

THE CURVATURE AND THE INTEGRABILITY OF ALMOST-KÄHLER MANIFOLDS: A SURVEY

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1. INTRODUCTION

Situated at the intersection of Riemannian, symplectic and complex geometry, Kähler geometry is a subject that has been intensively studied in the past half-century. The overlap of just Riemannian and symplectic geometry is the wider area of almost-Kähler geometry. Although problems specific to almost-Kähler geometry have been considered in the past (see e.g. [42, 44, 77]), it is fair to say that our knowledge beyond the Kähler context is still small at this point. In recent years there is a renewed interest in almost-Kähler geometry, mainly motivated by the rapid advances in the study of symplectic manifolds.

An *almost-Kähler structure* on a manifold M of (real) dimension $n = 2m$ is a blend of three components: a symplectic form Ω , an almost-complex structure J and a Riemannian metric g , which satisfy the compatibility relation

$$\Omega(\cdot, \cdot) = g(J\cdot, \cdot). \quad (1)$$

Any one of these three components is completely determined by the remaining two via (1). If, additionally, the almost-complex structure J is integrable, i.e., is induced by a complex coordinate atlas on M , then the triple (g, J, Ω) is a *Kähler structure* on M . In real dimension 2, the notions of almost-Kähler and Kähler structure coincide, but this is no longer true in higher dimensions. Throughout the paper *strictly* almost-Kähler will mean that the corresponding almost-complex structure is *not* integrable, equivalently, that the almost-Kähler structure is not Kähler.

Depending on whether the Riemannian metric, or the symplectic form are initially given, there are slightly different perspectives towards studying almost-Kähler structures. Given a Riemannian metric g , it is a difficult question to decide if g admits any compatible almost-Kähler structure at all; in other words, we ask for the existence of an almost-complex structure J such that Ω determined by (1) is a closed 2-form. It is well known that the holonomy group determines if the given metric admits a compatible Kähler structure. Comparatively, the almost-Kähler problem is considerably more intricate, even locally. For example, until recently it was still an open question whether or not metrics of constant negative

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sectional curvature admit compatible almost-Kähler structures. Now the answer is known to be negative as a consequence of the results in [13, 75, 77].

Starting with a symplectic manifold (M, Ω) , there are many pairs (g, J) which satisfy the compatibility relation (1) and we shall refer to any such pair as an Ω -compatible almost-Kähler structure on M . The space of all Ω -compatible almost Kähler structures — equivalently, the space of all Ω -compatible metrics (or almost-complex structures) — will be denoted by $\mathbf{AK}(M, \Omega)$ and is well known to be an infinite dimensional, contractible Fréchet space. The choice of an *arbitrary* (or generic) metric in this space serves as an useful tool in studying the symplectic geometry and topology of the manifold. This is manifested in the theory of pseudo-holomorphic curves initiated by Gromov [45], and more recently, in Taubes' characterization of Seiberg-Witten invariants of symplectic 4-manifolds [84, 85]. Rather than dealing with generic Ω -compatible almost-Kähler metrics, the present survey is centered around the problem of identifying and studying “*distinguished*” Riemannian metrics in $\mathbf{AK}(M, \Omega)$.

One particularly nice situation would be to have an Ω -compatible Kähler metric g . In this case, the Levi-Civita connection ∇ preserves the other two structures, i.e., $\nabla J = 0$ and $\nabla \Omega = 0$. This leads to useful symmetry properties of basic geometric operators (like Laplacian, curvature, etc.) with respect to the complex structure, which further determine geometric and topological consequences on the manifold. However, it is now well known that in dimension higher or equal to four, most compact symplectic manifolds do not admit Kähler metrics. Thus one should require less of a compatible metric.

A good candidate for privileged Riemannian metric on a given manifold is an *Einstein* metric, i.e., a Riemannian metric g for which

$$\text{Ric} = \lambda g, \tag{2}$$

where Ric is the Ricci tensor and λ is a constant equal, up to a factor $\frac{1}{\dim_{\mathbb{R}}(M)}$, to the scalar curvature of g . Although a remarkable amount of research in Riemannian geometry has been done to study Einstein metrics (see e.g. [16, 65]), it seems that the almost-Kähler aspects of the problem remain in its infancy. A long standing, still open conjecture of Goldberg [42] affirms that there are no *Einstein*, strictly almost-Kähler metrics on a compact symplectic manifold. Indirectly, the Goldberg conjecture predicts that compatible Einstein metrics are very rare on compact symplectic manifolds. The conjecture is still far from being solved, but there are cases when it has been confirmed: Sekigawa [81] proved that the conjecture is true if the scalar curvature is non-negative (see Theorem 2 below) and there are further positive partial results in dimension four under other additional curvature assumptions [4, 12, 13, 75]. Moreover, a number of subtle topological restrictions to the existence of Einstein metrics on compact 4-manifolds are now known [55, 61, 62, 63], and these can be thought as further support for the conjecture.

Although the Goldberg conjecture is global in nature, it is the merit of John Armstrong [11] to point out that the local aspects of the problem will most likely

determine the global ones as well. His idea was to apply Cartan-Kähler theory to prove local existence of strictly almost-Kähler Einstein structures. Because of mounting algebraic difficulties he did not achieve this goal, but he proved several other interesting results and it became clear that Cartan-Kähler theory is a very useful tool for various problems in almost-Kähler geometry. For example, it turns out that the Cartan-Kähler theory is not uniquely relevant to Einstein almost-Kähler metrics, but is also suited for the study of a more general class of almost-Kähler metrics arising from a variational approach [17, 18]. This approach is reviewed in section 2. After we introduce some notations and basic facts of almost-Kähler geometry in section 3, section 4 casts some of the known integrability results under the unifying framework of Cartan-Kähler theory. In section 5 we then present a number of local examples of strictly almost-Kähler manifolds with interesting curvature properties, in particular some recent local constructions of Einstein strictly almost-Kähler metrics. The material in section 6 illustrates how additional conditions on the curvature often lead to completely solvable equations. We present some recent local classification results in dimension four, which can be used to obtain further integrability results. The last section is inspired by the recent works of Donaldson [32] and C. LeBrun [63], which naturally lead to the study of Einstein-like conditions with respect to the canonical Hermitian connection. We present a brief discussion on the relevant curvature conditions and highlight some directions for further research.

2. A VARIATIONAL APPROACH TO ALMOST-KÄHLER MANIFOLDS

Recall that [16] on a compact manifold M , the Einstein condition is the Euler-Lagrange equation of the *Hilbert functional* \mathbf{S} , the integral of the scalar curvature, acting on the space of all Riemannian metrics on M of a given volume.

A “symplectic” setting of this variational problem was proposed by Blair and Ianus [18]: restricting the Hilbert functional to the space $\mathbf{AK}(M, \Omega)$ of compatible metrics of a given compact symplectic manifold (M, Ω) , the critical points are the almost-Kähler metrics (g, J) whose Ricci tensor Ric is J -invariant, i.e., satisfies

$$\text{Ric}(J \cdot, J \cdot) = \text{Ric}(\cdot, \cdot). \quad (3)$$

The Euler-Lagrange equation (3) is thus a weakening of both the Einstein and the Kähler conditions. Furthermore, Blair [17] observed that for any almost-Kähler metric (g, J) the following relation holds:

$$\frac{1}{4} \int_M |\nabla J|^2 dv + \mathbf{S}(g) = \frac{4\pi}{(m-1)!} (c_1 \cdot [\Omega]^{\wedge(m-1)})(M), \quad (4)$$

where ∇ is the Levi-Civita connection of g , $|\cdot|$ is the point-wise norm induced by g , and c_1 and $dv = \frac{1}{m!} \Omega^{\wedge m}$ are respectively the first Chern class and the volume form of (M, Ω) . It follows that \mathbf{S} is directly related to the *Energy functional* which acts on the space of Ω -compatible almost-complex structures by

$$\mathbf{E}(J) = \int_M |\nabla J|^2 dv.$$

From this point of view, almost-Kähler metrics satisfying (3) have been recently studied in [59], and have been called there *harmonic* almost-Kähler metrics. However, in this paper we adopt the definition:

Definition 1. An almost-Kähler metric (g, J) is called *critical* if it satisfies (3).

It follows from (4) that the functional \mathbf{E} (resp. \mathbf{S}) is bounded from below (resp. from above), the Kähler metrics being minima of \mathbf{E} (resp. maxima of \mathbf{S}). However, a direct variational approach to finding extrema seems not to be easily applicable since it may happen that the infimum of \mathbf{E} be zero, although M does not carry Kähler structures at all.

Example 1. [59] Let $M = S^1 \times (Nil^3/\Gamma)$, where Nil^3 is the three-dimensional Heisenberg group and Γ is a co-compact lattice of Nil^3 ; as first observed by Thurston [86], M is a smooth four-dimensional manifold which admits a symplectic structure but does not admit Kähler metrics at all. Specifically, M carries an invariant complex structure J with trivial canonical bundle, and thus comes equipped with a holomorphic symplectic structure. To see this we write

$$Nil^3 = \{A \in GL(3, \mathbb{R}) \mid A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R}\}.$$

Then the 1-forms $dt, dx, dy, dz - xdy$ are left-invariant on $S^1 \times Nil^3$; we define a left-invariant complex structure J on this manifold by

$$J(dx) := dy; \quad J(dz - xdy) := dt,$$

and a holomorphic symplectic form ω by

$$\omega = (dx \wedge (dz - xdy) - dy \wedge dt) + i(dx \wedge dt + dy \wedge (dz - xdy)).$$

Being left-invariant, both J and ω descend to M to define a so-called Kodaira complex surface (M, J, ω) , cf. e.g. [14]. Note that the first Betti number of M is equal to 3, showing that no Kähler metric exists on M . On the other hand

$$\Omega = \text{Im}(\omega) = dx \wedge dt + dy \wedge (dz - xdy)$$

is a left-invariant symplectic form. Following [59], we consider a 2-parameter family $\mathbf{AK}(a, b), a > 0, b > 0$, of Ω -compatible (left-invariant) almost-Kähler structures $(g_{a,b}, I_{a,b}, \Omega)$, defined by

$$g_{a,b} = \frac{1}{a^2} dx^{\otimes 2} + \frac{1}{b^2} dy^{\otimes 2} + a^2 dt^{\otimes 2} + b^2 (dz - xdy)^{\otimes 2};$$

$$I_{a,b}(dx) := a^2 dt; \quad I_{a,b}(dy) := b^2 (dz - xdy).$$

One calculates directly

$$\text{Ric}(g_{a,b}) = -\frac{a^2 b^4}{2} \left(\frac{1}{a^2} dx^{\otimes 2} + \frac{1}{b^2} dy^{\otimes 2} - b^2 (dz - xdy)^{\otimes 2} \right),$$

so that the scalar curvature is

$$s_{a,b} = -\frac{a^2 b^4}{2}.$$

It follows that on the space $\mathbf{AK}_{a,b}$ we have $\sup(\mathbf{H}) = \inf(\mathbf{E}) = 0$, while M does not carry Kähler metrics at all. \diamond

