\alpha(N_{X,Y}), \tag{12}$$

where $d^J = i(\bar{\partial} - \partial)$; thus, d^J acts on real functions by $d^J f = Jdf$, on real 1-forms by $(d^J\alpha)_{X,Y} = -d(J\alpha)_{JX,JY}$, etc. Observe that $d^2 = 0$ does not imply $dd^J + d^J d = 0$, unless $N \equiv 0$. For example, using (12) one obtains on functions

$$(dd^J + d^J d)f = 2(dd^J f)'' = (d^J f)_{N(\cdot,\cdot)}. \tag{13}$$

In particular, $dd^J f$ is not, in general, a $(1, 1)$-form; however, for an almost Kähler manifold its projection to the Ω-factor of (6) is a second order linear differential operator which, up to a sign, is nothing else than the Riemannian Laplacian of (M, g):

$$\Delta f = -\langle dd^J f, \Omega\rangle. \tag{14}$$

For an almost Kähler manifold (M, g, J, Ω), the vanishing of N is equivalent to Ω being parallel with respect to the Levi-Civita connection ∇ of g; specifically, for any almost Kähler manifolds the following identity holds (cf. *e.g.* [30]):

$$(\nabla_X \Omega)(\cdot, \cdot) = \frac{1}{2}\langle JX, N(\cdot, \cdot)\rangle. \tag{15}$$

Using the above relation, we shall often think of N as a T^*M-valued 2-form, or as a $\Lambda^2 M$-valued 1-form, by tacitly identifying N with $\nabla\Omega$. Further, since Ω is closed and N is a J-anti-invariant 2-form with values in TM, one easily deduces that $\nabla\Omega$ is, in fact, a section of the vector bundle $[\![\Lambda^{0,1}M \otimes \Lambda^{0,2}M]\!]$, i.e. the following relation holds:

$$\nabla_{JX} J = -J(\nabla_X J), \quad \forall X \in TM. \tag{16}$$

Note that, in four dimensions, any section of $[\![\Lambda^{0,1}M \otimes \Lambda^{0,2}M]\!]$ has a non-trivial kernel; then, at any point $x \in M$, we define the sub-space

$$D_x = \{X \in T_x M : (\nabla_X J)_x = 0\},$$

which we call *Kähler nullity* of (g, J); thus, $\dim_{\mathbb{R}}(D) = 4$ or 2, depending on whether or not N vanishes at the given point. Assuming that $N \neq 0$, we let $D^{\perp}$ be the orthogonal complement of D in TM. Relation (16) identifies the complex line bundle $D^{\perp}$ with the *canonical* bundle $K_J(M) \cong \Lambda^{2,0}M$; consequently, we have the following relations for the corresponding first Chern classes:

$$c_1(D^{\perp}) = -c_1(TM) = -c_1(J)\ , \quad c_1(D) = c_1(TM) - c_1(D^{\perp}) = 2c_1(J).$$

On the open subset of points where $N \neq 0$, we shall also consider the opposite almost complex structure I, which is equal to J on D, but to $-J$ on $D^{\perp}$; clearly I is compatible with the metric but yields the opposite orientation to the one of (M, J). We mention below the following observation due to J. Armstrong [8] and C. LeBrun [38]:

Lemma 2. *Let (M, g, J, Ω) be an almost-Kähler 4-manifold for which the Nijenhuis tensor nowhere vanishes. Then, the Chern classes of the almost complex structures J and I satisfy $c_1(I) = 3c_1(J)$; if, moreover, M is compact, then the Euler characteristic $\chi(M)$ and signature $\sigma(M)$ of M are related by $5\chi(M) = -6\sigma(M)$.*

Proof. We clearly have

$$c_1(I) = c_1(D) - c_1(D^{\perp}) = 3c_1(J).$$

By Wu's formula we know that $c_1^2(M, J) = 2\chi(M) + 3\sigma(M)$ and $c_1^2(M, I) = 2\chi(M) - 3\sigma(M)$, so that we get $-9(2\chi(M) + 3\sigma(M)) = 2\chi(M) - 3\sigma(M)$; the claim follows. □

For a Kähler structure (g, J, Ω) some of the curvature components defined in Section 2.2 identically vanish: indeed, in this case $\mathrm{Ric}'' = 0$, $W_2^+ = 0$, $W_3^+ = 0$ and the scalar function κ is nothing else than the scalar curvature s. This suggests that for an almost Kähler structure those components depend directly on the

Nijenhuis tensor (and its covariant derivative). To see this, consider the Ricci identity for Ω:

$$(\nabla^2_{X,Y} - \nabla^2_{Y,X})(\Omega) = [J, R_{X,Y}], \tag{17}$$

where $R_{X,Y}$ is viewed as an endomorphism of TM and $[\cdot,\cdot]$ denotes the commutator. A contraction of this relation leads to

$$-\sum_{i=1}^{4}(\nabla^2_{e_i,Z}\Omega)(e_i, X) = \mathrm{Ric}(JX, Z) - \mathrm{Ric}^*(JX, Z). \tag{18}$$

Symmetrising (18) in X and Z, one obtains the relation between Ric'' and the Nijenhuis tensor:

$$(J\mathrm{Ric}'')_{X,Z} = \frac{1}{2}\sum_{i=1}^{4}\Big((\nabla^2_{e_i,Z}\Omega)(e_i, X) + (\nabla^2_{e_i,X}\Omega)(e_i, Z)\Big).$$

Anti-symmetrising (18) in X and Z and using $d\Omega = 0$, one gets

$$\frac{1}{2}\nabla^*\nabla\Omega = \rho_* - \rho, \tag{19}$$

which can be recognized as being nothing but the usual Weitzenböck formula [11] applied to the harmonic 2-form Ω. From (19) we also get immediately

$$2\rho''_* = (\nabla^*\nabla\Omega)'', \quad \rho_0 = (\rho'_*)_0 \text{ and}$$

$$s^* - s = |\nabla\Omega|^2, \text{ where } s^* = \frac{2\kappa + s}{3} \tag{20}$$

is the trace of the star-Ricci tensor Ric^*.

Finally, the component W_3^+ is determined by the $[[\Lambda^{0,2}M]]$-component of the relation (17).

As $\nabla\Omega$ is a section of the bundle $[[\Lambda^{0,1}M\otimes\Lambda^{0,2}M]]$, it will be very convenient at times to express the higher jets of Ω in terms of a (local) section $\phi \in [[\Lambda^{0,2}M]]$ of square-norm 2. Observe that we have an S^1-freedom for the choice of ϕ and, for this reason, such a form ϕ will be called a *gauge*.

Thus, assuming that we have made a choice for the gauge ϕ, we can write:

$$\nabla\Omega = a \otimes \phi - Ja \otimes J\phi. \tag{21}$$

The gauge-dependent 1-form a is dual to a vector field in $D^\perp$ and satisfies $|\nabla\Omega|^2 = 4|a|^2$, which shows that $|a|^2$ is a gauge-independent quantity. As $\{\frac{1}{\sqrt{2}}\Omega, \frac{1}{\sqrt{2}}\phi, \frac{1}{\sqrt{2}}J\phi\}$ is an orthonormal basis of Λ^+M, the covariant derivatives of ϕ and $J\phi$ are given by

$$\nabla\phi = -a \otimes \Omega + b \otimes J\phi; \quad \nabla J\phi = Ja \otimes \Omega - b \otimes \phi, \tag{22}$$

for some gauge-dependent 1-form b. In fact, if we change the gauge by

$$\phi' = (\cos\theta)\phi + (\sin\theta)J\phi,$$

the corresponding 1-forms change as follows

$$a' = (\cos\theta)a - (\sin\theta)Ja;\ \ b' = b + d\theta.$$

From (21) and (22), we get

$$\nabla^2|_{\Lambda^2 M}\Omega = (da - Ja \wedge b) \otimes \phi - (d(Ja) + a \wedge b) \otimes J\phi,$$

so, the Ricci identity for Ω, (17), can be rewritten as

$$da - Ja \wedge b = -R(J\phi);\ \ d(Ja) + a \wedge b = -R(\phi). \tag{23}$$

From the Ricci identity for ϕ

$$(\nabla^2_{X,Y} - \nabla^2_{Y,X})(\phi) = [\phi, R_{X,Y}],$$

after using (21) and (22), one gets the gauge independent relation

$$db = a \wedge Ja - R(\Omega) = a \wedge Ja - \rho_*. \tag{24}$$

Note that the closed, gauge independent 2-form

$$\gamma_J = -db = R(\Omega) - a \wedge Ja \tag{25}$$

is a deRham representative of $2\pi c_1(M, J)$, calculated with respect to the *first canonical connection* of (M, g, J)

$$\nabla^1_X Y = \nabla_X Y - \frac{1}{2}J(\nabla_X J)(Y),$$

see e.g. [21]. We shall refer to γ_J as the *canonical Chern form* of (M, g, J).

