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Symplectic obstructions to the existence of ω-compatible Einstein metrics

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Abstract

It is shown that the existence of an ω-compatible Einstein metric on a compact symplectic manifold (M,ω) imposes certain restrictions on the symplectic Chern numbers. Examples of symplectic manifolds which do not satisfy these restrictions are given. The results offer partial support to a conjecture of Goldberg.

MSC: 53C25

Keywords: Almost Kähler structures; Einstein metrics

1. Introduction

This note is motivated by the following still open conjecture of Goldberg:

Conjecture 1 [11]. *On a compact symplectic manifold (M^{2n},ω) any Einstein ω-compatible metric is Kähler Einstein.*

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A Riemannian metric g is said to be compatible with a symplectic form ω, or shortly, *ω-compatible*, if there exists a g-orthogonal almost complex structure J such that

$$\omega(\cdot,\cdot) = g(J\cdot,\cdot).$$

Such a triple (g, J, ω) is called an *almost Kähler structure*.

Given a symplectic form ω on a compact manifold M^{2n}, the space of almost Kähler metrics compatible with ω is well known to be infinite dimensional and contractible. The latter fact implies that the Chern classes $c_k \in \mathbf{H}^{2k}(M,\mathbb{R})$ are independent of the choice of a compatible almost complex structure. As ω induces a non-trivial cohomology class $[\omega] \in \mathbf{H}^2(M,\mathbb{R})$, we define numerical symplectic invariants, which we call *symplectic Chern numbers*, by taking cup products of the Chern classes c_k with appropriate powers of $[\omega]$. The symplectic Chern numbers $(c_1 \vee [\omega]^{n-1})(M)$ and $(c_1^2 \vee [\omega]^{n-2})(M)$ will play an important role in this note.

It is now well known that Kähler metrics exist only very rarely on compact symplectic manifolds. Indirectly, the Goldberg conjecture predicts that ω-compatible Einstein metrics are even scarcer. Although the conjecture is still wide open, this prediction can be confirmed in certain cases and our purpose is to bring further support to its validity.

First, let us mention that for compact 4-manifolds there are known topological obstructions to the existence of Einstein metrics. For instance, the Hitchin–Thorpe inequality $3|\sigma(M)| \leqslant 2\chi(M)$ must hold, where $\sigma(M)$, $\chi(M)$ are the signature and the Euler number of M^4, respectively. Important refinements of this inequality were proved by LeBrun [14,15], using Seiberg–Witten theory. There are now known many examples of compact symplectic manifolds which violate the Hitchin–Thorpe inequality or its refinements and, hence, do not admit *any* Einstein metrics (compatible or not). This provides indirect support to the 4-dimensional Goldberg conjecture. In higher dimensions there are no known topological obstructions to the existence of Einstein metrics.

There are results directly supporting the Goldberg conjecture. Most notably, Sekigawa proved in [20] that the conjecture is true provided that the scalar curvature is assumed to be non-negative. Other positive partial results have been obtained in dimension 4 under various additional curvature assumptions [1,5,6,17,18]. However these partial results do not provide obstructions to the existence of Einstein compatible metrics, because of the Riemannian nature of the additional assumptions imposed.

It was observed in [8], that Sekigawa's result can be slightly improved by replacing the assumption $s \geqslant 0$ with the weaker condition $(c_1 \vee [\omega]^{n-1})(M) \geqslant 0$. As we need its proof later on, we incorporate this remark as part of our main result. Furthermore, in dimension 4, Armstrong proved that integrability holds even when one replaces the symplectic condition $(c_1 \vee [\omega])(M) \geqslant 0$, with, the essentially topological one, that the manifold admits a metric of everywhere positive scalar curvature (see [4, Corollary 2.3.5]).