In general, for a given compact symplectic manifold (M^{2m}, Ω) , it is tempting to compute the symplectic invariant

$$\mathbf{E}(\Omega) = \inf_g \int_M |\nabla J|^2 d\mu,$$

where the infimum is taken over all almost-Kähler structures (g, J) compatible with Ω . A natural approach to this problem is to study the (global) behavior of the solutions of the corresponding (negative) *gradient flow* equation

$$\frac{d}{dt}g_t = -(\text{Ric}^t)'',$$

where (g_t, J_t) is a path of almost-Kähler structures in $\mathbf{AK}(M, \Omega)$, Ric^t is the Ricci tensor of g_t and $(\text{Ric}^t)''$ is the gradient of \mathbf{E} , i.e.,

$$(\text{Ric}^t)''(\cdot, \cdot) = \frac{1}{2} \left(\text{Ric}^t(\cdot, \cdot) - \text{Ric}^t(J_t \cdot, J_t \cdot) \right).$$

This approach has been recently explored in [59]. However, as Example 1 suggests, technical difficulties appear when studying global properties of the flow g_t . One reason for this is perhaps hidden in the fact that even locally the equation (3) is difficult to be solved, and our next goal is to highlight some of these “local obstructions”.

3. THE CURVATURE AND THE NIJENHUIS TENSOR: FIRST OBSTRUCTIONS

3.1. Type-decompositions of forms and vectors. Let (M, g, J, Ω) be an almost-Kähler manifold of real dimension $n = 2m$. We denote by: TM the (real) tangent bundle of M ; T^*M the (real) cotangent bundle; $\Lambda^r M, r = 1, \dots, n$ the bundle of real r -forms; $S^\ell M$ the bundle of symmetric ℓ -tensors; (\cdot, \cdot) the inner product induced by g on these bundles (or on their tensor products).

Using the metric, we shall implicitly identify vectors and covectors and, accordingly, a 2-form ϕ with the corresponding skew-symmetric endomorphism of the tangent bundle TM , by putting: $(\phi(X), Y) = \phi(X, Y)$ for any vector fields X, Y .

The almost-complex structure J gives rise to a type decomposition of complex vectors and forms. By convention, J acts on the cotangent bundle T^*M by $(J\alpha)_X = -\alpha_{JX}$, so that J commutes with the Riemannian duality between TM and T^*M . We shall use the standard decomposition of the complexified cotangent bundle

$$T^*M \otimes \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M,$$

given by the $(\pm i)$ -eigenspaces of J , the type decomposition of complex 2-forms

$$\Lambda^2 M \otimes \mathbb{C} = \Lambda^{1,1}M \oplus \Lambda^{2,0}M \oplus \Lambda^{0,2}M,$$

and the type decomposition of symmetric (complex) bilinear forms

$$S^2 M \otimes \mathbb{C} = S^{1,1}M \oplus S^{2,0}M \oplus S^{0,2}M.$$

Besides the type decomposition of complex forms, we shall also consider the splitting of real 2-forms into J -invariant and J -anti-invariant ones; for any real section

of $\Lambda^2 M$ (resp. of $S^2 M$), we shall use the super-script $'$ to denote the projection to the real sub-bundle $\Lambda_{\mathbb{R}}^{1,1} M$ (resp. $S_{\mathbb{R}}^{1,1} M$) of J -invariant 2-forms (resp. symmetric 2-tensors), while the super-script $''$ stands for the projection to the bundle $[\Lambda^{0,2} M]$ (resp. $[S^{0,2} M]$) of J -anti-invariant ones; here and henceforth $[\cdot]$ denotes the real vector bundle underlying a given complex bundle. Thus, for any section ψ of $\Lambda^2 M$ (resp. of $S^2 M$) we have the orthogonal splitting $\psi = \psi' + \psi''$, where

$$\psi'(\cdot, \cdot) = \frac{1}{2}(\psi(\cdot, \cdot) + \psi(J\cdot, J\cdot)) \quad \text{and} \quad \psi''(\cdot, \cdot) = \frac{1}{2}(\psi(\cdot, \cdot) - \psi(J\cdot, J\cdot)).$$

The real bundle $[\Lambda^{0,2} M]$ (resp. $[S^{0,2} M]$) inherits a canonical complex structure, still denoted by J , which is given by

$$(J\psi)(X, Y) := -\psi(JX, Y), \quad \forall \psi \in [\Lambda^{0,2} M],$$

so that $([\Lambda^{0,2} M], J)$ becomes isomorphic to the complex bundle $\Lambda^{0,2} M$. We adopt a similar definition for the action of J on $[S^{0,2} M]$. Notice that, using the metric g , $[S^{0,2} M]$ can be also viewed as the bundle of symmetric, J -anti-commuting endomorphisms of TM .

We finally define the $U(m)$ -decomposition (with respect to J) of real 2-forms

$$\Lambda^2 M = \mathbb{R} \cdot \Omega \oplus \Lambda_0^{1,1} M \oplus [\Lambda^{0,2} M], \quad (5)$$

where $\Lambda_0^{1,1} M$ is the sub-bundle of the *primitive* (1,1)-forms, i.e., the J -invariant 2-forms which are point-wise orthogonal to Ω .

3.2. Decompositions of algebraic curvature tensors. Following [16], we denote by R the curvature with respect to the Levi-Civita connection ∇ of g . Classically, R is considered as a real (1,3)-tensor on M , but, using the metric, we shall rather think of R as a (0,4)-tensor, or as a section of the bundle $S^2(\Lambda^2 M)$ of symmetric endomorphisms of $\Lambda^2 M$, depending on the context. We adopt the convention $R_{XYZV} = (\nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z, V)$; the action of R on $\Lambda^2 M$ is then defined by

$$R(\alpha \wedge \beta)_{X,Y} := R_{\alpha\sharp, \beta\sharp, X, Y},$$

where the super-script \sharp denotes the corresponding (Riemannian) dual vector fields. Thus, the curvature R can be viewed as a section of the symmetric tensor product $\Lambda^2 M \odot \Lambda^2 M$, and satisfies the *algebraic Bianchi identity*

$$R_{XYZ} + R_{ZXY} + R_{YZX} = 0.$$

It follows that R is actually a section of the sub-bundle $\mathcal{R}(M) = \ker(\epsilon)$, where $\epsilon : \Lambda^2 M \odot \Lambda^2 M \rightarrow \Lambda^4 M$ is given by $\epsilon(\phi \odot \phi) = \phi \wedge \phi$. The bundle $\mathcal{R}(M)$ is called the bundle of *algebraic curvature tensors*.

Recall that the *Ricci* contraction, Ric , associates to R the symmetric bilinear form

$$\text{Ric}(X, Y) = \sum_{i=1}^n R_{X, e_i, Y, e_i}.$$

It can be thought as a linear map $\mathfrak{r} : \mathcal{R}(M) \rightarrow S^2 M$. For $n \geq 3$ this map is surjective and its adjoint, \mathfrak{r}^* , is injective. We thus obtain the orthogonal decomposition

$$\mathcal{R}(M) = \mathfrak{r}^*(S^2 M) \oplus \mathcal{W}(M),$$

where the sub-bundle $\mathcal{W}(M)$ is the kernel of \mathfrak{r} and is called the bundle of *algebraic Weyl tensors*. Accordingly, the curvature R splits as $R = \widehat{\text{Ric}} + W$, where the *Weyl curvature*, W , is the component of R in $\mathcal{W}(M)$ while $\widehat{\text{Ric}}$ is identified with the Ricci tensor by the property $\mathfrak{r}(\widehat{\text{Ric}}) = \text{Ric}$. Specifically, starting from the splitting of Ric into its trace

$$s = \sum_{i=1}^n \text{Ric}(e_i, e_i),$$

the scalar curvature of (M, g) , and its trace-free part $\text{Ric}_0 = \text{Ric} - \frac{s}{n}g$, we have that

$$\widehat{\text{Ric}} = h \wedge g,$$

where $h = \frac{s}{2n(n-1)}g + \frac{1}{(n-2)}\text{Ric}_0$ is the *normalized Ricci tensor* of (M, g) and \wedge denotes the Kulkarni-Nomizu product of bilinear forms. We finally obtain the splitting of R into three orthogonal pieces:

$$R = \frac{s}{n(n-1)}\text{Id}|_{\Lambda^2 M} + \frac{1}{(n-2)}(\text{Ric}_0 \wedge g) + W. \quad (6)$$

Each of the three terms in the right-hand side of (6) is an element of $\mathcal{R}(M)$; the corresponding sub-bundle of $\mathcal{R}(M)$ is attached to an irreducible representation of the orthogonal group $O(n)$ acting on $\mathcal{R}(\mathbb{R}^n)$, namely the trivial representation, the Cartan product $\mathbb{R}^n \odot \mathbb{R}^n$ and the product $\mathfrak{o}(n) \odot \mathfrak{o}(n)$ where the Lie algebra $\mathfrak{o}(n)$ is identified with $\Lambda^2 \mathbb{R}^n$. Note that the *Einstein equations* for a Riemannian metric g correspond to the vanishing of the component $\frac{1}{(n-2)}(\text{Ric}_0 \wedge g)$ of R ; equivalently, the Ricci tensor satisfies $\text{Ric} = \frac{s}{n}g$, in which case the scalar curvature s is necessarily constant. Recall also that a manifold with zero Weyl tensor is conformally flat.

In dimension four, one can further refine the splitting (6) by fixing an orientation on (M, g) and considering the *Hodge* $*$ -operator which acts on the bundle $\Lambda^2 M$ as an involution, thus giving rise to an orthogonal splitting of $\Lambda^2 M$ into the sum of the \pm -eigenspaces, the bundles of self-dual and anti-self-dual 2-forms. Consequently, the Weyl tensor also decomposes as $W^+ + W^-$, where $W^+ = \frac{1}{2}(W + W \circ *)$ is the part acting on self-dual forms, while $W^- = \frac{1}{2}(W - W \circ *)$ is the restriction of W to the bundle of anti-self-dual forms. An oriented Riemannian 4-manifold (M, g) is called *self-dual* (resp. *anti-self-dual*) if its Weyl tensor is self-dual (resp. anti-self-dual), i.e., if $W^- \equiv 0$ (resp. $W^+ \equiv 0$).