2.4. The differential Bianchi identity for almost Kähler 4-manifolds

For a Riemannian 4-manifold (M, g) the differential Bianchi identity can be written as $\delta W = C$, where W is the Weyl tensor, C denotes the *Cotton-York* tensor of (M, g) and the co-differential δ acts on W as on a $\Lambda^2 M$-valued 2-form. Recall that C is defined by

$$C_{X,Y,Z} = -(d^\nabla h)(X, Y, Z) = -(\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z),$$

where $h = \frac{1}{2}\mathrm{Ric}_0 + \frac{s}{24}g$ denotes the *normalized Ricci tensor* and d^∇ is the Riemannian differential acting on T^*M valued 1-forms. Both δW and C are sections of the bundle $\Lambda^1 M \otimes \Lambda^2 M$ and because of the splitting $\Lambda^2 M = \Lambda^+ M \oplus$

$\Lambda^- M$, the differential Bianchi identity splits in two halves, selfdual and anti-selfdual:

$$\delta W^+ = C^+, \quad \delta W^- = C^-,$$

where the $\pm$-superscript denotes the selfdual, resp. anti-selfdual component of the corresponding tensor. We would like to express the above formulae in terms of the various $U(2)$ curvature components of an almost Kähler 4-manifold (M, g, J, Ω). We start with the Cotton-York tensor C.

Contracting the differential Bianchi identity $\delta W = C$, we obtain

$$\delta(\mathrm{Ric}_0 - \frac{s}{4} g) = 0$$

and this, together with the fact that the fundamental form Ω is harmonic, implies:

$$-\delta(\rho_0 - \frac{s}{4}\Omega) = J\delta(\mathrm{Ric}''). \tag{26}$$

Taking the Hodge star operator of both sides, the above relation is equivalent to

$$d\rho = *(J\delta\mathrm{Ric}'').$$

In particular, if $\mathrm{Ric}'' = 0$, then ρ is closed [19].

For a given vector field Z, we denote by C_Z the section of $\Lambda^2 M$, defined by $C_Z(X, Y) := C(X, Y, Z)$, and similarly we define C_Z^+ and C_Z^-. Denote also by A_Z the $\Lambda^2 M$-valued 1-form given by

$$A_Z = (d^\nabla \mathrm{Ric}'')_Z + \iota_{JZ}(d\rho),$$

where ι stands for interior derivative, and let $A_Z^\pm$ be the selfdual and anti-selfdual components of A_Z.

Lemma 3. *Let (M, g, J, Ω) be an almost Kähler 4-manifold. Then, for any vector field Z, the Cotton-York tensor C_Z is given by*

$$2C_Z = \nabla_{JZ}\rho_0 - \frac{1}{4}((d^J s)_Z)\Omega + \nabla_{\rho_0(Z)}\Omega + \frac{1}{6} ds \wedge Z^\flat - A_Z, \tag{27}$$

where $Z^\flat$ denotes the g-dual 1-form of Z.

Proof. Since $h = \frac{1}{2}\mathrm{Ric} - \frac{s}{12} g$, the Cotton-York tensor is written as:

$$C_Z = -\frac{1}{2}(d^\nabla \mathrm{Ric}'')_Z - \frac{1}{2}(d^\nabla \mathrm{Ric}')_Z + \frac{1}{12} ds \wedge Z^\flat. \tag{28}$$

Taking into account that $\rho(\cdot, \cdot) = \mathrm{Ric}^{'}(J\cdot, \cdot)$, we have

$$\begin{aligned}(d^{\nabla}\mathrm{Ric}^{'})(X, Y, Z) &= (\nabla_X \mathrm{Ric}^{'})(Y, Z) - (\nabla_Y \mathrm{Ric}^{'})(X, Z)\\ &= (d^{\nabla}\rho)(X, Y, JZ) + \Big(\rho(Y, (\nabla_X J)Z) - \rho(X, (\nabla_Y J)Z)\Big).\end{aligned} \tag{29}$$

For the term $d^{\nabla}\rho$ we have

$$\begin{aligned}(d^{\nabla}\rho)(X, Y, JZ) &= (\nabla_X \rho)(Y, JZ) - (\nabla_Y \rho)(X, JZ)\\ &= -(\nabla_{JZ}\rho)(X, Y) + (d\rho)(X, Y, JZ)\\ &= -(\nabla_{JZ}\rho)(X, Y) + \iota_{JZ}(d\rho)(X, Y).\end{aligned} \tag{30}$$

To refine the last term of (29), note that as an algebraic object, $\nabla_X J$ is a skew-symmetric endomorphism of TM, associated (by g-duality) to the section $\nabla_X \Omega$ of the bundle $[[\Lambda^{0,2}M]]$. Thus $\nabla_X J$ anti-commutes with J, and commutes with any skew-symmetric endomorphism associated to a section of $\Lambda^{-}M$; in particular, it commutes with the endomorphism corresponding to ρ_0 via the metric (which is still denoted by ρ_0). We thus obtain

$$\begin{aligned}\rho(Y, (\nabla_X J)(Z)) - \rho(X, (\nabla_Y J)(Z)) &= \frac{s}{4}\Big((\nabla_Y \Omega)(X, JZ) - (\nabla_X \Omega)(Y, JZ)\Big)\\ &\quad + (\nabla_X \Omega)(Y, \rho_0(Z)) - (\nabla_Y \Omega)(X, \rho_0(Z))\\ &= \frac{s}{4}(\nabla_{JZ}\Omega)(X, Y) - (\nabla_{\rho_0(Z)}\Omega)(X, Y),\end{aligned} \tag{31}$$

where for the last equality we used the closedness of Ω.

Substituting (30) and (31) in (29), we get

$$\begin{aligned}(d^{\nabla}\mathrm{Ric}^{'})_Z &= -\nabla_{JZ}\rho_0 - \nabla_{\rho_0(Z)}\Omega\\ &\quad + \frac{1}{4}((d^J s)_Z)\Omega + \iota_{JZ}(d\rho).\end{aligned} \tag{32}$$

From (28) and (32), we obtain relation (27) claimed in the statement. $\square$

Lemma 4. *Let (M, g, J, Ω) be an almost Kähler 4-manifold. Then the differential Bianchi identities $\delta W^+ = C^+$, $\delta W^- = C^-$ are equivalent to:*

$$\begin{aligned}0 &= -\frac{1}{4}(d^J(\kappa - s))_Z \Omega + \frac{1}{6}(d(\kappa - s) \wedge Z^{\flat})^+ + (\delta\rho''_*)_Z \Omega\\ &\quad + \frac{\kappa}{4}\nabla_{JZ}\Omega - \nabla_{\rho_0(Z)}\Omega + \nabla_{\rho''_*(Z)}\Omega + \nabla_{JZ}\rho''_* + 2(\delta W_3^+)_Z + A_Z^+\ ;\end{aligned} \tag{33}$$

$$0 = \nabla_{JZ}\rho_0 + \frac{1}{6}(ds \wedge Z^{\flat})^- - 2\delta W_Z^- - A_Z^-\ . \tag{34}$$

Proof. Relation (34) follows from $\delta W^- = C^-$ by simply taking the anti-selfdual component of (27). Similarly, for (33) we take the selfdual component of (27), but we also use the relations

$$(\delta W_1^+)_Z = -\frac{1}{8}(d^J\kappa)_Z \Omega + \frac{\kappa}{8}\nabla_{JZ}\Omega + \frac{1}{12}(d\kappa \wedge Z^\flat)^+, \tag{35}$$

$$(\delta W_2^+)_Z = \frac{1}{2}\nabla_{JZ}(\rho_*)'' + \frac{1}{2}\nabla_{(\rho_*)''(Z)}\Omega + \frac{1}{2}(\delta\rho_*'')(Z)\Omega, \tag{36}$$

which easily follow from (11) and (8). □

It will be useful to further determine the Ω-component and the $[\![\Lambda^{0,2}M]\!]$-component of (33) in accordance with the decomposition (7) of $\Lambda^+ M$. These are given by