The main goal of this note is to investigate the case $(c_1 \vee [\omega]^{n-1})(M) < 0$. We prove that the existence of an Einstein ω-compatible metric imposes certain inequalities between the symplectic Chern numbers $(c_1 \vee [\omega]^{n-1})(M)$ and $(c_1^2 \vee [\omega]^{n-2})(M)$, which are not satisfied by all symplectic manifolds. The following theorem summarizes our main results:

Theorem 1. *Let (M^{2n}, ω) be a $2n$-dimensional compact symplectic manifold. Assume that M admits an ω-compatible Einstein metric g.*

A. *If $(c_1 \vee [\omega]^{n-1})(M) \geqslant 0$, then g is a Kähler–Einstein metric. In particular, $c_1 \in \mathbb{R}_+[\omega]$.*

B. *If* $(c_1 \vee [\omega]^{n-1})(M) < 0$, *then the following inequalities hold*:

$$\big(c_1^2 \vee [\omega]^{n-2}(M)\big) \cdot \big([\omega]^n(M)\big) < k_1\big(c_1 \vee [\omega]^{n-1}(M)\big)^2, \tag{1}$$

where $k_1 = 25/9$ *if* $2n \geqslant 6$ *and* $k_1 = 9/4$ *if* $2n = 4$;

$$\big(c_1^2 \vee [\omega]^{n-2}(M)\big) \cdot \big([\omega]^n(M)\big) > k_2\big(c_1 \vee [\omega]^{n-1}(M)\big)^2, \tag{2}$$

where $k_2 = \frac{n-(25/9)}{n-1}$ *if* $2n \geqslant 6$ *and* $k_2 = 2/3$ *if* $2n = 4$.

Part A of Theorem 1 leads to first examples of compact symplectic manifolds of any dimension which do not admit compatible Einstein metrics. Indeed, any symplectic manifold (M, ω) with $(c_1 \vee [\omega]^{n-1})(M) \geqslant 0$, but $c_1 \notin \mathbb{R}[\omega]$ will have this property. Concerning part B, the constants k_1, k_2 are most likely not optimal. In fact, I recently learned from Claude LeBrun [16] that in dimension 4 inequality (2) still holds for $k_2 = 3/4$. One would hope the result to be valid with k_1, k_2 as close to 1 as possible. Nevertheless, even with the current constants, in Section 4 we give examples of symplectic manifolds which violate (1) or (2) and thus cannot admit compatible Einstein metrics.

2. Preliminaries

Assume for the beginning that (M^{2n}, g, J, ω) is only an almost Hermitian manifold, i.e., that the fundamental form ω is not necessarily closed. We shall use the following notations: ∇ is the Levi-Civita connection, R, Ric, s are respectively the curvature tensor, the Ricci tensor and the scalar curvature of ∇; $\sigma = \frac{\omega^n}{n!}$ is the volume form and $(,)$ is the pointwise inner product induced by the metric g on various bundles of tensors and forms.

The almost complex structure J induces an involution on the bundle of real 2-forms, by

$$\Lambda^2 M \ni \xi(\cdot,\cdot) \to \xi(J\cdot, J\cdot) \in \Lambda^2 M.$$

The ± 1-eigenspaces of this involution, which we denote by $\Lambda_{\mathbb{R}}^{1,1}M$ and $[\![\Lambda^{0,2}M]\!]$, are the bundles of J-invariant, respectively, J-anti-invariant 2-forms. The notation is explained by the correspondence with the usual type decomposition of complex 2-forms: J-invariant 2-forms are nothing but real forms of complex type $(1,1)$, while J-anti-invariant 2-forms are real parts of complex 2-forms of type $(0,2)$ (equivalently, of type $(2,0)$). The fundamental form ω is J-invariant and we denote by $\Lambda_0^{1,1}M \subset \Lambda_{\mathbb{R}}^{1,1}M$ the sub-bundle of *primitive* real $(1,1)$-forms, i.e., J-invariant 2-forms which are point-wise orthogonal to ω. Thus we have

$$\Lambda^2 M = \Lambda_{\mathbb{R}}^{1,1}M \oplus [\![\Lambda^{0,2}M]\!] = (\mathbb{R}\omega \oplus \Lambda_0^{1,1}M) \oplus [\![\Lambda^{0,2}M]\!], \tag{3}$$

and the components of a section $\xi \in \Lambda^2 M$ with respect to this decomposition are

$$\xi = \xi' + \xi'' = \frac{1}{n}(\xi,\omega)\omega + \xi_0' + \xi''.$$

Here and throughout the paper we use the superscripts $'$ and $''$ to denote respectively the J-invariant and J-anti-invariant components and the subscript 0 for the primitive part.