We now assume that $n = 2m$ is even and that (M^{2m}, g) is endowed with a g -orthogonal almost-complex structure J , i.e., (g, J) is an almost-Hermitian structure on M . Representation theory of the unitary group $U(m)$ implies the splitting of the bundle $\mathcal{R}(\mathbb{C}^m)$ into irreducible factors, each of them defining a *curvature component* of R with respect to J , cf. [89]. Thus, for $n \geq 8$, R has 10 different curvature components; for $n = 6$, we have 9 curvature components, while for $n = 4$ one gets 7 different curvature components. Note that the most natural special types of almost-Hermitian manifolds correspond to the vanishing of some of these components.

The following $U(m)$ -invariant sub-bundles of $\mathcal{R}(M)$ arise naturally [44, 89]:

- $\mathcal{K}(M) = \{R \in \mathcal{R}(M) : R_{JXJY} = R_{XY}\}$ is the sub-bundle of elements acting trivially on $[\Lambda^{2,0}M]$. The curvature of any Kähler metric belongs to $\mathcal{K}(M)$, so $\mathcal{K}(M)$ is usually referred to as the bundle of *kählerian algebraic curvature tensors*.
- $\mathcal{R}_2(M) = \{R \in \mathcal{R}(M) : R_{XYZW} - R_{JXJYZW} = R_{JXYJZW} + R_{JXJYZJW}\}$ is the sub-bundle of elements preserving the type decomposition (5) of complex 2-forms; equivalently, these are the elements of $\mathcal{R}(M)$ which preserve the splitting (5) and commute with the complex structure of $[\Lambda^{0,2}M]$.
- $\mathcal{R}_3(M) = \{R \in \mathcal{R}(M) : R_{JXJYJZJT} = R_{XYZT}\}$ is the sub-bundle of elements of $\mathcal{R}(M)$ preserving the type decomposition (5) of real 2-forms.

Note that $\mathcal{K}(M) \subset \mathcal{R}_2(M) \subset \mathcal{R}_3(M)$, and for any almost-Hermitian manifold (M, g, J) whose curvature tensor belongs to $\mathcal{R}_3(M)$, the Ricci tensor satisfies (3).

Each of the above sub-bundles can be further decomposed into irreducible factors. For instance, the bundle $\mathcal{K}(M)$ splits into three pieces, associated respectively to the trivial representation of $U(m)$, to the Cartan product $\mathbb{C}^m \odot \mathbb{C}^m$, and to $\mathfrak{u}(m) \odot \mathfrak{u}(m)$. Corresponding to the trivial factor is the curvature of a complex space form (so that this component can be identified to the scalar curvature), the second factor is identified to the traceless Ricci tensor, while the third component is the so-called *Bochner curvature*; for more details see [89, 36].

We further denote by W'' the component of the curvature operator defined by

$$W''_{XYZT} = \frac{1}{8} \left(R_{XYZT} - R_{JXJYZT} - R_{XYJZJT} + R_{JXJYJZJT} \right. \\ \left. - R_{XJYZJT} - R_{JXYZJT} - R_{XJYJZT} + R_{JXYJZT} \right).$$

It follows that

$$W''_{Z_1Z_2Z_3Z_4} = R_{Z_1Z_2Z_3Z_4} = W_{Z_1Z_2Z_3Z_4} \quad \forall Z_i \in T^{1,0}M,$$

which explains the notation.

Let us also introduce the following Ricci-type curvature tensors of an almost-Hermitian manifold:

- the invariant and the anti-invariant parts of the Ricci tensor with respect to J , denoted by Ric' and Ric'' respectively. We also put $\rho(\cdot, \cdot) = \text{Ric}'(J\cdot, \cdot)$ to be the (1,1)-form corresponding to the J -invariant part of Ric , which will be called *Ricci form* of (M, g, J) .
- the *twisted Ricci form* $\rho^* = R(\Omega)$ which is not, in general, J -invariant. We will consequently denote by $(\rho^*)'$ and $(\rho^*)''$ the corresponding projections onto the bundles $\Lambda_{\mathbb{R}}^{1,1}M$ and $[\Lambda^{0,2}M]$, respectively. Following [89], the trace of ρ^* , $s^* = 2(R(\Omega), \Omega)$, will be called **-scalar curvature*.

For a Kähler manifold $\rho = \rho^*$ is the usual Ricci form which is closed, of type (1,1) and represents (up to a scaling factor $\frac{1}{2\pi}$) the de Rham cohomology of the first Chern class of (M, J) . Neither of these properties remains true for a generic almost-Kähler manifold.

3.3. The Nijenhuis tensor. The *Nijenhuis tensor* (or *complex torsion*) of an almost-complex structure J is defined by

$$N_{X,Y} = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

Thus, N is a 2-form with values in TM , which by Newlander-Nirenberg theorem [72] vanishes if and only if J is integrable. Alternatively, the Nijenhuis tensor can be viewed as a map from $\Lambda^{1,0}M$ to $\Lambda^{0,2}M$ in the following manner: given a complex (1,0)-form ψ , we define by $\partial\psi$ and $\bar{\partial}\psi$ the projectors of $d\psi$ to $\Lambda^{2,0}M$ and $\Lambda^{1,1}M$, respectively. In general, $d \neq \partial + \bar{\partial}$, as $d\psi$ can also have a component of type (0,2), which we denote by $N(\psi)$; writing $\psi = \alpha + iJ\alpha$ where α is a real 1-form, one calculates

$$N(\psi)_{X,Y} = \frac{1}{2}\alpha(N_{X,Y}) = \frac{1}{4}\psi(N_{X,Y}), \quad \forall X, Y \in T^{0,1}M.$$

For an almost-Kähler manifold the vanishing of N is equivalent to Ω being parallel with respect to the Levi-Civita connection ∇ of g ; specifically, the following identity holds (cf. e.g. [53]):

$$(\nabla_X \Omega)(\cdot, \cdot) = \frac{1}{2}(JX, N(\cdot, \cdot)).$$

Using the above relation, we shall often think of N as a T^*M -valued 2-form, or as a $\Lambda^2 M$ -valued 1-form, by tacitly identifying N with $\nabla\Omega$. Further, since Ω is closed and N is a J -anti-invariant 2-form with values in TM , one easily deduces that $\nabla\Omega$ is, in fact, a section of the vector bundle $[[\Lambda^{0,1}M \otimes \Lambda^{0,2}M]]$, i.e., the following relation holds:

$$\nabla_{JX} J = -J(\nabla_X J), \quad \forall X \in TM. \quad (7)$$

3.4. The Weitzenböck decomposition of the symplectic form. Let δ be the codifferential of g , acting on (real) r -tensors by

$$(\delta T)_{X_1, \dots, X_{r-1}} = - \sum_{i=1}^n (\nabla_{e_i} T)_{e_i, X_1, \dots, X_{r-1}},$$

where $\{e_i\}_{i=1}^n$ is any orthonormal frame. Note that when acting on r -forms, δ is the formal adjoint operator to the exterior differential d ; the two operators are then related via the Hodge star-operator $*_g$:

$$\delta = - *_g \circ *_g d.$$

Then, the operator $\Delta = d\delta + \delta d$, which acts on the smooth sections of $\Lambda^r M$, is the *Hodge-de Rham Laplacian* of (M, g) ; we also consider the *rough Laplacian*, $\nabla^* \nabla$, which for any smooth r -form, ψ , gives $\nabla^* \nabla \psi := \delta(\nabla\psi)$, again an r -form. The two Laplacians are related by a *Weitzenböck decomposition*

$$\Delta = \nabla^* \nabla + A(R),$$

where A depends linearly on the curvature R of (M, g) , see e.g. [16]. For example, the Weitzenböck decomposition on 2-forms reads as:

$$\begin{aligned} \Delta\psi - \nabla^*\nabla\psi &= [\text{Ric}(\psi, \cdot) - \text{Ric}(\cdot, \psi)] - 2R(\psi) \\ &= \frac{2(m-1)}{m(2m-1)}s\psi - 2W(\psi) \\ &\quad + \frac{(m-2)}{(m-1)}[\text{Ric}_0(\psi, \cdot) - \text{Ric}_0(\cdot, \psi)]. \end{aligned} \quad (8)$$

The symplectic form Ω is a real harmonic 2-form of type $(1, 1)$ with respect to any compatible almost-Kähler metric (g, J) , i.e.

$$\Omega(J\cdot, J\cdot) = \Omega(\cdot, \cdot), \quad d\Omega = 0 \text{ and } \delta\Omega = 0.$$

Applying relation (8) to Ω , we obtain

$$\nabla^*\nabla\Omega = 2R(\Omega) - [\text{Ric}(J\cdot, \cdot) - \text{Ric}(\cdot, J\cdot)]. \quad (9)$$

Formula (9) is a measure of the difference of the two types of Ricci forms on an almost-Kähler manifold:

$$\rho^* - \rho = \frac{1}{2}(\nabla^*\nabla\Omega). \quad (10)$$

Taking the inner product with Ω of the relation (10), we obtain the difference of the two types of scalar curvatures:

$$s^* - s = |\nabla\Omega|^2 = \frac{1}{2}|\nabla J|^2. \quad (11)$$

Formulae (10) and (11) can be interpreted as “obstructions” to the (local) existence of a strictly almost-Kähler structure J , compatible with a given metric g . Indeed, if we denote by P the curvature type operator acting on Λ^2M by

$$P(\psi) = \frac{2(m-1)}{m(2m-1)}s\psi - 2W(\psi),$$

then, by (8) and (11), $(P(\Omega), \Omega) = -|\nabla\Omega|^2 \leq 0$. This shows that Riemannian metrics for which P is semi-positive definite do not admit even locally compatible strictly almost-Kähler structures [47]. Note that the latter curvature condition is implied by the non-negativity of the *isotropic* sectional curvatures. The following compilation of results is an illustration of this criterion of non-existence:

Proposition 1. *Let (M, g) be a Riemannian manifold of dimension $n = 2m$, which satisfies one of the following conditions:*

- (i) *M is four-dimensional and oriented, and g is anti-self-dual metric of non-negative scalar curvature;*
- (ii) *(M, g) is a conformally-flat manifold of non-negative scalar curvature;*
- (iii) *(M, g) is a symmetric space of compact type.*

Then, (M, g) does not admit even locally defined strictly almost-Kähler structures (which are also compatible with the fixed orientation in the case (i)).