Corollary 1. *For any almost Kähler 4-manifold the following identities hold:*

$$\begin{aligned} 0 = &-\frac{1}{3}(d^J(\kappa - s))_Z + 2\langle(\delta W_3^+)_Z, \Omega\rangle \\ &+2(\delta\rho_*'')_Z - \langle\rho_*'', \nabla_{JZ}\Omega\rangle + \langle A_Z^+, \Omega\rangle\ ; \end{aligned} \tag{37}$$

$$\begin{aligned} 0 = &\frac{1}{6}(d(\kappa - s) \wedge Z^\flat)'' + \frac{\kappa}{4}\nabla_{JZ}\Omega - \nabla_{\rho_0(Z)}\Omega + 2(\delta W_3^+)_Z'' \\ &+\nabla_{\rho_*''(Z)}\Omega + (\nabla_{JZ}\rho_*'')'' + (A_Z^+)''. \end{aligned} \tag{38}$$

2.5. A Weitzenböck formula and unique continuation of the Nijenhuis tensor

Recall that the *weak unique continuation property* for a map $u : M \to E$ between connected Riemannian manifolds, M and E, says that if u is a constant map on an open subset $U \subset M$, then u is constant everywhere; for the *strong* unique continuation property, the condition that u is constant on an open subset is replaced with the assumption that at a given point, u has a contact of infinite order with the constant map (see e.g. [29]).

In this subsection we shall show that the (strong) unique continuation property holds for the Nijenhuis tensor of an almost Kähler 4-manifold which satisfies certain curvature conditions. In fact, we shall concentrate our attention on $\nabla\Omega$, which is identified to the Nijenhuis tensor via (15). As typically happens in Riemannian geometry, the unique continuation property appears as a consequence of a Weitzenböck formula. In our case, we shall use a general Weitzenböck formula of Bourguignon [14] applied to $\nabla\Omega$, seen as a section of the bundle of $\Lambda^2 M$-valued 1-forms. The role of the “constant map” in this setting is played by the zero section.

Proposition 1. *Let (M, g, J, Ω) be an almost Kähler 4-manifold whose curvature satisfies the third Gray condition* (3)*. Then, the following identity holds:*

$$\nabla^*\nabla(\nabla\Omega) + \frac{3s}{4}\nabla\Omega + \nabla_{\mathrm{Ric}_0(\cdot)}\Omega = 0, \tag{39}$$

In particular, if M is connected, then the (strong) unique continuation property holds for $\nabla\Omega$ (hence for the Nijenhuis tensor N as well).

Proof. We shall in fact establish a formula for $\nabla^*\nabla(\nabla\Omega)$ on an arbitrary almost Kähler 4-manifold, and then relation (39) will be an immediate consequence. The starting point is the general Weitzenböck formula for a $\Lambda^2 M$-valued 1-form V (e.g. see [14], p.282):

$$\Big((d^\nabla\delta^\nabla + \delta^\nabla d^\nabla)V\Big)_X = \Big(\nabla^*\nabla V\Big)_X + V_{\mathrm{Ric}(X)} + (R\cdot V)_X. \tag{40}$$

In the above formula

- d^∇ is the Riemannian differential acting on $\Lambda^2 M$-valued k-forms;
- δ^∇ is the formal adjoint of d^∇;
- the curvature action is given by:

$$(R\cdot V)_X = \sum_{i=1}^{4}[R_{X,e_i}, V_{e_i}],$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis of TM, the 2-forms R_{X,e_i}, V_{e_i} are freely identified with the corresponding skew-symmetric endomorphism of TM and $[\cdot,\cdot]$ denotes the commutator. We apply (40) to $V = \nabla\Omega$; using (19), (20) and (17) we get

$$\delta^\nabla(\nabla\Omega) = \nabla^*\nabla\Omega = 2(\rho_* - \rho) = 2\rho_*'' + \frac{(\kappa - s)}{3}\Omega\ , \tag{41}$$

$$(d^\nabla(\nabla\Omega))_{X,Y} = (\nabla^2_{X,Y}\Omega) - (\nabla^2_{Y,X}\Omega) = [J, R_{X,Y}]. \tag{42}$$

From (41) we derive immediately

$$(d^\nabla\delta^\nabla(\nabla\Omega))_X = 2\nabla_X\rho_*'' + \frac{1}{3}d(\kappa - s)_X\Omega + \frac{(\kappa - s)}{3}\nabla_X\Omega\ , \tag{43}$$

whereas a short computation starting from (42) leads to

$$(\delta^\nabla d^\nabla(\nabla\Omega))_X = -(R\cdot(\nabla\Omega))_X + \Big((\nabla_{e_i}R)_{e_i,X}(J\cdot,\cdot) + (\nabla_{e_i}R)_{e_i,X}(\cdot,J\cdot)\Big)\ . \tag{44}$$

Using the differential Bianchi identity and (32), we get

$$\begin{aligned}\Big((\nabla_{e_i}R)_{e_i,X}(J\cdot,\cdot)+(\nabla_{e_i}R)_{e_i,X}(\cdot,J\cdot)\Big)&=(d^{\nabla}\mathrm{Ric})_X(J\cdot,\cdot)+(d^{\nabla}\mathrm{Ric})_X(\cdot,J\cdot)\\&=A_X(J\cdot,\cdot)+A_X(\cdot,J\cdot)+2(\nabla_{\mathrm{Ric}_0'(X)}\Omega)(\cdot,\cdot)\\&=-2(J(A_X^+)'')(\cdot,\cdot)+2(\nabla_{\mathrm{Ric}_0'(X)}\Omega)(\cdot,\cdot)\ .\end{aligned}\tag{45}$$

From the relations (43-45) and (40), we obtain

$$\begin{aligned}\nabla^*\nabla(\nabla\Omega)&=2\nabla\rho_*''-2J(A^+)''-\nabla_{\mathrm{Ric}(\cdot)}\Omega\\&\quad-2R\cdot(\nabla\Omega)+2\nabla_{\mathrm{Ric}_0'(\cdot)}\Omega+\frac{1}{3}d(\kappa-s)\otimes\Omega+\frac{(\kappa-s)}{2}\nabla\Omega\ .\end{aligned}\tag{46}$$

It remains to detail the expression of the term $R\cdot(\nabla\Omega)$. For this, we shall use the $U(2)$-decomposition of the curvature (10) and compute the action of each component on $\nabla\Omega$. It is clear that the action of W^- is zero, and for the action of the component $\widetilde{\mathrm{Ric}_0''}$ we will not attempt to do any simplification, as $\mathrm{Ric}_0''=0$ for almost Kähler 4-manifolds satisfying (3). For the other components though, from their definitions (see (11), (8), (9)), one computes successively:

$$\begin{aligned}\Big(\frac{s}{12}\mathrm{Id}_{|\Lambda^2M}\cdot(\nabla\Omega)\Big)_X&=\frac{s}{12}(\nabla_X\Omega);\\\Big(W_1^+\cdot(\nabla\Omega)\Big)_X&=\frac{\kappa}{6}(\nabla_X\Omega);\\\Big(W_2^+\cdot(\nabla\Omega)\Big)_X&=-(\nabla_{J\rho_*''(X)}\Omega)-\frac{1}{2}\langle\rho_*'',\nabla_X\Omega\rangle\Omega;\\\Big(W_3^+\cdot(\nabla\Omega)\Big)_X&=\langle(\delta W_3^+)_{JX},\Omega\rangle\Omega;\\\Big(\widetilde{\mathrm{Ric}_0'}\cdot(\nabla\Omega)\Big)_X&=\nabla_{\mathrm{Ric}_0'(X)}\Omega\ .\end{aligned}$$

Finally, using the relations above, back in (46), we obtain:

$$\begin{aligned}\Big(\nabla^*\nabla(\nabla\Omega)\Big)_X&=-\nabla_{\mathrm{Ric}_0'(X)}\Omega-\frac{3s}{4}\nabla_X\Omega\\&\quad+\Big(\frac{1}{3}d(\kappa-s)_X-2\langle(\delta W_3^+)_{JX},\Omega\rangle\Big)\Omega\\&\quad+2(\nabla_X\rho_*'')''+2\nabla_{J\rho_*''(X)}\Omega\\&\quad-\nabla_{\mathrm{Ric}_0''(X)}\Omega-2J(A_X)''-2\Big(\widetilde{\mathrm{Ric}_0''}\cdot(\nabla\Omega)\Big)_X\ .\end{aligned}\tag{47}$$