For any $\xi \in \Lambda^2 M$, easy computations imply:

$$\xi \wedge \omega^{n-1} = \frac{1}{n}(\xi, \omega)\omega^n = (n-1)!(\xi, \omega)\sigma, \tag{4}$$

$$\begin{aligned}\xi \wedge \xi \wedge \omega^{n-2} &= (n-2)!\left[\frac{n-1}{n}(\xi, \omega)^2 - |\xi_0'|^2 + |\xi''|^2\right]\sigma \\ &= (n-2)!\left[(\xi, \omega)^2 - |\xi'|^2 + |\xi''|^2\right]\sigma.\end{aligned} \tag{5}$$

From now on we assume that (g, J, ω) is an almost Kähler structure, i.e., that ω is closed. It is well known that for an almost Kähler structure, $\nabla\omega$ is identified with the Nijenhuis tensor N of J by (cf., e.g., [13]):

$$(\nabla_X \omega)(\cdot, \cdot) = \frac{1}{2}\big(JX, N(\cdot, \cdot)\big). \tag{6}$$

Since $N(J\cdot, \cdot) = N(\cdot, J\cdot) = -JN(\cdot, \cdot)$, the identification (6) implies that for any tangent vectors X, Y, Z

$$(\nabla_X \omega)(JY, JZ) = -(\nabla_X \omega)(Y, Z), \tag{7}$$

$$(\nabla_{JX} \omega)(JY, Z) = -(\nabla_X \omega)(Y, Z). \tag{8}$$

Relation (8) is sometimes called the *quasi-Kähler condition*. The trace in X, Y of (8) leads to the (again well-known) fact that ω is also co-closed and hence harmonic with respect to g.

The standard Weitzenböck formula for 2-forms

$$\Delta\xi - \nabla^*\nabla\xi = \big[\mathrm{Ric}(\xi\cdot, \cdot) - \mathrm{Ric}(\cdot, \xi\cdot)\big] - 2R(\xi),$$

specialized to $\xi = \omega$, gives

$$\frac{1}{2}\nabla^*\nabla\omega = R(\omega) - \frac{1}{2}\big[\mathrm{Ric}(J\cdot, \cdot) - \mathrm{Ric}(\cdot, J\cdot)\big] = \rho_* - \rho. \tag{9}$$

Formula (9) is a measure of the difference of two types of Ricci forms. For an arbitrary almost Kähler structure the Ricci tensor is in general not J-invariant, but taking its J-invariant part Ric', we can define the *Ricci form*, $\rho(\cdot, \cdot) = \mathrm{Ric}'(J\cdot, \cdot)$. The 2-form defined by $\rho_* = R(\omega)$ is called the $*$-*Ricci form*; this is in general not J-invariant. In fact, it follows from (9) that $\rho_*'' = \frac{1}{2}(\nabla^*\nabla\omega)''$. As for the J-invariant part of (9), taking the covariant derivative ∇_W of the relation (7) and then taking the trace in W, X, we obtain $(\nabla^*\nabla\omega)' = \psi$, where ψ is the semi-positive 2-form given by

$$\psi(X, Y) = \sum_{i=1}^{2n}\big((\nabla_{e_i} J)JX, (\nabla_{e_i} J)Y\big).$$

Here and throughout $\{e_i\}_{i=1,2n}$ denotes an orthonormal basis with respect to g. A J-invariant 2-form $\xi \in \Lambda^{1,1}_{\mathbb{R}} M$ is called *semi-positive* if $\xi(X, JX) \geqslant 0$, $\forall X \in TM$.

The inner product with ω of the relation (9) yields the difference of the two types of scalar curvatures:

$$s^* - s = |\nabla\omega|^2 = \frac{1}{2}|\nabla J|^2, \tag{10}$$

where $s^* = 2(R(\omega), \omega)$, is the so-called $*$-*scalar curvature*.