Remark. The situation dramatically changes if one considers symmetric spaces of non-compact type, see Proposition 4 below. \diamond

4. CARTAN-KÄHLER THEORY: FURTHER OBSTRUCTIONS

In this section, which is inspired from [11] (see also [4]), we shall consider the problem of local existence of Einstein almost-Kähler metrics. By Darboux's theorem, it is enough to look for Einstein metrics

$$g = \sum_{i,j=1}^{2m} g_{ij}(x) dx_i \otimes dx_j$$

compatible with the standard symplectic form

$$\Omega_0 = \sum_{i=1}^m dx_i \wedge dx_{m+i}$$

on an open set of $\mathbb{R}^{2m} = \{x = (x_1, \dots, x_{2m})\}$. By a result of DeTurck and Kazdan [31], any Einstein metric is real analytic (in suitable local coordinates), so that it is also natural to require the analyticity of the components $g_{ij}(x)$ of the metric.

The Ricci curvature can be thought as a non-linear second-order differential operator

$$\text{Ric} : \mathbf{AK}(\Omega_0) \ni g \rightarrow \text{Ric}(g) \in \Gamma(S^2)$$

acting on the space of Ω_0 -compatible almost-Kähler metrics. Here and henceforth, S^l stands for the bundle of symmetric l -tensors on \mathbb{R}^{2m} . For a fixed $J \in \mathbf{AK}(\Omega_0)$, let $E = \llbracket S^{0,2} \rrbracket$ denote the corresponding bundle of J -anti-invariant elements of S^2 (see section 2.1). Then the space of smooth sections of E is naturally identified with the tangent space of $\mathbf{AK}(\Omega_0)$ at (g, J) , and the *principal symbol* $\sigma(\text{Ric})$ of the Ricci operator is a bundle-map

$$\sigma(\text{Ric}) : S^2 \otimes E \longrightarrow S^2$$

given by

$$\sigma(\text{Ric})(C)_{X,Y} = \frac{1}{2} \sum_{i=1}^{2m} \left[C_{e_i, X, e_i, Y} + C_{e_i, Y, e_i, X} - C_{e_i, e_i, X, Y} - C_{X, Y, e_i, e_i} \right]$$

for any section C of $S^2 \otimes E$ (see e.g. [16]). One can easily check that this map is surjective. This shows that for any symmetric $(2m \times 2m)$ -matrix r_0 with constant coefficients there are metrics g which are compatible with Ω_0 and such that, at a given point $x_0 \in \mathbb{R}^{2m}$, we have $\text{Ric}(g)_{x_0} = r_0$; in other words, at any given point x_0 and for any $r \in \Gamma(S^2)$, one can always find *point-wise* (or *algebraic*) solutions to the equation $\text{Ric}(g)_{x_0} = r_{x_0}$. Similarly, at any point x_0 and for any value λ , one can find a metric $g \in \mathbf{AK}(\Omega_0)$ for which $\text{Ric}(g)_{x_0} = \lambda g_{x_0}$. The next question is whether or not a point-wise solution can be extended to a real analytic solution defined in a neighborhood of the point. A coordinate-free realization of the Cauchy-Kowalewski theorem, known as the Cartan-Kähler theory, gives a method of answering this question and we refer the reader to [22] for details and references relevant to this method. We recall here that the basic idea behind Cartan-Kähler theory is a simple one: one builds up order by order real analytic solutions to the given system of differential equations, by using the point-wise solutions.

Although Cartan-Kähler theory is usually used as a tool of proving existence results, in [11] a different aspect of the theory is emphasized. It also provides a method to find non-obvious conditions — which we shall refer to in this paper as “obstructions”, but which are called “torsion” in [22] — that solutions of a differential equation must satisfy. These obstructions encapsulate in an invariant manner the elementary fact that derivatives in different coordinate directions must commute. We shall write down in an explicit way such local obstructions to finding critical almost-Kähler metrics and use them to prove the integrability of the almost-complex structure in the compact case, provided that certain additional curvature conditions are satisfied. Thus, Cartan-Kähler theory will be used to prove *non-existence* results. On the other hand, because of algebraic difficulties in applying the theory to its end, no general existence result has been proven so far.

Suppose that one wishes to apply the Cartan-Kähler theory (as described in [37]) in order to prove that strictly almost-Kähler Einstein metrics exist. We have already seen that one can always find examples of 2-jets of compatible metrics which satisfy $\text{Ric}(g) = \lambda g$; the next question is whether or not one can find algebraic examples of 3-jets satisfying this equation and its first derivative — i.e. whether or not one can find algebraic solutions to the first prolongation of the problem. The symbol of the first prolongation, $\sigma_1(\text{Ric})$, is a bundle map

$$\sigma_1(\text{Ric}) : S^3 \otimes E \rightarrow T^* \otimes S^2.$$

By calculating the dimension of the image of $\sigma_1(\text{Ric})$, one sees that this map is not onto. To see this directly, recall that the *differential Bianchi identity* implies $\delta \text{Ric} = -\frac{1}{2} ds$, where, we recall, δ is the co-differential of g and $s = \text{trace}_g(\text{Ric})$ is the scalar curvature. If we can extend a 2-jet solution of the Einstein equation to a 3-jet solution, we must have $d\lambda = 0$; we thus have found an obstruction to extending the 2-jet solution to a 3-jet solution, which simply tells us that an Einstein metric has constant scalar curvature. If we denote by $\mathfrak{b} : T^* \otimes S^2 \rightarrow T^*$ the equivariant map

$$\mathfrak{b}(C)_X = \sum_{i=1}^{2m} (C_{e_i, e_i, X} - \frac{1}{2} C_{X, e_i, e_i}),$$

then we have the exact sequence:

$$0 \rightarrow S^3 \otimes E \xrightarrow{\sigma_1(\text{Ric})} T^* \otimes S^2 \xrightarrow{\mathfrak{b}} T^* \rightarrow 0.$$

This tells us that this is the only such obstruction.

Exactly the same obstruction arises for the more general problem of finding Einstein metrics when one does not insist that the metric is compatible with the symplectic form. It turns out that if one is looking for metrics not necessarily compatible with a symplectic form, then there are no further obstructions and the Cartan-Kähler theorem allows one to conclude that there is a germ of local solutions to the general Einstein equations.

Theorem 1. [37] *Let R_0 be a given algebraic curvature tensor at the origin in \mathbb{R}^n , $n \geq 3$ (see section 3.2), and let g_0 be an algebraic (Riemannian) metric such that the Ricci tensor of R_0 with respect to g_0 is equal to λg_0 for some constant λ .*

Then there is a real analytic metric g defined in a neighborhood of the origin, such that

- (i) $g_{x=0} = g_0$ and $(R^g)_{x=0} = R_0$;
- (ii) $\text{Ric}(g) = \lambda g$.

Turning back to our initial problem of finding Einstein almost-Kähler metrics, one should notice that if we could extend the above result to the case of metrics compatible with Ω_0 , then strictly almost-Kähler solutions would automatically exist by taking any 2-jet solution (g_0, J_0) of the Einstein equations, for which the curvature R_0 does not belong to the space of kählerian algebraic tensors with respect to J_0 ; for instance, just by calculating dimensions, one sees that there are algebraic 2-jet solutions to the Einstein equation such that $W_0'' \neq 0$ (see section 3.2).

One then naturally wonders, are there any higher obstructions to extending ℓ -jet algebraic solutions of the Einstein equations to $(\ell + 1)$ -jet solutions, but this time in the setting of Ω_0 -compatible metrics? Unfortunately, it turns out that there are. Indeed, letting $\sigma_2(\text{Ric})$ be the symbol of the second prolongation, it turns out [11] that the sequence

$$0 \rightarrow S^4 \otimes E \xrightarrow{\sigma_2(\text{Ric})} S^2 \otimes S^2 \xrightarrow{\sigma_1(\mathfrak{b})} T^* \otimes T^* \rightarrow 0$$

is not exact. By calculating the dimension of the image of $\sigma_2(\text{Ric})$ one can see that there must be an equivariant bundle-map $\mathfrak{c} : S^2 \otimes S^2 \rightarrow \mathbb{R}$ such that

$$0 \rightarrow S^4 \otimes E \xrightarrow{\sigma_2(\text{Ric})} S^2 \otimes S^2 \xrightarrow{\sigma_1(\mathfrak{b}) \oplus \mathfrak{c}} T^* \otimes T^* \oplus \mathbb{R} \rightarrow 0$$

is exact. Thus, there is some obstruction to extending 3-jet solutions of the Einstein equations in the almost-Kähler setting.

The above theory does not lead in a particularly simple way to finding the obstruction in an explicit form. It merely tells us that there is such an obstruction and that we have to examine the 4-jet of the metric to find it — even though eventually the obstruction takes the form of a condition on the 3-jet.

The explicit calculations for obtaining the obstruction in most general form have been carried out in [34] (in the four dimensional case) and have been later extended in [10] for the case of higher dimensional almost-Kähler manifolds; as a final result, we obtain a general identity which holds for any almost-Kähler manifold. In particular cases this identity was found in [4, 11, 75, 81]. To state our result, we introduce

$$\phi(X, Y) = (\nabla_{JX}\Omega, \nabla_Y\Omega), \quad (12)$$

which, by (7), is a semi-positive definite $(1, 1)$ form. We then have

Proposition 2. [10] *For any almost-Kähler structure (g, J, Ω) the following relation holds (notations of section 3):*

$$\begin{aligned} \delta(J\delta(J\text{Ric}'')) &= -\frac{1}{4}\Delta(|\nabla\Omega|^2) + 2\delta((\rho^*, \nabla \cdot \Omega)) + \frac{1}{2}|\text{Ric}''|^2 - |(\rho^*)''|^2 \quad (13) \\ &\quad - 2|W''|^2 - \frac{1}{4}|(\nabla^*\nabla\Omega)'|^2 - \frac{1}{4}|\phi|^2 + (\rho, \phi) - (\rho, \nabla^*\nabla\Omega). \end{aligned}$$

In the case of an Einstein manifold, the above relation reduces to

$$8\delta((\rho^*, \nabla \cdot \Omega)) - \Delta(|\nabla\Omega|^2) = 8|W''|^2 + 4|(\rho^*)''|^2 + |(\nabla^*\nabla\Omega)'|^2 + |\phi|^2 + \frac{s}{n}|\nabla\Omega|^2;$$

this condition on the 3-jet of an Einstein, almost-Kähler manifold is precisely the obstruction referred to earlier.