For an almost Kähler 4-manifold which satisfies the third Gray condition, the terms in the last two lines of the above equation all vanish, as they contain

Ric'' and ρ_*'', see Lemma 1. In fact, the expression of the second line is also 0, as a consequence of the Bianchi relation (37). Thus, under the assumption $\mathrm{Ric}'' = 0$ & $\rho_*'' = 0$, the general formula (47) reduces to (39); the fact that the relation (39) implies the (strong) unique continuation property for $\nabla\Omega$ (and therefore also for the Nijenhuis tensor N), follows from the classical result of Aronszajn [7]. ◻

Remark 1. (i) Note that solely for the purpose of getting the unique continuation property for $\nabla\Omega$, we need not make the effort to compute the action of each of the curvature components on $\nabla\Omega$. Indeed, to apply the result of Aronszajn [7], it is enough to obtain an estimate of the form

$$|\nabla^*\nabla(\nabla\Omega)|^2 \leq M\Big(|\nabla(\nabla\Omega)|^2 + |\nabla\Omega|^2\Big) ,$$

and relation (46) already implies such an estimate, once we also note that

$$\frac{2}{3}|d(\kappa - s)| = |d(|\nabla\Omega|^2)| \leq 2|\nabla(\nabla\Omega)||\nabla\Omega| = \frac{4}{3}(\kappa - s)|\nabla(\nabla\Omega)|.$$

More generally, from (47) (or equally well from (46)), one can deduce that the (strong) unique continuation property for $\nabla\Omega$ holds for almost Kähler 4-manifolds for which two of the three tensors, Ric'', ρ_*'', and W_3^+ vanish. Indeed, in the case when $\rho_*'' = 0$ & $W_3^+ = 0$, the differential Bianchi identity (38) can be written as

$$-(A_Z^+)'' = \frac{1}{6}(d(\kappa - s) \wedge Z^\flat)'' + \frac{\kappa}{4}\nabla_{JZ}\Omega - \nabla_{\rho_0(Z)}\Omega.$$

Putting the above relation back in (46), one again obtains the needed estimate to apply [7]. One can proceed similarly in the case $\mathrm{Ric}'' = 0$ & $W_3^+ = 0$. In fact, in this latter case, employing a different estimate, it was shown in [2] that an even stronger version of the unique continuation property of $\nabla\Omega$ holds; namely, if $\nabla\Omega$ vanishes at one point, then it must be identically zero.

(ii) One would expect formula (47) to eventually lead to the integrability of compact almost Kähler 4-manifolds satisfying certain curvature conditions. This is indeed the case: along these lines it was proved in [20] that on a compact symplectic 4-manifold there are no critical, strictly almost Kähler structures of everywhere non-negative scalar curvature.

3. The examples of almost Kähler 4-manifolds satisfying the third Gray condition

We give here more details about the explicit construction given in Theorem 1; the material in this section has appeared in [3].

Let (Σ, g_Σ) be any oriented Riemann surface and $h = w + iv$ be a holomorphic function on Σ, such that $w = \Re e(h)$ is a positive (harmonic) function on Σ. We can then locally write the metric g_Σ as

$$g_\Sigma = e^u w(dx^{\otimes 2} + dy^{\otimes 2}),$$

where x, y are (local) isotherm coordinates of (Σ, g_Σ) and u is a function on Σ. We consider the metric

$$\begin{aligned} g &= g_\Sigma + wdz^{\otimes 2} + \frac{1}{w}(dt + vdz)^{\otimes 2} \\ &= e^u w(dx^{\otimes 2} + dy^{\otimes 2}) + wdz^{\otimes 2} + \frac{1}{w}(dt + vdz)^{\otimes 2}, \end{aligned} \tag{48}$$

defined on $M = \Sigma \times \mathbb{R}^2$, where (z, t) are the canonical coordinates of $\mathbb{R}^2$. Clearly, $K_1 = \frac{\partial}{\partial z}$ and $K_2 = \frac{\partial}{\partial t}$ are two commuting Killing fields, so that g is an $\mathbb{R}^2$-invariant metric compatible with the product structure in the sense of [28].

As observed in [35], the metric (48) carries a compatible Kähler structure, I, whose fundamental form is

$$\overline{\Omega} = \Omega_\Sigma + dz \wedge (dt + vdz) = (e^u w)dx \wedge dy + dz \wedge dt,$$

where Ω_Σ denotes the volume form of (Σ, g_Σ). Equivalently, I is defined by its action on 1-forms:

$$I(dx) = dy;\ \ I(dz) = \frac{1}{w}(dt + vdz). \tag{49}$$

Thus, I is compatible with the product structure as well, and induces an orientation on M and on each of the factors Σ and $\mathbb{R}^2$. Besides I, one can consider the almost complex structure J, which coincides with I on Σ, but which is equal to $-I$ on $\mathbb{R}^2$, i.e.

$$J(dx) = dy;\ \ J(dz) = -\frac{1}{w}(dt + vdz). \tag{50}$$

Thus, J is compatible with g and yields on M the orientation opposite to the one induced by I. Furthermore, the fundamental form of J is given by

$$\Omega = \Omega_\Sigma - dz \wedge (dt + vdz) = \Omega_\Sigma - dz \wedge dt,$$

and is clearly closed, meaning that (g, J) is an almost Kähler structure. It is easily seen that J is integrable (i.e. (g, J) is Kähler) if and only if h is constant (i.e. g is a product metric), a possibility which we exclude in what follows below.

An important feature of the construction comes from the following observation: at any point where $dw \neq 0$, the Kähler nullity $D = \{TM \ni X : \nabla_X J = 0\}$ of J is a two dimensional subspace of TM, which is tangent to the surface

Σ (see [3]); thus, the Kähler structure I and the almost Kähler structure J are equivalently related by

$$J_D = I_D \;\; J_{D^\perp} = -I_{D^\perp}.$$

The almost Kähler structure (g, J, Ω) is sufficiently explicit to make it straightforward, though tedious, to check that its curvature satisfies the third Gray condition (3); see our previous paper [3] for detailed calculations. But the heart of [3] consists of the following abstract characterization of the almost Kähler structures given by (48-50):

Theorem 3. [3, Theorem 2] *Let (M, g, J, Ω) be a strictly almost Kähler 4-manifold whose curvature satisfies* (3). *At each point $x \in M$ where the Nijenhuis tensor of J does not vanish, consider the orthogonal almost complex structure I which is equal to J on the Kähler nullity $D_x \subset T_xM$, but to $-J$ on the orthogonal complement of D_x. If we suppose that (g, I) is Kähler on an open subset U, then for every sufficiently small neighborhood of $x \in U$, (g, J, I) is given by* (48-50), *up to a homothety.*

The key point for obtaining the above result in [3] was to show that the distribution $D^\perp$ is (locally) spanned by commuting hamiltonian Killing fields, a subject to a straightforward verification of the integrability condition of a relevant Frobenius system; then, according to [35], the Kähler metric (g, I) can be put (locally) in the form (48-49) and Theorem 3 follows easily.

On the other hand, it was shown in [9] that when (g, J) is an Einstein, strictly almost Kähler structure satisfying (3), (g, I) is necessarily Kähler ; the same was later established for strictly almost Kähler 4-manifolds satisfying the second curvature condition of Gray (see [3]). In the next section we generalize these results.

4. Proof of Theorem 1

The following proposition, which via Proposition 1 and Theorem 3 implies Theorem 1, is the technical core of the present paper.