Unlike the Kähler case, the Levi-Civita connection cannot be used directly to provide representatives for the Chern classes c_k. Instead, one uses the so called *Hermitian* or *first canonical* connection (see, e.g.,

[10]), defined by:

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J)(Y).$$

If $\widetilde{R}$ denotes the curvature tensor of $\widetilde{\nabla}$, then

$$\tilde{\rho}(X,Y) = \frac{1}{2}\sum_{i=1}^{2n} (\widetilde{R}_{X,Y} e_i, J e_i)$$

is a closed 2-form which is a deRham representative of $2\pi c_1$ in $H^2(M,\mathbb{R})$. One easily finds the explicit relation between the curvature tensors $\widetilde{R}$ and R, of $\widetilde{\nabla}$ and ∇. We will only need the relationship of the Ricci forms:

$$\tilde{\rho} = \rho^* - \frac{1}{2}\phi, \tag{11}$$

where ϕ is the J-invariant, semi-positive 2-form given by $\phi(X,Y) = (\nabla_{JX}\omega, \nabla_Y \omega)$.

Hence, by (4), (5), (10) and (11) we have

$$\frac{4\pi}{(n-1)!}\big(c_1 \vee [\omega]^{n-1}\big)(M) = \int_M \frac{1}{2}(s^* + s)\sigma = \int_M \left(s + \frac{1}{2}|\nabla\omega|^2\right)\sigma, \tag{12}$$

$$\frac{4\pi^2}{(n-2)!}\big(c_1^2 \vee [\omega]^{n-2}\big)(M) = \int_M \left[\frac{(s^*+s)^2}{16} - \left|\rho'_* - \frac{1}{2}\phi\right|^2 + |\rho''_*|^2\right]\sigma. \tag{13}$$

The formula (12) is due to Blair [7], who first noted that the integral $\int_M (s^*+s)\sigma$ is a symplectic invariant. We let the reader observe that formulas (12) and (13) reduce to the well known ones in the Kähler case.

We close this section with the following classical result of Apte about the Chern numbers $(c_1^2 \vee [\omega^{n-2}])(M)$ and $(c_1 \vee [\omega^{n-1}])(M)$ in the Kähler case:

Proposition 1 [3]. *Let M^{2n} be a compact manifold and let ω be a symplectic form on M which admits a compatible Kähler metric. Then*

$$\big(c_1^2 \vee [\omega]^{n-2}\big)(M) \cdot \big([\omega]^n\big)(M) \leqslant \Big(\big(c_1 \vee [\omega]^{n-1}\big)(M)\Big)^2, \tag{14}$$

with equality iff $c_1 \in \mathbb{R}[\omega]$.

To sketch a proof, slightly different than the original one of [3] (see also [19]), note that for a Kähler manifold the decomposition (3) descends to cohomology. In view of (5), the bilinear form $b(c,d) = (c \vee d \vee [\omega]^{n-2})(M)$ has Lorenz signature $(+,-,\ldots,-)$ when restricted to $\mathbf{H}^{1,1}_{\mathbb{R}} \times \mathbf{H}^{1,1}_{\mathbb{R}}$, where $\mathbf{H}^{1,1}_{\mathbb{R}}$ denotes the subset of $\mathbf{H}^2(M,\mathbb{R})$ consisting of cohomology classes represented by real harmonic 2-forms of type $(1,1)$. This fact is part of the so-called Hodge–Riemann bilinear relations (see [12, p. 123]). For any $c \in \mathbf{H}^{1,1}_{\mathbb{R}}$, we then have the following "opposite" Cauchy–Schwarz inequality:

$$b(c,c)\cdot b\big([\omega],[\omega]\big) \leqslant \big(b\big(c,[\omega]\big)\big)^2,$$

with equality iff $c \in \mathbb{R}[\omega]$. It is well known that for a Kähler manifold the first Chern class c_1 belongs to $\mathbf{H}^{1,1}_{\mathbb{R}}$.