Suppose M is a compact Einstein, almost-Kähler manifold of positive or zero scalar curvature. After integrating the above relation over the manifold, we obtain the following result due to Sekigawa.

Theorem 2. [81] *A compact Einstein, almost-Kähler manifold of non-negative scalar curvature is necessarily Kähler .*

More generally, (13) provides a non-trivial relation on the 3-jet of any critical metric, which can be also viewed as an obstruction for the solvability of Equation (3). In the case of a (connected) critical almost-Kähler 4-manifold whose curvature component W'' identically vanishes, this relation implies a strong maximum principle for the Nijenhuis tensor N [4]: if N vanishes at one point, then N is identically zero on M . It follows that on any (connected) compact critical strictly almost-Kähler 4-manifold with $W'' = 0$, the complex rank 2 bundle $L = \Lambda^{0,1}M \otimes \Lambda^{0,2}M$ has a nowhere vanishing (real) smooth section, the Nijenhuis tensor N (see section 3.3). As observed by J. Armstrong [13], the latter in turn implies strong topological consequences via the Chern-Weil-Wu formulae:

$$0 = c_2(L)(M) = (2c_1^2 + c_2)(M) = (5\chi + 6\sigma)(M),$$

where $c_2(L)$, c_1 and c_2 are the Chern classes of the (rank 2) complex bundles L and TM , while χ and σ are the Euler characteristic and the signature of M . Combining with the Hitchin-Thorpe inequality $(2\chi \pm 3\sigma)(M) \geq 0$ in the Einstein case [48], we obtain the following integrability result.

Theorem 3. [4] *Let (g, J) be a critical almost-Kähler metric on a compact symplectic 4-manifold (M, Ω) . Suppose that $W'' = 0$ (see section 3.2). Then (g, J) is Kähler provided that $(5\chi + 6\sigma)(M) \neq 0$. In particular, a compact Einstein, almost-Kähler 4-manifold is Kähler if and only if $W'' = 0$.*

One might further ask if there are any other obstructions to finding almost-Kähler Einstein metrics. For that we consider the third order differential operator

$$O_1(g) = \Delta(|\nabla\Omega|^2) - 8\delta((\rho^*, \nabla \cdot \Omega)) + 8|W''|^2 + 4|(\rho^*)''|^2 + |(\nabla^*\nabla\Omega)'|^2 + |\phi|^2 + \frac{s}{n}|\nabla\Omega|^2$$

and try to solve the modified system

$$\text{Ric}(g) - \lambda g = 0, \quad O_1(g) = 0.$$

J. Armstrong [11] showed that there are at least two further obstructions arising in this manner. The first one is an obstruction to lifting the 4-jet to the 5-jet solution of the above system, which in the case of 4-manifolds corresponds to a bundle-map

$$\mathfrak{d} : S^3M \otimes S^2M \oplus S^2M \rightarrow \Lambda^{1,1}M \cong \Lambda_0^{1,1}M \oplus \mathbb{R}.$$

Finding the above mentioned obstructions in an explicit form seems to be a difficult task, even in the case of 4-manifolds. However, along the lines of [11], one sees that a certain component of \mathfrak{d} ,

$$\mathfrak{d}_{\mathbb{R}} : S^2M \otimes S^2M \rightarrow \mathbb{R},$$

should correspond to a relation expressing the term $(\nabla^*\nabla N, N)$ with lower jets of (g, J) , where N is the Nijenhuis tensor of J , also identified to $\nabla\Omega$ via the metric, see section 3.3. We can write down the relation corresponding to $\mathfrak{d}_{\mathbb{R}}$ by specifying the Weitzenböck decomposition of TM -valued 2-forms for the particular section N . This is done in [34] (see also [6]).

Proposition 3. [34] *For any almost-Kähler 4-manifold (M, g, J, Ω) the following relation holds*

$$\begin{aligned} \delta(J\delta(J\text{Ric}'')) &= -\frac{1}{4}\Delta(|\nabla\Omega|^2) - \delta\delta(\text{Ric}'') - \delta((\rho^*, \nabla \cdot \Omega)) + \frac{1}{2}|\text{Ric}''|^2 + |(\rho^*)''|^2 \\ &\quad - \frac{1}{2}|(\nabla^2\Omega)_0^{\text{sym}}|^2 - \frac{3}{32}|\nabla\Omega|^4 - |W''|^2 - \frac{s}{4}|\nabla\Omega|^2 - (\rho, \phi), \end{aligned}$$

where $(\nabla^2\Omega)_0^{\text{sym}}$ is the image of $\nabla^2\Omega$ under the bundle map $(T^*M)^{\otimes 2} \otimes \Lambda^2M \rightarrow S_0^2M \otimes \Lambda^2M$, which acts as the identity on the second factor and in the first factor takes the trace free part of the symmetrization.

Combining the two obstructions given by Propositions 2 and 3, and taking into account that in 4-dimension $(\nabla^*\nabla\Omega)' = \frac{|\nabla\Omega|^2}{2}\Omega$ and $|\phi|^2 = \frac{|\nabla\Omega|^4}{4}$, one derives the following integrability result.

Theorem 4. [34] *Let (M, Ω) be a compact, 4-dimensional symplectic manifold with $(c_1 \cdot [\Omega])(M) \geq 0$. Let (g, J) be an Ω -compatible, critical almost-Kähler metric and assume that one of the following is satisfied:*

- (a) *the scalar curvature s is non-negative;*
- (b) *the scalar curvature s is constant.*

Then (g, J) is necessarily a Kähler structure.

Remarks. 1. Note that if the scalar curvature is non-negative, then the condition $(c_1 \cdot [\Omega])(M) \geq 0$ is automatically satisfied because of (4). For the case (b), it is *a priori* open the possibility that the scalar curvature is a negative constant, although *a posteriori*, the result shows that this cannot be the case, again because of (4). It would be interesting to see if the result is still valid without any of the conditions (a) or (b).

2. The above theorem should be seen in the vein of the recent classification results of [58, 68, 74]; accordingly, any compact symplectic 4-manifold which admits a metric of positive scalar curvature is deformation equivalent to a rational or a ruled Kähler surface. It is also known [49, 83, 91] that these manifolds admit Kähler structures of positive scalar curvature.

3. As observed in [21], the integrability results stated in theorems 2 and 4 hold true for compact almost-Kähler orbifolds. \diamond

5. EXAMPLES OF CRITICAL ALMOST-KÄHLER METRICS

In this section we review some (mostly non-compact) examples of critical (Einstein) strictly almost-Kähler metrics. We first make the remark that given such a manifold we can take its product with any Kähler (Einstein) manifold to produce examples of higher dimension. We will be therefore interested in irreducible (i.e. non-product) examples.

5.1. The Bérard Bergery construction. Starting from a critical (Einstein) strictly almost-Kähler manifold (M^{2m}, g, J, Ω) , there is a method of producing higher dimensional non-product examples, due to L. Bérard Bergery [15] (see also [16, Th.9.129]). We take an S^1 -bundle P over M , whose curvature is a multiple of Ω . Then the almost-Kähler structure on M induces a K -contact structure on P (see e.g. [21]) and further, a critical strictly almost-Kähler structure on the “symplectisation” $N^{2(m+1)} = P \times \mathbb{R}$ of P (see also [59]); the metric on N is complete provided that M is compact. Furthermore, if M is Einstein, then the strictly almost-Kähler metric on N is also Einstein (with scalar curvature smaller than the one of M). The reader is referred to [15] for more details on the construction.

By applying this procedure to the next examples, one can construct many strictly almost-Kähler critical/Einstein manifolds.

5.2. Twistorial examples. For a given oriented Riemannian 4-manifold (M, g) , the set of all almost-Hermitian structures compatible with the metric and the orientation can be naturally identified with the set of sections of the sphere bundle $S(\Lambda^+M)$, by identifying each positively oriented almost-Hermitian structure (g, J) with its (normalized) fundamental 2-form $\frac{1}{\sqrt{2}}g(J\cdot, \cdot)$. The total space Z of $S(\Lambda^+M)$ is called (positive) *twistor space* of (M, g) . Thus, Z can be viewed as an $SO(4)/U(2)$ fibre-bundle associated to the canonical principal $O(4)$ -bundle P of (M, g) , i.e., $Z \cong P \times_{O(4)}(SO(4)/U(2))$. Denote by $p : Z \rightarrow M$ the natural projection; the vertical distribution $\mathcal{V} = \text{Ker}(p_*)$ inherits a canonical complex structure $J^\mathcal{V}$ coming from the natural complex structure of the fibre $\mathbb{C}\mathbb{P}^1 = SO(4)/U(2)$. Moreover, the Levi-Civita connection ∇ of (M, g) induces a splitting $TZ = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of Z into horizontal and vertical components, so that $\mathcal{H} \cong p_*^{-1}TM$ acquires a tautological complex structure $J^\mathcal{H}$ given by $J_{x,j}^\mathcal{H} = j$ for $x \in M$ and $j \in p^{-1}(x) \cong \mathbb{C}\mathbb{P}^1$. Following [35], we define an almost-complex structure J on Z , by

$$J = J^\mathcal{H} - J^\mathcal{V}.$$

Using the splitting of the tangent bundle of Z into horizontal and vertical components, one also determines a family of Riemannian metrics h on Z

$$h_t = p^*g + tg_{\mathbb{C}\mathbb{P}^1}, \quad t > 0,$$

where $g_{\mathbb{C}\mathbb{P}^1}$ is the standard metric (of constant curvature 1) on the fibre $\mathbb{C}\mathbb{P}^1$. Clearly, h_t is compatible with J . We then have the following

Theorem 5. [28] *The almost-Hermitian 6-manifold (Z, h_t, J) is almost-Kähler if and only if (M, g) is Einstein, anti-self-dual 4-manifold of negative scalar curvature $s = -\frac{12}{t}$. Moreover, in this case $(h_{-\frac{12}{s}}, J)$ is strictly almost-Kähler structure whose curvature tensor belongs to $\mathcal{R}_2(Z)$; in particular, it is a critical, strictly almost-Kähler metric on Z .*

Example 2. The only known examples of compact Einstein self-dual 4-manifolds of negative scalar curvature are compact quotients of the real hyperbolic space \mathbb{RH}^4 , or of the complex hyperbolic space \mathbb{CH}^2 , where the quotients of \mathbb{CH}^2 are considered with the non-standard orientation (so that the induced metric is *anti-self-dual*). Let (M, g) be such a quotient and by rescaling the metric, suppose that the scalar curvature is equal to -12 . According to Theorem 5 the twistor space (Z, h_1, J) is then a compact locally-homogeneous critical almost-Kähler 6-manifold whose curvature belongs to $\mathcal{R}_2(Z)$. In fact, it turns out that Z is an example of a compact, locally 3-symmetric almost-Kähler 6-manifold in the sense of [43]. \diamond

Remarks. 1. Note that the fundamental group of Z is equal to that of M . When M is a compact quotient of \mathbb{RH}^4 , it follows that the smooth 6-manifold underlying the twistor space Z admits no Kähler structures at all [26]. The analogous assertion for the twistor space of a compact quotient of \mathbb{CH}^2 is an open and intriguing problem [56].