Proposition 2. *Let (M, g, J, Ω) be a strictly almost Kähler 4-manifold satisfying the third Gray condition* (3) *and let (g, I) be the opposite almost Hermitian structure associated to (g, J) as in Theorem 3. Then, on the open set $U \subset M$, where $|\nabla J| \neq 0$, (g, I) is a Kähler structure.*

There are three main steps in the proof, corresponding roughly to Lemmas 5, 6 and 7 below. In Lemma 5, we first establish that under the third Gray condition (3) for (g, J), the opposite almost Hermitian structure (g, I) is *almost Kähler* and we specifically find, in accordance with Lemma 2, the exact form representing the difference $\gamma_I - 3\gamma_J$, where γ_J and γ_I are the canonical Chern forms of (g, J) and

(g, I) respectively (see Section 2.3). Under the *second* Gray condition studied in [3], or with the Einstein assumption made in [9], Lemma 5 it was shown that (g, I) is Kähler. Assuming the (weaker) third Gray condition only, the same conclusion is still true, but not so immediate. The idea of the proof is to proceed by contradiction, thus assuming that (g, I) is *not* Kähler and to try to iterate the construction. This is somehow complicated by the fact that, *a priori*, (g, I) does not satisfy the third Gray condition (3). Nevertheless, in Lemma 6 we show that the curvature has the necessary features to allow the iteration of the construction. The last step is essentially done in Lemma 7: We consider $(g, \tilde{J})$, the opposite almost Hermitian structure associated to (g, I) and, analogously to Lemma 5, we get an explicit expression for the exact form representing $\gamma_{\tilde{J}} - 3\gamma_I$, where $\gamma_{\tilde{J}}$ denotes the canonical Chern form of $(g, \tilde{J})$. But one immediately notices that, in view of Lemma 5, the "opposite of the opposite" almost complex structure $\tilde{J}$ is nothing else than $-J$, hence we also have $\gamma_{\tilde{J}} = -\gamma_J$. Therefore, the expression for $\gamma_{\tilde{J}} - 3\gamma_I$ obtained in Lemma 7 provides another relation between γ_J and γ_I, *different* than the one obtained in Lemma 5 (see also Lemma 2). The combination of the two leads to the desired contradiction.

This is the outline of the proof of Proposition 2. Before we proceed to detail each of the three mentioned steps, let us first recall some notations from Section 2 and introduce some new ones, for convenience. As mentioned in Section 2.3, it is useful to make computations using a gauge ϕ; recall that ϕ is a local section of norm $\sqrt{2}$ of the bundle $[\![\Lambda^{0,2}M]\!]$. We thus choose an arbitrary gauge ϕ and fix it. By formulae (21) and (22), we define the local 1-forms a and b corresponding to ϕ. We shall denote the (local) endomorphisms of TM induced by ϕ and $J\phi$ by J_1 and, respectively, J_2; note that the triple $(J, J_1, J_2 = JJ_1)$ is a local almost quaternionic structure on the manifold. We also make the convention to denote a tensor (or form) defined by the opposite almost complex structure (g, I) with the same symbol as the corresponding one defined by (g, J), but with a "bar" sign on top. Thus, for instance, the fundamental form of (g, I) is denoted by $\overline{\Omega}$, the Ricci form of (g, I) by $\bar{\rho}$, etc.

Because of the existence of the opposite almost Hermitian structure (g, I), the decomposition (6) of the bundle of 2-forms further refines to:

$$\Lambda^2 M = \mathbb{R} \cdot \Omega \oplus [\![\Lambda_J^{0,2} M]\!] \oplus \mathbb{R} \cdot \overline{\Omega} \oplus [\![\Lambda_I^{0,2} M]\!],$$

where $[\![\Lambda_J^{0,2}M]\!]$, $[\![\Lambda_I^{0,2}M]\!]$ denote the underlying real bundles of the anti-canonical bundles of (g, J), (g, I), respectively. We shall keep the convention made earlier to use the superscripts $'$ and $''$ for the J-invariant, respectively, J-anti-invariant parts of a 2-form, or symmetric 2-tensor. But we shall need occasionally to also consider $[\![\Lambda_I^{0,2}M]\!]$-components of 2-forms and we make the convention to use the notation $(\cdot)''_I$ for this. Thus, for instance, $(\bar{\rho}_*)''_I$ will denote the $[\![\Lambda_I^{0,2}M]\!]$-component of the $*$-Ricci form $\bar{\rho}_*$ of (g, I). In addition to d^J, we shall also use

d^I, the complex differential with respect to I, acting on functions by $d^I f = Idf$ (see Sect. 2.3).

The Lemma below was essentially proved in our previous work [3].

Lemma 5. *Let (M, g, J, Ω) be a strictly almost Kähler 4-manifold satisfying* (3)*. Then, on the open set U where $N \neq 0$, the Kähler nullity D of (g, J), and its orthogonal complement, $D^\perp$, are both involutive distributions. Furthermore, on U, (g, I) is an almost Kähler structure and the Kähler nullity of (g, I) contains the distribution $D^\perp$. Moreover, the following relation between the canonical Chern forms γ_I and γ_J of (g, I) and (g, J) holds:*

$$\gamma_I = 3\gamma_J - dd^J(\ln|\nabla\Omega|^2). \tag{51}$$

Proof. The opposite almost complex structure I is defined (on U) through the distributions D and $D^\perp$, which in turn are determined by the 1-jet of J. To obtain information about the 1-jet of I, we thus need to study the 2-jet of J (equivalently, the 2-jet of Ω). For this purpose, let us define the 1-forms $m_i, n_i,\ i = 1, 2$, by the first relation below, while the other three relations follow easily from the first one and (21), (22).

$$\begin{aligned}
\nabla a &= m_1 \otimes a + n_1 \otimes Ja + m_2 \otimes J_1 a + n_2 \otimes J_2 a;\\
\nabla(Ja) &= -n_1 \otimes a + m_1 \otimes Ja + (a - n_2) \otimes J_1 a + (m_2 - Ja) \otimes J_2 a;\\
\nabla(J_1 a) &= -m_2 \otimes a + (n_2 - a) \otimes Ja + m_1 \otimes J_1 a + (b - n_1) \otimes J_2 a;\\
\nabla(J_2 a) &= -n_2 \otimes a + (Ja - m_2) \otimes Ja + (n_1 - b) \otimes J_1 a + m_1 \otimes J_2 a.
\end{aligned} \tag{52}$$

Note that for a strictly almost Kähler 4-manifold (M, g, J, Ω) satisfying (3), the identity (37) becomes

$$0 = \frac{1}{6}(d^J(\kappa - s))_Z - \langle(\delta W_3^+)_Z, \Omega\rangle.$$

Using (20), (9) and (21), this can be further written as

$$\frac{1}{4}d|\nabla\Omega|^2 = d|a|^2 = -J\langle\delta W_3^+, \Omega\rangle = -2JW_3^+(\phi)(a). \tag{53}$$

Hence the 1-form m_1 is immediately determined to be:

$$m_1 = \frac{1}{2}d(\ln|\nabla\Omega|^2) = -\frac{1}{|a|^2}JW_3^+(\phi)(a). \tag{54}$$

As a consequence, observe that $m_1^\sharp \in D$, where here and henceforth the superscript $\sharp$ denotes the dual vector field of a 1-form, through the metric. The forms n_1, m_2, n_2 are determined by the relations (23). Indeed, under the condition (3), the 2-forms $R(\phi)$ and $R(J\phi)$ are J-anti-invariant, thus, according to (23), we have

$$(da - Ja \wedge b)' = 0, \quad (d(Ja) + a \wedge b)' = 0.$$

These relations combined with (52) lead to

$$\begin{aligned} n_1 &= -b - Jm_1 = -b - \frac{1}{2}d^J(\ln|\nabla\Omega|^2);\\ m_2 &= \frac{1}{2}Ja + Jm_0;\\ n_2 &= -Jm_2 = \frac{1}{2}a + m_0, \end{aligned} \tag{55}$$

where m_0 is a 1-form dual to a vector field in D.

Thus, formulae (52), (54) and (55) imply

$$\begin{aligned} da(X,X') = d(Ja)(X,X') = 0,\ \forall X, X' \in D,\\ d(J_1a)(Y,Y') = d(J_2a)(Y,Y') = 0,\ \forall Y, Y' \in D^{\perp}. \end{aligned}$$

As the vector fields dual to a, Ja generate $D^{\perp}$ and the vector fields dual to J_1a, J_2a generate D, the above relations show that $D^{\perp}$ and D are involutive distributions.

To show that (g, I) is almost Kähler, observe first that the fundamental forms of (g, I) and (g, J) are related by $\overline{\Omega} = \Omega - \frac{2}{|a|^2}a \wedge Ja$. We take the covariant derivative of this relation and use (21) and (52-55) to get

$$\nabla\overline{\Omega} = 2m_0 \otimes \bar{\phi} - 2Im_0 \otimes I\bar{\phi}, \tag{56}$$

where $\bar{\phi} \in [[\Lambda_I^{0,2}M]]$ is the natural gauge for (g, I), determined by ϕ through the relation

$$\bar{\phi} = \phi + \frac{2}{|a|^2}Ja \wedge J_2a. \tag{57}$$

Then, $d\overline{\Omega} = 0$ is immediate from (56), and so is the claim about the Kähler nullity of (g, I) since $\bar{a} = 2m_0$ is dual to a vector field in D.