The proposition is no longer true in the non-Kähler case. One can find examples of symplectic forms which do not satisfy the conclusion of the Proposition 1, and hence do not admit compatible Kähler metrics (see [9] and Proposition 4 below).

3. Proof of Theorem 1

We start by recalling the remarkable integral formula of Sekigawa, which is valid on an arbitrary compact almost Kähler manifold. The original proof of this formula [20] is based on Chern–Weil theory. An alternative approach, based on Weitzenböck formulae, was described in [2].

Proposition 2 [20]. *For any compact almost Kähler manifold* (M^{2n}, g, J, ω), *the following integral formula holds*:

$$0 = \int_M \left[\frac{1}{2}|\mathrm{Ric}''|^2 - |\rho_*''|^2 - 2|W''|^2 + (\rho, \phi - \psi) - \frac{1}{4}|\psi|^2 - \frac{1}{4}|\phi|^2\right]\sigma. \tag{15}$$

The notations are those from Section 2; we should add that W'' is a certain component of the Weyl part of the curvature (for more details see [2]). For our purposes here, all that matters is that $|W''|^2$ is a non-negative quantity.

According to (3),

$$\rho = \frac{s}{2n}\omega + \rho_0, \qquad \phi = \frac{|\nabla\omega|^2}{2n}\omega + \phi_0, \qquad \psi = \frac{|\nabla\omega|^2}{n}\omega + \psi_0,$$

hence (15) becomes

$$0 = \int_M \left[\frac{1}{2}|\mathrm{Ric}''|^2 - |\rho_*''|^2 - 2|W''|^2 + (\rho_0, \phi_0 - \psi_0) - \frac{s}{4n}|\nabla\omega|^2 - \frac{5}{16n}|\nabla\omega|^4 - \frac{1}{4}|\psi_0|^2 - \frac{1}{4}|\phi_0|^2\right]\sigma. \tag{16}$$

In the Einstein case this implies

$$\int_M \left(-s|\nabla\omega|^2\right)\sigma \geqslant \frac{5}{4}\int_M |\nabla\omega|^4\sigma$$

and concludes the proof of Sekigawa's theorem that compact almost Kähler Einstein manifolds with $s \geqslant 0$ are necessarily Kähler Einstein [20]. Making no assumption on the sign of the (constant) scalar curvature and using Schwarz inequality, one obtains

$$-s\,\mathrm{vol}(M)\int_M |\nabla\omega|^2\sigma \geqslant \frac{5}{4}\left(\int_M |\nabla\omega|^2\sigma\right)^2.$$

Assuming now that the manifold is *not* Kähler, this leads to

$$-s\,\mathrm{vol}(M) \geqslant \frac{5}{4}\int_M |\nabla\omega|^2\sigma,$$

and, further, using Blair's formula (12), to

$$\big(c_1 \vee [\omega]^{n-1}\big)(M) > \frac{(n-1)!}{4\pi} s \operatorname{vol}(M) \geqslant \frac{5}{3}\big(c_1 \vee [\omega]^{n-1}\big)(M). \tag{17}$$

In particular, $(c_1 \vee [\omega]^{n-1})(M) < 0$, hence part A of Theorem 1 follows by contra-position.

The constant 5/3 in (17) can be lowered in the 4-dimensional case. In this dimension, the bundle of 2-forms also decomposes $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$, into the sub-bundles of self-dual and anti-self-dual 2-forms. This is related to the type decomposition (3) by

$$\Lambda^+ M = \mathbb{R}\omega \oplus [\![\Lambda^{0,2} M]\!], \qquad \Lambda^- M = \Lambda_0^{1,1} M.$$

One then immediately concludes that $(\nabla^*\nabla\omega)'$ must be a multiple of ω. Also, using (8) and the fact that the sub-bundle $[\![\Lambda^{0,2} M]\!]$ has dimension 2, it follows that the symmetric 2-tensor $(\nabla_\cdot\omega, \nabla_\cdot\omega)$ has a double eigenvalue 0 and a double eigenvalue $\frac{|\nabla\omega|^2}{2}$. Hence, in dimension 4 we have

$$\psi_0 = 0, \qquad |\phi_0|^2 = \frac{1}{8}|\nabla\omega|^4. \tag{18}$$

Using these in (16) and following the path described above, we obtain that a 4-dimensional Einstein strictly almost Kähler manifold satisfies

$$\big(c_1 \vee [\omega]\big)(M) > \frac{1}{4\pi} s \operatorname{vol}(M) \geqslant \frac{3}{2}\big(c_1 \vee [\omega]\big)(M). \tag{19}$$

Part B of Theorem 1 is a consequence of the following proposition, which may be of interest in its own.