2. Theorem 5 was generalized in [3] for twistor spaces of quaternion-Kähler manifolds of negative scalar curvature; considering compact quotients of locally-symmetric quaternion-Kähler $4k$ -manifolds of non-compact type, one obtains compact examples of critical strictly almost-Kähler manifolds of dimension $n = 4k + 2$. \diamond

5.3. Homogeneous examples via Kähler geometry. The examples given in this section are studied in [10], and rely on the following two simple lemmas. For their proofs, we refer the reader to [10].

Lemma 1. *Let (M, g, I) be a Kähler manifold whose Ricci tensor (considered as a symmetric endomorphism of the tangent bundle via the metric g) has constant eigenvalues $\lambda < \mu$. Denote by E_λ and E_μ the corresponding I -invariant eigenspaces and define an almost-complex structure J by $J|_{E_\lambda} = I|_{E_\lambda}$; $J|_{E_\mu} = -I|_{E_\mu}$. Then (g, J) is an almost-Kähler structure; it is Kähler if and only if (M, g, I) is locally product of two Kähler-Einstein manifolds of scalar curvatures λ and μ , respectively.*

Lemma 2. *With the notations from the previous lemma, let (g, I) be a Kähler structure whose Ricci tensor has constant eigenvalues, and consider the 1-parameter family of metrics $g_t = g|_{E_\lambda} + tg|_{E_\mu}$, $t > 0$, where $g|_{E_\lambda}$ (resp. $g|_{E_\mu}$) is the restriction of g to the eigenspace E_λ (resp. to E_μ). Then, for any $t > 0$, the metric g_t is Kähler with respect to I , almost-Kähler with respect to J and has the same Ricci tensor as the metric $g = g_1$.*

Lemma 2 shows that one can normalize any Kähler metric with Ricci tensor having two distinct constant eigenvalues to one whose Ricci tensor has eigenvalues equal to $-1, 0$ or $+1$. In particular, we get

Corollary 1. *On a complex manifold (M^{2m}, I) there is a one-to-one correspondence between Kähler metrics (g, I) with Ricci tensor having constant eigenvalues $\lambda < \mu$ with $\lambda\mu > 0$ and Kähler-Einstein metrics (\tilde{g}, I) of scalar curvature $2m\lambda$ carrying an almost-Kähler structure J which commutes with and differs from $\pm I$; in this correspondence J is compatible also with g and coincides (up to sign) with the almost-Kähler structure defined in Lemma 1; moreover, J is integrable precisely when (g, I) (and (\tilde{g}, I)) is locally product of two Kähler-Einstein metrics.*

Combining Corollary 1 with Theorem 2, it follows that a compact Kähler manifold whose Ricci tensor has two distinct constant *positive* eigenvalues is always locally the product of two Kähler-Einstein manifolds. In fact, using Proposition 2 for the almost-Kähler structure (g, J) , one can refine the latter result:

Theorem 6. [10] *Let (M, g, I) be a compact Kähler manifold whose Ricci tensor has two distinct constant non-negative eigenvalues λ and μ . Then the universal cover of (M, g, I) is the product of two simply connected Kähler-Einstein manifolds of scalar curvatures λ and μ , respectively.*

The above theorem can be equally seen as an integrability result for the almost-Kähler metric (g, J) given by Lemma 1; it is thus rather disappointing from the point of view of the search for critical strictly almost-Kähler metrics on compact manifolds. In dimension four, the situation is even more rigid, since we can further improve the above integrability result, by using the Kodaira classification of compact complex surfaces and some consequences of the Seiberg-Witten theory, recently established in [54, 66, 78].

Theorem 7. [10] *Let (M, g, I) be a compact Kähler surface whose Ricci tensor has two distinct constant eigenvalues. Then one of the following alternatives holds:*

- (i) *(M, g, I) is locally symmetric, i.e., locally is the product of Riemann surfaces of distinct constant Gauss curvatures;*
- (ii) *if (M, g, I) is not as described in (i), then the eigenvalues of the Ricci tensor are both negative and (M, J) must be a minimal surface of general type with ample canonical bundle and with even and positive signature. Moreover, in this case, reversing the orientation, the manifold would admit an Einstein, strictly almost-Kähler metric.*

At this time, we do not know if the alternative (ii) may really hold. As pointed out, if it does, it also provides a counter-example to the four-dimensional Goldberg conjecture.

While Lemma 1 does not seem to be a particularly useful tool of constructing compact examples of critical almost-Kähler metrics, it does provide complete ones, by considering irreducible homogeneous Kähler manifolds having Ricci tensor with two distinct eigenvalues. Indeed, according to the structure theorem for homogeneous Kähler manifolds [33, 41], there are such examples of any complex dimension $m \geq 2$. To see this, recall that any homogeneous Kähler manifold admits

a holomorphic fibering over a homogeneous bounded domain whose fiber, with the induced Kähler structure, is isomorphic to a direct product of a flat homogeneous Kähler manifold and a simply connected compact homogeneous Kähler manifold; in this structure theorem an important role is played by the Ricci tensor whose kernel corresponds to the flat factor; thus, when the Ricci tensor is non-negative, the manifold splits as the product of a flat homogeneous manifold (corresponding to the kernel of Ric) and a compact homogeneous Kähler manifold (and thus having positive Ricci form). Considering non-trivial (Kähler) homogeneous fibrations over bounded homogeneous domains, we obtain examples of (deRham) irreducible homogeneous Kähler manifolds with two distinct eigenvalues of the Ricci tensor. In complex dimension 2, there is only one such manifold which is described below.

Example 3. There is an irreducible homogeneous Kähler surface (M, I, g) whose Ricci tensor (after normalization) has eigenvalues $(-1, 0)$. Following [90], M can be written as

$$M = \frac{\mathbb{R}^2 \cdot Sl_2(\mathbb{R})}{SO(2)},$$

equipped with a left-invariant Kähler structure (g, I) . To make the description of the Kähler metric more explicit, one observes that the four-dimensional (real) Lie group $G = \mathbb{R}^2 \cdot Sol_2$ acts on M simply transitively, so that one can identify M with G . The corresponding Lie algebra \mathfrak{g} is then generated by X_1, X_2, E_1, E_2 with

$$[X_1, X_2] = [X_1, E_2] = 0; [E_1, E_2] = 2E_2; [X_2, E_2] = -2X_1;$$

$$[X_1, E_1] = -E_1; [X_2, E_1] = X_2.$$

The left-invariant complex structure I is defined by $I(X_1) = X_2; I(E_1) = E_2$, while the Kähler form is $\Omega_I = X^1 \wedge X^2 + E^1 \wedge E^2$. This example is studied in detail in [90] where it is referred to as the \mathbf{F}_4 -geometry. As a matter of fact, M does not admit any compact quotients, but it does admit quotients of finite volume. The corresponding critical strictly almost-Kähler structure (g, J) via Lemma 1 is isomorphic to the unique *proper 3-symmetric space* in four dimensions (see [43, 57]); it follows by [43] that its curvature tensor belongs to $\mathcal{R}_2(M)$. The uniqueness of this example, as well as its coordinate realization, will be discussed in the next section. \diamond

Remark. It is erroneously stated in [10] and [7, Example 3] that the homogeneous Kähler surface $M = (SU(2) \cdot Sol_2)/U(1)$ appearing in the classification of Shima [82] is deRham irreducible. In fact, a more careful investigation of the construction in [82] shows that any invariant Kähler metric (g, I) on $M = (SU(2) \cdot Sol_2)/U(1)$ is the Kähler product of a metric of constant Gauss curvature on $\mathbb{C}\mathbb{P}^1$ with a metric of constant Gauss curvature on $\mathbb{C}\mathbb{H}^1$. In particular, the almost-Kähler structure J obtained via Lemma 1 is integrable for this example. The issue of existence of local examples of strictly almost-Kähler 4-manifolds with pointwise constant Lagrangian sectional curvature, raised in [7], should be therefore considered an open problem. \diamond

We are now going to provide (non-compact) examples of deRham irreducible homogeneous Kähler manifolds (M, g, I) with Ricci tensor having two negative

eigenvalues $\lambda < \mu < 0$. According to Corollary 1, these will also provide *complete* examples of Einstein strictly almost-Kähler manifolds. Since Ric is negative definite, (M, I) must be a bounded homogeneous domain; it is a result of Vinberg, Gindikin and Piatetskii-Shapiro [41] that any such domain has a realization as a *Siegel domain of type II*, i.e, a domain $D = \{(z, w) \in \mathbb{C}^p \times \mathbb{C}^q : \text{Im}z - H(w, w) \in \mathfrak{C}\}$, where \mathfrak{C} is an open convex cone (containing no lines) in \mathbb{R}^p and $H : \mathbb{C}^q \times \mathbb{C}^q \mapsto \mathbb{C}^p$ is a Hermitian map which is \mathfrak{C} -positive in the sense that

$$H(w, w) \in \overline{\mathfrak{C}} - \{0\} \quad \forall w \neq 0.$$

In the particular case when $q = 0$, we have $D = \mathbb{R}^p + i\mathfrak{C}$; such domains are called *Siegel domains of type I* or *tube domains*. The following observation made in [10] provides the needed examples:

Proposition 4. *Every irreducible Hermitian symmetric space of non-compact type which admits a realization as a tube domain carries a strictly almost-Kähler structure commuting with the standard Kähler structure.*

Example 4. Explicit examples of Hermitian symmetric spaces that admit strictly almost-Kähler structures are

$$M^{2m} = \frac{SO(2, m)}{SO(2) \times SO(m)}, \quad m \geq 3.$$

Indeed, it is well known that these spaces admit realizations as tube domains, cf. e.g. [80]. \diamond

Remarks. 1. By using Corollary 1, one can also find *non-symmetric* homogeneous Kähler-Einstein manifolds of complex dimension greater than three, which admit strictly almost-Kähler structures. In complex dimension two and three, however, any bounded homogeneous domain is symmetric [27]. Therefore, according to Corollary 3 in section 6, no four-dimensional examples of Einstein, strictly almost-Kähler metrics arise from the above construction.