Finally, taking the covariant derivative of (57) and using (22) and (52-55), one finds

$$\nabla\bar{\phi} = \bar{b} \otimes I\bar{\phi} - \bar{a} \otimes \overline{\Omega}, \tag{58}$$

where $\bar{a} = 2m_0$ and

$$\bar{b} = 3b + d^J(\ln|\nabla\Omega|^2). \tag{59}$$

Since $\gamma_I = -d\bar{b}$ and $\gamma_J = -db$ (see (25)), relation (51) follows. □

The next step is to gather more information on the curvature components of the almost Kähler metric (g, I).

Lemma 6. *Let* (M, g, J) *be a strictly almost Kähler 4-manifold satisfying* (3) *and* (g, I) *be the opposite almost Kähler structure as above. Then, the trace-free Ricci form* ρ_0 *of* (g, J) *and the* $[[\Lambda_I^{2,0}M]]$*-component of the star-Ricci form* $\bar{\rho}_*$ *of* (g, I) *are given by:*

$$\rho_0 = \frac{1}{4}\left(s + \frac{|\nabla\overline{\Omega}|^2}{4}\right)\overline{\Omega} + \overline{\Psi}, \quad (\bar{\rho}_*)''_I = \overline{\Psi}, \quad \textit{where } \overline{\Psi} = \frac{1}{2}\nabla_{d(\ln|\nabla\Omega|^2)}\overline{\Omega}. \tag{60}$$

Moreover, the identity (34) *reads as*

$$\frac{1}{4}d|\nabla\overline{\Omega}|^2 = d|\bar{a}|^2 = -I\langle\delta W_3^-, \overline{\Omega}\rangle = -2IW_3^-(\bar{\phi})(\bar{a}) \ . \tag{61}$$

Proof. We first prove the equalities in (60). With respect to the gauge $\bar{\phi}$ from the previous lemma, the Ricci identity for $\overline{\Omega}$ takes the form (compare with (23))

$$d\bar{a} - I\bar{a}\wedge\bar{b} = -R(I\bar{\phi}); \ d(I\bar{a}) + \bar{a}\wedge\bar{b} = -R(\bar{\phi}). \tag{62}$$

We proved in Lemma 5 that $\bar{a}^\sharp \in D$ and that D is an involutive distribution. It thus follows that the left-hand sides of the relations (62) vanish on $X \wedge JX$ with $X \in D^\perp$. This is equivalent with the fact that the $[[\Lambda_I^{0,2}M]]$-components of $R(\Omega)$ and $R(\overline{\Omega})$ are equal. Thus

$$(\rho_0)''_I = (\bar{\rho}_*)''_I = \overline{\Psi},$$

and it remains to obtain the claimed expressions of $\overline{\Psi}$ and the $\overline{\Omega}$-component of ρ_0. For this we turn to relation (51). Using (24) and its analogue for $\bar{\phi}$

$$d\bar{b} = \bar{a}\wedge I\bar{a} - R(\overline{\Omega}),$$

relation (51) is equivalent to

$$\frac{3}{2}R(\Omega) - \frac{1}{2}R(\overline{\Omega}) - \frac{1}{2}dd^J(\ln|\nabla\Omega|^2) - \frac{3}{2}a\wedge Ja + \frac{1}{2}\bar{a}\wedge J\bar{a} = 0. \tag{63}$$

Taking the $[[\Lambda_I^{0,2}M]]$-component of (63) we obtain

$$\overline{\Psi} = \frac{1}{2}(dd^I(\ln|\nabla\Omega|^2))''_I = \frac{1}{2}\nabla_{d(\ln|\nabla\Omega|^2)}\overline{\Omega},$$

where the first equality uses that $(d(\ln|\nabla\Omega|^2))^\sharp \in D$ (see (53)), hence $d^J(\ln|\nabla\Omega|^2) = d^I(\ln|\nabla\Omega|^2)$, and the second equality follows from (13) and (15).

Since $(d^J(\ln|\nabla\Omega|^2))^\sharp \in D$ and D is involutive, we also have

$$\langle dd^J(\ln|\nabla\Omega|^2), (\Omega - \overline{\Omega})\rangle = 0.$$

Thus, taking the inner product of relation (63) with $1/2(\Omega - \overline{\Omega})$ and using (20) and its analogue for the almost Kähler structure (g, I), we derive

$$\langle R(\Omega), \overline{\Omega} \rangle = \frac{s}{2} + \frac{1}{8} |\nabla \overline{\Omega}|^2,$$

and the desired form of the $\overline{\Omega}$-component of ρ_0 follows.

Now we prove the relation (61). The starting point for this is the identity (34) established in Lemma 4. Taking into account that the Ricci tensor is J-invariant and that the anti-selfdual Weyl tensor decomposes into $W^- = W_1^- + W_2^- + W_3^-$ (with respect to the almost complex structure I), relation (34) can be written as

$$0 = \nabla_{JZ}\rho_0 + \frac{1}{6}(ds \wedge Z^\flat)^- - 2(\delta W_1^-)_Z - 2(\delta W_2^-)_Z - 2(\delta W_3^-)_Z\ .$$

We consider the $\overline{\Omega}$-component of this relation, taking into account that for $(\delta W_1^-)_Z$, $(\delta W_2^-)_Z$ we have similar expressions to the ones for $(\delta W_1^+)_Z$, $(\delta W_2^+)_Z$ given in (35) and (36). We thus get

$$\begin{aligned} 0 = \langle \nabla_{JZ}\rho_0, \overline{\Omega} \rangle &- \frac{1}{6}(d^I(\bar{\kappa} - s))_Z + \frac{1}{2}(d^I \bar{\kappa})_Z \\ &- \langle \nabla_{IZ}(\bar{\rho}_*)_I'', \overline{\Omega} \rangle - 2(\delta(\bar{\rho}_*)_I'')_Z - 2\langle (\delta W_3^-)_Z, \overline{\Omega} \rangle\ . \end{aligned}$$

We plug the expressions for ρ_0 and $(\bar{\rho}_*)_I''$ given by (60) into the above formula to eventually obtain

$$\begin{aligned} 0 = 2(d^I |\bar{a}|^2)_Z &- \frac{1}{2}(d^J |\bar{a}|^2)_Z - \frac{1}{2}(d^I s - d^J s)_Z \\ &- 2(\delta \overline{\Psi})_Z - 2\langle (\delta W_3^-)_Z, \overline{\Omega} \rangle\ . \end{aligned} \tag{64}$$

To derive the latter formula we have also used that $\langle \overline{\Psi}, \nabla_{IZ-JZ}\overline{\Omega} \rangle = 0$ which follows by Lemma 5, since $IZ - JZ \in D^\perp$.

Substituting the expression for ρ_0 found in (60) into (26), and applying J, we get

$$d^I s - d^J s = -d^I |\bar{a}|^2 + 4\delta \overline{\Psi}.$$

Using this last relation back in (64) to replace the term $\frac{1}{2}(d^I s - d^J s)_Z$, we obtain

$$0 = 2(d^I |\bar{a}|^2)_Z - \frac{1}{2}(d^J |\bar{a}|^2 + d^I |\bar{a}|^2)_Z - 2\langle (\delta W_3^-)_Z, \overline{\Omega} \rangle\ . \tag{65}$$

Since $\bar{a}^\sharp \in D$, the 1-form

$$\langle \delta W_3^-, \overline{\Omega} \rangle = 2W_3^-(\bar{\phi})(\bar{a})$$

is dual to a vector field in $D^\perp$, and then relation (65) is equivalent with the relation (61). $\square$

The next result is an “iteration” of Lemma 5.

Lemma 7. *Let (M, g, J) be a strictly almost Kähler 4-manifold satisfying* (3) *and let U denote the open set of points where $\nabla J \neq 0$. Let (g, I) be the opposite almost Kähler structure on U, associated to (g, J) and assume that on a non-empty open subset $U_0 \subset U$ the opposite almost Kähler structure (g, I) is not Kähler. Then, on U_0, the opposite almost Hermitian structure $(g, \tilde{J})$ associated to (g, I) is $(g, -J)$. Moreover, the following relation between the canonical Chern forms of (g, I) and $(g, \tilde{J})$ holds*

$$\gamma_{\tilde{J}} = -\gamma_J = 3\gamma_I - dd^I(\ln |\nabla\overline{\Omega}|^2) - 2dd^J(\ln |\nabla\Omega|^2). \tag{66}$$

Proof. From the assumption that (g, I) is not Kähler on U_0 and from Lemma 5, it follows that the Kähler nullity of (g, I) is precisely the distribution $D^\perp$ on U_0; the opposite almost complex structure $\tilde{J}$ associated to (g, I) is then defined by $\tilde{J} = I$ on $D^\perp$ and $\tilde{J} = -I$ on D, i.e., $\tilde{J} = -J$.