Proposition 3. *Let (M^{2n}, g, J, ω) be a compact almost Kähler manifold. Then the following lower estimates of the L^2-norm of the Ricci tensor hold*:

$$\int_M |\mathrm{Ric}|^2\sigma \geqslant \frac{8\pi^2}{(n-1)!}\left(\frac{n(c_1 \vee [\omega]^{n-1}(M))^2}{[\omega]^n(M)} - (n-1)\big(c_1^2 \vee [\omega]^{n-2}\big)(M)\right), \tag{20}$$

$$\int_M |\mathrm{Ric}|^2\sigma \geqslant \frac{8\pi^2}{(n-1)!}\big(c_1^2 \vee [\omega]^{n-2}\big)(M). \tag{21}$$

Equality holds in (20) if and only if (g, J, ω) is Kähler with constant scalar curvature and equality holds in (21) if and only if (g, J, ω) is Kähler Einstein.

Proof. Note first that using (9), we have

$$\left|\rho'_* - \frac{1}{2}\phi\right|^2 = \left|\rho + \frac{1}{2}(\psi - \phi)\right|^2 = |\rho|^2 - \langle\rho, \phi\rangle + \langle\rho, \psi\rangle + \frac{1}{4}|\psi - \phi|^2.$$

With this, Sekigawa's formula (15) can also be written as

$$0 = \int_M \left[\frac{1}{2}|\mathrm{Ric}|^2 - 2|W''|^2 - \frac{1}{2}\langle\psi, \phi\rangle - |\rho''_*|^2 - \left|\rho'_* - \frac{1}{2}\phi\right|^2\right]\sigma. \tag{22}$$

Using (22) to successively substitute terms in (13), we get the following alternative expressions for the symplectic Chern number $(c_1^2 \vee [\omega]^{n-2})(M)$:

$$\frac{4\pi^2}{(n-2)!}\big(c_1^2 \vee [\omega]^{n-2}\big)(M) = \int_M \left[\frac{(s^*+s)^2}{16} + 2|\rho_*''|^2 - \frac{1}{2}|\mathrm{Ric}|^2 + 2|W''|^2 + \frac{1}{2}\langle\psi,\phi\rangle\right]\sigma, \tag{23}$$

$$\frac{4\pi^2}{(n-2)!}\big(c_1^2 \vee [\omega]^{n-2}\big)(M)$$
$$= \int_M \left[\left(1-\frac{2}{n}\right)\frac{(s^*+s)^2}{16} + \frac{1}{2}|\mathrm{Ric}|^2 - 2\left|\left(\rho_*' - \frac{1}{2}\phi\right)_0\right|^2 - 2|W''|^2 - \frac{1}{2}\langle\psi,\phi\rangle\right]\sigma. \tag{24}$$

Since both ϕ and ψ are semi-positive 2-forms, relations (23) and (24) imply immediately the following inequalities:

$$\frac{4\pi^2}{(n-2)!}\big(c_1^2 \vee [\omega]^{n-2}\big)(M) \geqslant \int_M \left[\frac{(s^*+s)^2}{16} - \frac{1}{2}|\mathrm{Ric}|^2\right]\sigma, \tag{25}$$

$$\frac{4\pi^2}{(n-2)!}\big(c_1^2 \vee [\omega]^{n-2}\big)(M) \leqslant \int_M \left[\left(1-\frac{2}{n}\right)\frac{(s^*+s)^2}{16} + \frac{1}{2}|\mathrm{Ric}|^2 - 2\left|\left(\rho_*' - \frac{1}{2}\phi\right)_0\right|^2\right]\sigma. \tag{26}$$

The estimate (20) follows from (25), using (12) and Schwarz inequality. The estimate (21) follows from (26) $-(1-2/n)$ (25).