2. The reader may ask whether *compact* Einstein strictly almost-Kähler examples can be found as (smooth) quotients of the homogeneous examples presented. The answer is negative. Indeed, it is well-known that if a homogeneous space $M = G/H$ admits a compact smooth quotient M/Γ , then G must be unimodular (see e.g. [79]); it then follows by [20, 46] that a bounded homogeneous domain admits compact quotients if and only if it is symmetric. On the other hand, the maximal connected subgroup G' in G of isometries of the almost-Kähler structure arising from Proposition 4 is a proper closed subgroup of the full group of isometries G of the Hermitian symmetric space (M, g) . To see this, observe that according to Lemma 2 there is at least a one-parameter family of G' -invariant Kähler metrics on M , while for an irreducible Hermitian symmetric space the Bergman metric g is the unique G -invariant metric. It is known [19] that no co-compact lattice Γ of G can be contained in a connected proper closed subgroup of G , so that the almost-Kähler structure J does not descend to any compact quotient.

3. The first homogeneous examples of Einstein, strictly almost-Kähler metrics have been found by D. Alekseevsky [1] (see also [16, 14.100]) on certain solvable

Lie groups of real dimension $n = 4k, k \geq 6$. The Alekseevsky examples appear in the framework of quaternion-Kähler geometry and differ from the ones arising from homogeneous Kähler manifolds via Corollary 1. Note that the Alekseevsky spaces do not admit compact quotients either [2]. \diamond

5.4. Four-dimensional examples. The following construction is discussed in [5, 6] and generalizes the examples described in [73] and [13].

Proposition 5. *Let $(\Sigma, g_\Sigma, \omega_\Sigma)$ be an oriented Riemann surface with metric g_Σ and volume form ω_Σ , and let $h = w + iv$ be a non-constant holomorphic function on Σ , whose real part w is everywhere positive. On the product of Σ with $\mathbb{R}^2 = \{(z, t)\}$ consider the symplectic form*

$$\Omega = \omega_\Sigma - dz \wedge dt \quad (14)$$

and the compatible Riemannian metric

$$g = g_\Sigma + w dz^{\otimes 2} + \frac{1}{w} (dt + v dz)^{\otimes 2}; \quad (15)$$

Then, (g, Ω) defines a strictly almost-Kähler structure whose curvature belongs to $\mathcal{R}_3(M)$; in particular, g is a critical almost-Kähler metric.

Clearly, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial z}$ are two commuting hamiltonian Killing vector fields for the almost-Kähler structure of Proposition 5. Thus, the metric (15) is endowed with an \mathbb{R}^2 -isometric action which is *surface-orthogonal*. This suggests that the construction is local in nature and will not lead to compact examples (see Theorem 9 below).

The almost-Kähler structure (14-15) is sufficiently explicit to make it straightforward, though tedious, to check the following assertions:

- g is Einstein if and only if (Σ, g_Σ) is a flat surface, in which case g is a Gibbons-Hawking self-dual *Ricci-flat* metric [40] with respect to a translation-invariant harmonic function w (see [12, 73]);
- the curvature of g belongs to $\mathcal{R}_2(M)$ (equivalently, $W'' = 0$) if and only if (Σ, g_Σ) is hyperbolic, and, in the half-plane realization $(\mathbb{RH}^2, \frac{dx^2 + dy^2}{x^2})$, the holomorphic function h is given by $h = x + iy$; in this case (14-15) is homogeneous and isometric to the four-dimensional proper 3-symmetric space described in Example 3, see [5].

We also note here that an important feature of the above construction comes from the following observation: at any point where $dw \neq 0$, the Kähler nullity $D = \{TM \ni X : \nabla_X J = 0\}$ of J is a two dimensional subspace of TM , which is tangent to the surface Σ . If we define a new almost-complex structure I by

$$J_D = I_D, \quad J_{D^\perp} = -I_{D^\perp},$$

where D^\perp is the g -orthogonal complement of D , then one immediately sees that (g, I) is a Kähler structure. From this point of view, Proposition 5 provides a construction similar to the one coming from Lemma 1. In fact, at each point, the Ricci tensor of (15) has eigenvalues equal to $(0, \frac{s}{2})$ and these are constant precisely when s is constant. If we introduce local coordinates on (Σ, g_Σ) , such

that $g_\sigma = e^u w(dx^{\otimes 2} + dy^{\otimes 2})$ for some function $u(x, y)$, then the latter condition reads

$$u_{xx} + u_{yy} = (sw)e^u$$

for a constant s . Generic (local) solutions, u , to the above equation provide *non-homogeneous* metrics. Hence there are non-homogeneous examples satisfying the conditions of Lemma 1.

The next proposition is proved in [7].

Proposition 6. *Let g_Σ be a metric on a 2-manifold Σ with volume form ω_Σ , β be a 1-form on Σ with $d\beta = w\omega_\Sigma$ for an arbitrary holomorphic function $h = w + iv$ on Σ , whose real part w is everywhere positive. Then*

$$g = \frac{w}{z}(z^2 g_\Sigma + dz^{\otimes 2}) + \frac{z}{w} \left(dt + \frac{v}{z} dz + \beta \right)^{\otimes 2}, \quad (16)$$

$$\Omega = zw\omega_\Sigma + dz \wedge \left(dt + \frac{v}{z} dz + \beta \right), \quad (17)$$

defines a critical almost-Kähler metric which is Kähler if and only if the function h is constant.

Note that the almost-Kähler metric (16-17) is endowed with a hamiltonian Killing vector field $K = \frac{\partial}{\partial t}$. Furthermore, one can directly compute the curvature of (16-17). It turns out that it does not belong to $\mathcal{R}_3(M)$, showing that the solutions are different from the ones given by Proposition 5. We thus can get new examples of self-dual *Ricci-flat* strictly almost-Kähler 4-manifolds. Indeed, let (g, Ω) be given by (16-17) and suppose that $g_{\mathbb{C}\mathbb{P}^1}$ is the standard metric on an open subset Σ of $\mathbb{C}\mathbb{P}^1$; let $h = w + iv$ be a non-constant holomorphic function on Σ with positive real part. If we write $z = r$, we see that the metric

$$g = \frac{w}{r}(dr^{\otimes 2} + r^2 g_{\mathbb{C}\mathbb{P}^1}) + \frac{r}{w}(dt + \alpha)^{\otimes 2}$$

is given by applying the Gibbons-Hawking Ansatz [40] applied with respect to the harmonic function w/r . This class of Gibbons-Hawking metrics has been studied in [24] and [25]. The reader is referred to these references for more information. The observation that these metrics are almost-Kähler was made in [7].

6. LOCAL CLASSIFICATION RESULTS IN DIMENSION 4 AND FURTHER INTEGRABILITY RESULTS.

We start this section with the following result recently proven in [6].

Theorem 8. *For any connected strictly almost-Kähler 4-manifold (M, g, Ω) whose curvature belongs to $\mathcal{R}_3(M)$ there exists an open dense subset U with the following property: in a neighborhood of any point of U , (g, Ω) is homothetic to an almost-Kähler structure given by Proposition 5.*

This theorem generalizes some previous results:

Corollary 2. [13] *There are no Einstein anti-self-dual strictly almost-Kähler 4-manifolds.*

Since any anti-self-dual Kähler surface is necessarily scalar-flat (see e.g. [38]), it also follows

Corollary 3. [13] *Let (M, g) be the four dimensional real hyperbolic space, or the two dimensional complex hyperbolic space but endowed with the non-standard orientation. Then (M, g) does not admit even a locally defined compatible almost Kähler structure.*

Remarks. 1. To the best of our knowledge, it is still an open problem whether the complex hyperbolic space $\mathbb{C}\mathbb{H}^2$ does admit a (locally defined) strictly almost-Kähler structure J which is compatible with the canonical metric and with the *standard* orientation. If it does, this would be the first example of a strictly almost-Kähler Einstein, self-dual metric with negative scalar curvature, and would constitute a major step in the development of the local theory. In the same vein, it is unknown if $\mathbb{C}\mathbb{H}^1 \times \mathbb{C}\mathbb{H}^1$ (locally) admits strictly almost-Kähler structures compatible with the product Einstein metric. By contrast, for the other symmetric four-dimensional spaces the answer is known: applying the curvature criterion for non-existence of strictly almost-Kähler structures, established in section 3.3, one concludes that a strictly almost-Kähler structure can possibly exist only on the space $\mathbb{R} \times \mathbb{R}\mathbb{H}^3$. Conversely, Oguro and Sekigawa showed [76] that this symmetric space does admit a strictly almost-Kähler structure.

2. Any real hyperbolic space $(\mathbb{R}\mathbb{H}^{2m}, \text{can}), m \geq 2$ does not admit even locally defined almost-Kähler structures. This result was proved for $m > 3$ in [77]. The 6-dimensional version is much more subtle and was proved only recently in [13]. \diamond

As another immediate consequence from Theorem 8 and the results in [43], we derive

Corollary 4. [5] *Let (M, g, J, Ω) be a complete, simply connected strictly almost-Kähler 4-manifold whose curvature tensor belongs to $\mathcal{R}_2(M)$. Then (M, g, J, Ω) is isometric to the proper 3-symmetric space described in Example 3.*

Since the homogeneous space of Example 3 does not admit any compact quotient (see [90]), we also get the following integrability result originally proven in [9].

Corollary 5. *Any compact almost-Kähler 4-manifold whose curvature tensor belongs to $\mathcal{R}_2(M)$ is Kähler .*

We can ask more generally whether there are *compact* strictly almost-Kähler 4-manifolds whose curvature belongs to $\mathcal{R}_3(M)$. Using the local structure established in Theorem 8 and some further global arguments, it was also shown in [6] that the answer is negative.

Theorem 9. [6] *Any compact almost-Kähler 4-manifold whose curvature tensor belongs to $\mathcal{R}_3(M)$ is Kähler .*

The above integrability result also implies

Corollary 6. *There are no critical, strictly almost-Kähler structures compatible with a compact locally-homogeneous Riemannian 4-manifold.*

Proof. Suppose for contradiction that (M, g) is a compact locally homogeneous 4-manifold which admits a compatible strictly almost-Kähler structure J such that the Ricci tensor Ric is J -invariant.