We now prove (66). As in Lemma 5, we need to first investigate the 2-jet of $\overline{\Omega}$ (equivalently, the 1-jet of $\bar{a}$), with the help of the Ricci relations (62) and (61). Let $p_i, q_i,\ i = 1, 2$ be the 1-forms defined by the first relation below. The other three relations follow by (21) and (22).

$$\begin{aligned}
\nabla\bar{a} &= p_1 \otimes \bar{a} + q_1 \otimes J\bar{a} + p_2 \otimes J_1\bar{a} + q_2 \otimes J_2\bar{a}\ ;\\
\nabla(J\bar{a}) &= -q_1 \otimes \bar{a} + p_1 \otimes J\bar{a} + (a - q_2) \otimes J_1\bar{a} + (p_2 - Ja) \otimes J_2\bar{a}\ ;\\
\nabla(J_1\bar{a}) &= -p_2 \otimes \bar{a} + (q_2 - a) \otimes J\bar{a} + p_1 \otimes J_1\bar{a} + (b - q_1) \otimes J_2\bar{a}\ ;\\
\nabla(J_2\bar{a}) &= -q_2 \otimes \bar{a} + (Ja - p_1) \otimes J\bar{a} + (q_1 - b) \otimes J_1\bar{a} + p_1 \otimes J_2\bar{a}\ .
\end{aligned} \tag{67}$$

From the formula (61) of Lemma 6, it follows immediately that the 1-form p_1 is given by

$$p_1 = \frac{1}{2}d(\ln |\nabla\overline{\Omega}|^2) = -\frac{1}{|\bar{a}|^2} I W_3^-(\bar{\phi})(\bar{a}). \tag{68}$$

An important point of this formula is that $p_1^\sharp \in D^\perp$. For the 1-forms q_1, p_2, q_2, one obtains

$$\begin{aligned}
q_1 &= -Ip_1 - \bar{b} - d^J(\ln |\nabla\Omega|^2);\\
p_2 &= -\frac{1}{2}I_2 J_1\bar{a} + \frac{1}{2}Ja;\\
q_2 &= Ip_2,
\end{aligned} \tag{69}$$

where I_1 and I_2 denote the (local) almost complex structures corresponding to $\bar{\phi}$ and $I\bar{\phi}$; recall that J_1, J_2 stand for the almost complex structures dual to ϕ and $J\phi$. Let us just briefly explain how the relations (69) are obtained. Since both D and $D^\perp$ are involutive distributions, we have

$$\begin{aligned}
(d\bar{a})(X, X') = (d(J\bar{a}))(X, X') = 0,\ \forall X, X' \in D^\perp\\
(d(J_1\bar{a}))(Y, Y') = (d(J_2\bar{a}))(Y, Y') = 0,\ \forall\, Y, Y' \in D.
\end{aligned}$$

and using (67), these can be seen to be equivalent to $q_2 = Ip_2$. For the remaining two relations we use (67) and $q_2 = Ip_2$ plugged into the identities (62). Taking the $1/2(\Omega + \overline{\Omega})$-components and the $[[\Lambda_J^{0,2}M]]$-components of the identities thus obtained, we eventually get

$$q_1 + Ip_1 + \bar{b} = -d^J(\ln|\nabla\Omega|^2),$$

which is the second relation of (69); the $[[\Lambda_I^{0,2}M]]$-components of the Ricci identities (62) imply

$$p_2^D = -\frac{1}{2}I_2J_1\bar{a},$$

or, equivalently,

$$p_2 = -\frac{1}{2}I_2J_1\bar{a} + p_0, \text{ with } p_0^\sharp \in D^\perp.$$

So far we have basically followed the track of the computations made in Lemma 5. However, now we can obtain even more by determining completely the 1-form p_0. For this, write the relation between Ω and $\overline{\Omega}$ as

$$\Omega = \frac{2}{|\bar{a}|^2}\bar{a} \wedge J\bar{a} - \overline{\Omega},$$

and take the covariant derivative of this relation, using (21), (56), (67-69). This eventually leads to $p_0 = \frac{1}{2}Ja$.

As we have already noticed, the opposite almost Hermitian structure $(g, \tilde{J})$ associated with (g, I) is not a new structure, but just $(g, -J)$. However, the iteration of the construction, imposes a natural new gauge $\tilde{\phi} \in [[\Lambda_J^{0,2}M]]$,

$$\tilde{\phi} = \bar{\phi} + \frac{2}{|\bar{a}|^2}I\bar{a} \wedge I\bar{\phi}(\bar{a}), \tag{70}$$

which may be different than the initial gauge ϕ. Taking the covariant derivative of (70) and using (58), (61) and (67-69), one finds, after a straightforward computation, that the 1-form $\tilde{b}$ of the structure $(g, \tilde{J})$ with respect to the gauge $\tilde{\phi}$ is given by (compare with (59)):

$$\tilde{b} = 3\bar{b} + d^I(\ln|\nabla\overline{\Omega}|^2) + 2d^J(\ln|\nabla\Omega|^2).$$

Now relation (66) immediately follows since $d\tilde{b} = -\gamma_{\tilde{J}} = \gamma_J$. □

Proof of Proposition 2. We assume as in Lemma 7 that (g, I) is *not* Kähler on an open set $U_0 \subset U$. Then on U_0 both relations (51) and (66) hold, and taking the appropriate linear combination of them, we obtain

$$5(\gamma_I - \gamma_J) = dd^I(\ln|\nabla\overline{\Omega}|^2).$$

On the other hand, starting from (25) and its companion relation giving γ_I, and by using (20) and the curvature information picked up in Lemma 6, we compute

$$\begin{aligned}\gamma_I - \gamma_J &= R(\overline{\Omega}) - \bar{a} \wedge I\bar{a} - R(\Omega) + a \wedge Ja \\ &= -\frac{|\nabla\Omega|^2}{8}(\Omega + \overline{\Omega}) - \frac{|\nabla\overline{\Omega}|^2}{16}(\Omega - \overline{\Omega}).\end{aligned}$$

From the last two identities we obtain the equality

$$-\frac{|\nabla\Omega|^2}{8}(\Omega + \overline{\Omega}) - \frac{|\nabla\overline{\Omega}|^2}{16}(\Omega - \overline{\Omega}) = \frac{1}{5}dd^I(\ln|\nabla\overline{\Omega}|^2), \tag{71}$$

which holds at every point of the open set U_0. Since $d^I(\ln|\nabla\overline{\Omega}|^2)$ is dual to a vector field in $D^{\perp}$ and $D^{\perp}$ is involutive, it follows that the right hand-side of (71) has a zero inner product with $\Omega + \overline{\Omega}$. For this to hold for the left hand-side of (71) as well, we must have $|\nabla\Omega| = 0$ at every point of U_0. But this is a contradiction, since U_0 is a subset of U, the set of points where $|\nabla\Omega| \neq 0$. Therefore, the assumption that the opposite structure (g, I) is not Kähler must be false, hence we proved Proposition 2. □

5. Proof of Theorem 2

As a consequence of Theorem 1 and Proposition 1, we first obtain the following global version of Proposition 2, which is essential for the proof of Theorem 2, but also presents interest in its own.

Proposition 3. *Any connected, strictly almost Kähler 4-manifold* (M, g, J, Ω) *satisfying the third Gray condition* (3) *admits a globally defined Kähler structure* (g, I) *which yields the opposite orientation to the one of* (M, J).