If g is Einstein, then by a well known result of Jensen [50] (M, g) must be a compact locally symmetric 4-manifold (see also [30, Prop.9] for a different proof). Using a case by case verification, it is shown in [71] that (M, g) does not admit strictly almost-Kähler structures. We give an alternative proof of this fact as follows. By Theorem 2, we can assume that (M, g) is a compact quotient of a symmetric Einstein 4-manifold of non-compact type; the latter are $\mathbb{R}\mathbb{H}^4$, $\mathbb{C}\mathbb{H}^2$ and $\mathbb{C}\mathbb{H}^1 \times \mathbb{C}\mathbb{H}^1$. By Corollary 3, we can further assume that (M, g) is locally isometric to $\mathbb{C}\mathbb{H}^2$ or $\mathbb{C}\mathbb{H}^1 \times \mathbb{C}\mathbb{H}^1$, and that J agrees with the standard orientation on these spaces. Then, being locally Hermitian symmetric, (M, g) admits both a Kähler structure and a strictly almost-Kähler structure compatible with the same orientation. This is impossible according to [8, Th.1].

Consider now the case when (M, g) is locally homogeneous but not Einstein. Since Ric is J -invariant, it has two distinct (constant) eigenvalues λ and μ , each one of multiplicity 2, such that the corresponding eigenspaces are invariant under the action of J ; it follows that J is uniquely determined (up to sign) on the oriented 4-manifold (M, g) , by acting as a rotation of angle $+\frac{\pi}{2}$ on each eigenspace of Ric . This shows that J must be invariant under the transitive group action, i.e., that (g, J) is a locally homogeneous almost-Kähler 4-manifold. In particular, the 2-form $(\rho^*)''$ defined in section 3.2 is invariant, and therefore of constant length. It is easily seen that $(\rho^*)'' = 0$ if and only if the curvature of (g, J) belongs to $\mathcal{R}_3(M)$ (see e.g. [6, Lemma 1]). Since we assumed J is non-integrable, by Theorem 9 we conclude that $(\rho^*)''$ is a nowhere vanishing smooth section of the bundle $[\Lambda^{0,2}M]$. In four dimensions, the induced action of J on $[\Lambda^{0,2}M]$ gives an isomorphism between this bundle and the canonical bundle of (g, J) . This shows that $c_1^{\mathbb{R}}(J) = 0$. Now, by Theorem 4, J must be integrable, a contradiction. \square

We end this section with the following characterization of the metrics given by (16-17).

Theorem 10. [7] *Let (M, g, J, Ω) be an almost-Kähler 4-manifold with J -invariant Ricci tensor and a non-vanishing hamiltonian Killing vector field K . Suppose that the pair $(\bar{g} = \mu^{-2}g, I)$ is Kähler, where μ is a momentum map for a nonzero multiple of K , and I is equal to J on $\text{span}(K, JK)$, but to $-J$ on the orthogonal complement of $\text{span}(K, JK)$. Then either J is integrable, or (g, Ω) is locally given by Proposition 6.*

7. CURVATURE CONDITIONS WITH RESPECT TO THE HERMITIAN CONNECTION

For an almost-Kähler manifold (M^{2m}, g, J, Ω) , besides the Levi-Civita connection ∇ of g , one can also consider the so called *Hermitian* or *first canonical* connection (see e.g. [39]), defined by :

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}J(\nabla_X J)(Y).$$

This is the unique connection satisfying the following three conditions:

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}J = 0, \quad (T^{\tilde{\nabla}})^{1,1} = 0,$$

where $(T^{\tilde{\nabla}})^{1,1}$ denotes the J -invariant part of the torsion $T^{\tilde{\nabla}}$ of $\tilde{\nabla}$, viewed as a real 2-form with values in TM . For any connection that has the first two of the above properties, the J -anti-invariant part of its torsion is canonically identified with the Nijenhuis tensor N of the almost-complex structure; it follows that among connections making both g and J parallel, the first canonical connection has torsion with minimal possible norm, cf. [39]. By Chern-Weil theory, the Chern classes of the manifold are directly related to the curvature of the connection $\tilde{\nabla}$. If we denote by \tilde{R} the curvature tensor of $\tilde{\nabla}$, then

$$\tilde{\rho}(X, Y) = \sum_{i=1}^m (\tilde{R}_{X,Y}e_i, Je_i)$$

is a closed 2-form which is a deRham representative of $2\pi c_1(M, J)$ in $H^2(M, \mathbb{R})$. Moreover, $\tilde{\rho}$ is related to the Ricci forms of ∇ introduced at the end of section 3.2 by

$$\tilde{\rho} = \rho^* - \frac{1}{2}\phi = \rho + \frac{1}{2}(\nabla^*\nabla\Omega - \phi),$$

where for the second equality we used (10) and ϕ is the $(1, 1)$ -form given by (12). The *Hermitian scalar curvature* is defined to be $\tilde{s} = 2(\tilde{R}(\Omega), \Omega)$; by (11), \tilde{s} is related to the scalar and *-scalar curvatures by

$$\tilde{s} = \frac{1}{2}(s^* + s) = s + \frac{1}{2}|\nabla\Omega|^2.$$

Further, we have

$$\int_M \tilde{s} \, dv = \frac{4\pi}{(m-1)!} (c_1 \cdot [\Omega]^{\wedge(m-1)})(M) = c \, vol(M),$$

where c is the constant $\frac{4\pi}{(m-1)!} (c_1 \cdot [\Omega]^{\wedge(m-1)})(M) / vol(M)$. Thus, almost-Kähler metrics which satisfy the conditions

$$\tilde{\rho} = \frac{\tilde{s}}{2m}\Omega, \quad \tilde{s} = c \tag{18}$$

are natural candidates for generalizing Kähler-Einstein metrics. They could exist only on compact symplectic manifolds for which the first Chern class c_1 is a multiple of the cohomology class of the symplectic form $[\Omega]$.

In the compact case, almost-Kähler metrics satisfying (18) can be equivalently characterized by the following three conditions:

$$(a) \quad \tilde{s} = c, \quad (b) \quad \tilde{\rho} \text{ is of type } (1, 1), \quad (c) \quad c_1 = \frac{c}{4m\pi}[\Omega]. \tag{19}$$

It is natural to drop the cohomological condition (c) and look for compact almost-Kähler manifolds satisfying just (a) and/or (b). For instance, in dimension four, almost-Kähler metrics which saturate the new curvature estimates of LeBrun [63] must satisfy (a) and (b).

The study of almost-Kähler metrics of constant Hermitian scalar curvature is motivated by some results and questions of Donaldson [32]. He uses a moment map approach to show that problems posed by Calabi about finding canonical Kähler metrics on complex manifolds have natural extensions in the symplectic context.

The first observation is that the Fréchet space $\mathbf{AK}(M, \Omega)$ of Ω -compatible metrics admits a formal Kähler structure which is preserved by the action of the connected component of the identity, $Symp_0(M, \Omega)$, of the symplectomorphism group of (M, Ω) . Assuming $H^1(M, \mathbb{R}) = 0$, the Lie algebra of this group is identified with the space of smooth functions on M of zero integral, $C_0^\infty(M)$, endowed with the Poisson bracket with respect to Ω . Then, it is shown in [32] that

$$\mu : \mathbf{AK}(M, \Omega) \rightarrow C_0^\infty(M), \quad \mu(g) = \tilde{s}_g - c,$$

is a moment map for the action of $Symp_0(M, \Omega)$ on $\mathbf{AK}(M, \Omega)$. The elements in the zero-set $\mu^{-1}(0)$ of the moment map are Ω -compatible almost-Kähler metrics with constant Hermitian scalar curvature. Having in mind the identification of quotients which holds in the finite dimensional theory of moment maps (see e.g. [52]), but which is largely conjectural for infinite dimensional problems, one can speculate that each “stable” leaf of the action on $\mathbf{AK}(M, \Omega)$ induced by the complexified Lie algebra of $Symp_0(M, \Omega)$, contains precisely one $Symp_0(M, \Omega)$ orbit of almost-Kähler metrics of constant Hermitian scalar curvature. There is supporting evidence for the validity of this scenario. Restricted to the space of Ω -compatible, *integrable* almost-Kähler structures and after an application of Moser’s lemma, Donaldson’s approach transforms exactly into the classical set up of Calabi regarding existence of Kähler metrics of constant scalar curvature (in particular, Kähler-Einstein metrics) on compact complex manifolds. The leaves of the complexified action correspond in this case to deformations of the Kähler form in a fixed cohomology class by Kähler potentials. A number of results in Kähler geometry partially confirm the quotient identification in the integrable case.

Of course, the ultimate goal would be to prove a general existence result in the symplectic context, but for the beginning it would be interesting to construct some particular compact examples of strictly almost-Kähler manifolds satisfying some of the conditions in (19). At this point, our knowledge of such examples is limited. Of course, all locally homogeneous almost-Kähler structures have constant Hermitian scalar curvature. Some satisfy the other conditions of (19). For instance, it is not difficult to check that the Kodaira-Thurston manifold presented in Example 1 satisfies $\tilde{\rho} = 0$. Using the curvature computation of [28], the same can be seen to be true for the Davidov-Muškarov twistorial examples presented in section 5.1, cf. [29].

Finally, it is worth mentioning another research problem suggested by Donaldson’s approach — the study of the critical points of the squared norm of the moment map, that is, the L^2 -norm of the Hermitian scalar curvature, seen as a functional on the space $\mathbf{AK}(M, \Omega)$. It follows from the moment map set-up that the critical points of this functional are precisely the almost-Kähler metrics for which the vector field dual to $Jd\tilde{s}$ is Killing with respect to g . Thus, we get a natural extension of the Calabi extremal Kähler metrics [23] to the almost-Kähler

case. It is interesting to see what parts of the theory of extremal Kähler metrics extend to the symplectic context. For example, similarly to the Kähler case [67], one immediately finds an obvious necessary condition for a compact symplectic manifold (M, Ω) to admit a compatible *extremal* almost-Kähler metric g of non-constant Hermitian scalar curvature: since the connected component of the identity of the isometry group of g is a compact subgroup of $Symp_0(M, \Omega)$ [69], it follows that $Symp_0(M, \Omega)$ contains an S^1 which acts with fixed points. In dimension four, combining this observation with the results in [70] and [51], one sees that any compact symplectic 4-manifold (M, Ω) possibly admitting an extremal almost-Kähler metric of non-constant Hermitian scalar curvature must be symplectomorphic to a rational or ruled Kähler surface endowed with an S^1 -isometric (and therefore holomorphic) action. These are complex surfaces possibly supporting extremal Kähler metrics as well (see e.g. [23, 64, 87, 88]).

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