Proof. We show that the opposite almost Kähler structure (g, I) defined as in Theorem 3 can be extended globally on M. Let $p \in M - U$ be any point of the zero locus of the Nijenhuis tensor N. By the unique continuation property of N (see Proposition 1), there exists a number $k \geq 2$ such that

$$(\nabla^{\ell}\Omega)(p) = 0, \ \forall\, 1 \leq \ell \leq (k-1); \quad (\nabla^k\Omega)(p) \neq 0.$$

As a consequence, we obtain

$$0 = \langle \nabla^*\nabla(\nabla^{k-1}\Omega), \nabla^{k-1}\Omega\rangle_p = -\frac{1}{2}\Delta(|\nabla^{k-1}\Omega|^2)(p) + |\nabla^k\Omega|^2(p),$$

which shows that $\Delta(|\nabla^{k-1}\Omega|^2) \neq 0$ at p, hence also in a small neighborhood of p. On U, the distribution $D^{\perp}$ is spanned by the commuting hamiltonian Killing fields $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial t}$ (see Theorem 1), so that $d|\nabla^{k-1}\Omega|^2$ is zero on $D^{\perp}$; equivalently,

$|\nabla^{k-1}\Omega|^2$ is a function on the Riemann surface Σ. It then follows that on U we have (see (14)):

$$dd^J(|\nabla^{k-1}\Omega|^2) = -\Delta(|\nabla^{k-1}\Omega|^2)\Omega_\Sigma, \tag{72}$$

where, recall, Ω_Σ denotes the volume form of the (locally defined) Riemann surface Σ; as a global object on U, Ω_Σ is given by the restriction of Ω to $D \subset TU$, i.e. $\Omega_\Sigma(\cdot,\cdot) = \Omega(\mathrm{pr}^D\cdot, \mathrm{pr}^D\cdot)$ where pr^D denotes the projection to D. We thus can define I in a small neighborhood of p by setting

$$\Omega_I = -\Omega - \frac{2}{\Delta(|\nabla^{k-1}\Omega|^2)} dd^J(|\nabla^{k-1}\Omega|^2).$$

By (72), this agrees with the definition of I on U. Since U is dense in M and (g, I) is Kähler on U, we conclude that I can be extended as a Kähler structure on whole M. $\square$

Now we are ready to prove Theorem 2.

Suppose for contradiction that (M, g, J, Ω) is a *compact* strictly almost Kähler 4-manifold whose curvature satisfies (3). According to Proposition 3, we can consider (M, g, I) as a compact Kähler surface which also admits an almost Kähler structure (g, J, Ω), compatible with the opposite orientation to the one of (M, I); equivalently, Ω is an *indefinite* Kähler structure on (M, I) (see [42]). Note that the distributions D and $D^\perp$ are then well defined on the whole M, respectively as the (-1) and the $(+1)$-eigenspace of the symmetric endomorphism $Q = J \circ I$ of TM. Moreover, since (g, I) is Kähler, the relation (60) of Lemma 6 implies that on U (and therefore, by continuity, on M) the Ricci tensor has the form

$$\mathrm{Ric} = \frac{s}{2} g_\Sigma, \tag{73}$$

where g_Σ is defined on whole M as the restriction of g to the distribution D. Alternatively, (73) can be directly derived from the explicit form of g given by (5) of Theorem 1. Using (25) and (73), it is easy to compute the Chern forms γ_I and γ_J of (g, I) and (g, J), respectively,

$$\gamma_I = \frac{s}{2}\Omega_\Sigma, \quad \gamma_J = \left(\frac{s}{2} + \frac{|\nabla\Omega|^2}{4}\right)\Omega_\Sigma. \tag{74}$$

It follows that γ_I and γ_J are both degenerate on M. From Wu's relations, we conclude

$$c_1^2(M, I) = 2e(M) + 3\sigma(M) = 0;\ c_1^2(M, J) = 2e(M) - 3\sigma(M) = 0,$$

hence the Euler characteristic $e(M)$ and the signature $\sigma(M)$ of (M, I) both vanish. From the Kodaira classification [10] and the results of [31] and [42], we conclude that (M, I) can be a ruled surface over an elliptic curve, a complex torus, a hyperelliptic surface, or a minimal properly elliptic surface which is a fibration over a curve of genus at least two, with no fibers of singular reduction. We can easily exclude the cases when (M, I) is a torus or a hyperelliptic surface. Indeed, by replacing M with a finite cover in the case when (M, I) is hyperelliptic, we can always assume that (M, I) is a complex torus. By results of Taubes [45,46], on the 4-torus any symplectic form is homotopic to an invariant one; i.e., $[\gamma_I] = 0$ and $[\gamma_J] = 0$, so that by (74) we obtain

$$0 = [\gamma_J - \gamma_I] \cdot [\Omega] = \frac{1}{4} \int_M |\nabla \Omega|^2 d\mu,$$

where $\cdot$ denotes the cup-product of $H^2(M, \mathbb{R})$. The latter equality contradicts the assumption that J is not integrable.

We thus have to analyze the remaining two cases, when (M, I) is a ruled surface or a minimal elliptic surface. Since $\frac{\partial}{\partial t}$ (and $\frac{\partial}{\partial z}$) is a (real) I-holomorphic vector field in $D^{\perp}$, we conclude that $D^{\perp}$ is a holomorphic, complex rank one distribution on (U, I), hence also on M, which gives rise to a non-singular *holomorphic* foliation $\mathcal{F}$ on the complex surface (M, I). From (74) we get

$$\frac{1}{2}(\gamma_I - \gamma_J) = -\frac{|\nabla \Omega|^2}{8} \Omega_\Sigma \in 2\pi c_1(T\mathcal{F}). \tag{75}$$

Our next aim is to show that after replacing M with a finite cover if necessary, $H^0(M, T\mathcal{F}) \neq 0$.

In the case when (M, I) admits an elliptic fibration (which also includes some ruled surfaces over an elliptic base), we can use an argument from [5]: since $e(M) = \sigma(M) = 0$, we know that the elliptic fibers are smooth or multiple; since $c_1(M, I) \neq 0$, we have by Kodaira's formula that $c_1(M, I)$ is a non-zero multiple K of the Poincaré-dual of any regular (elliptic) fiber F. Since $[F] \cdot [F] = 0$ in homology, from (74), (75) and using the Poincaré duality, we derive

$$0 = \frac{1}{2\pi} \int_M \gamma_I \wedge (\gamma_I - \gamma_J) = c_1(M, I) \cdot [\gamma_I - \gamma_J] = -\frac{K}{4} \int_F |\nabla \Omega|^2 \Omega_\Sigma.$$

Note that Ω_Σ is a semi-positive (1,1)-form on (M, I) whose kernel, at any point of U, is $T\mathcal{F}$, while F is a (smooth) holomorphic curve in (M, I); it then follows from the above equality that, at any point $x \in U$, the fiber F must be tangent to $\mathcal{F}$; since U is dense in M, we conclude that $\mathcal{F}$ is tangent to the fibers everywhere. But as observed in [5], by replacing M with a finite cover if necessary, (M, I) then admits a globally defined, non-trivial holomorphic field tangent to the fibers, which is also a holomorphic section of $T\mathcal{F}$.

Suppose now that (M, I) is a ruled surface over an elliptic base B. By the previous argument, we may also assume that (M, I) does not admit any elliptic fibration. According to the the classification of non-singular holomorphic foliations (see [15, Prop.6] and [24, Sec.3]), under the above assumptions for (M, I), the following two cases arise:

Case 1: $\mathcal{F}$ is tangent to the rational fibers; since (M, I) is not an elliptic fibration, according to [39] we have $H^0(M, T\mathcal{F}) \neq 0$ as claimed.

Case 2: $\mathcal{F}$ is a foliation transversal to the rational fibers, i.e. a *Riccati foliation* in the terminology of [24]. In this case $c_1(T\mathcal{F})$ is the pull-back of a class of $H^2(B, \mathbb{Z})$ (see [24]), so that $c_1(T\mathcal{F})$ is a multiple of the Poincaré-dual of any rational fiber F. Since $[F] \cdot [F] = 0$ in homology and in view of (75), we have

$$0 = \frac{1}{2} \int_F (\gamma_I - \gamma_J) = -\frac{1}{8} \int_F |\nabla \Omega|^2 \Omega_\Sigma.$$

Then, we conclude as in the case of elliptic fibrations that each fiber F must be tangent to $\mathcal{F}$, which is a contradiction.

As a final step of the proof of Theorem 2, note that $T\mathcal{F} = D^\perp$ lies in the kernel of the Ricci tensor of g (see (73)); then, by the well-known Bochner-Lichnerowicz argument, for any holomorphic section Ξ of $T\mathcal{F} \subset TM$ we get $\nabla \Xi = 0$. Since we already showed $H^0(M, T\mathcal{F}) \neq 0$, it follows that $T\mathcal{F}$ should be parallel, so must be then its orthogonal complement D. But then J must be parallel too, a contradiction. $\square$

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