For the equality statement, note first that equality holds in (25) or (26) if and only if the structure is Kähler. Indeed, assuming equality in either case, we must have $\langle\phi,\psi\rangle = 0$. Since both ϕ and ψ are semi-positive, it follows that for any $X \in TM$, $\phi(X, JX) = 0$ or $\psi(X, JX) = 0$. But $\phi(X, JX) = 0$ implies, by the definition of ϕ, that $\nabla_X\omega = 0$. The condition $\psi(X, JX) = 0$ leads to $(\nabla_Y\omega)(X, Z) = -(\nabla_Y\omega)(Z, X) = 0$, for any $Y, Z \in TM$. But, since ω is closed, this also leads to $\nabla_X\omega = 0$. Now further note that for (20) we also used Schwarz inequality, hence in the equality case we must have $s = \mathrm{const}$, while for (21) we neglected the last term of (26), which in the equality case implies $\mathrm{Ric}_0 = 0$. □

Remark. Note that the right-hand side of (20) is greater or smaller than the right-hand side of (21) depending on whether the inequality (14) holds or not. In the almost Kähler case either situation is possible as it will become clear in Section 4 (see also [9]).

Proof of Theorem 1, B. In case of dimension $2n \geqslant 6$, both inequalities (1) and (2) are now immediate. Indeed, assuming that (g, J, ω) is a non-Kähler, Einstein, almost Kähler structure, by Sekigawa's theorem and Theorem 1, part A, both s and $(c_1 \vee [\omega]^{n-1})(M)$ are negative numbers. The second part of (17) squared implies then

$$\frac{((n-1)!)^2}{16\pi^2} s^2\big(\mathrm{vol}(M)\big)^2 \leqslant \frac{25}{9}\Big(\big(c_1 \vee [\omega]^{n-1}\big)(M)\Big)^2.$$

Now combine this inequality with the (strict) inequalities (20) and (21) written in the Einstein case. Inequalities (1) and (2) follow, with $k_1 = 25/9$ and $k_2 = \frac{n-(25/9)}{n-1}$ as stated.

With the same arguments as above, the better constant $k_1 = 9/4$ for inequality (1) in the 4-dimensional case follows from (20) combined with (19).

To obtain the constant $k_2 = 2/3$ for inequality (2) in dimension 4, slightly more effort is required. With the Einstein assumption and taking into account (18), relation (23) becomes

$$4\pi^2 c_1^2(M) = \int_M \left[\frac{(s^* + s)^2}{16} - \frac{s^2}{8} + \frac{|\nabla\omega|^4}{8} + 2|\rho_*''|^2 + 2|W''|^2 \right] \sigma.$$

Using (10), the above can be written as

$$4\pi^2 c_1^2(M) = \int_M \left[\frac{1}{48}(s^* + s)^2 + \frac{1}{48}(2s^* - s)^2 + 2|\rho_*''|^2 + 2|W''|^2 \right] \sigma,$$

hence

$$4\pi^2 c_1^2(M) > \frac{1}{48} \int_M (s^* + s)^2 \sigma.$$

The inequality is strict because if not, the structure would be Kähler (see for, e.g., [5]) and we assumed otherwise. Further, using Schwarz inequality and (12), we get

$$\left(c_1^2(M)\right) \cdot \left([\omega]^2(M)\right) > \frac{2}{3}\left(\left(c_1 \vee [\omega]\right)(M)\right)^2,$$

which is the inequality claimed. □

4. Examples

We already remarked in the introduction that part A of Theorem 1 provides first examples of symplectic manifolds which do not admit compatible Einstein metrics. We now give such examples with $(c_1 \vee [\omega]^{n-1})(M) < 0$. The first source is the following proposition, which is essentially inspired from [9], but complements the results there.

Proposition 4. *Let (M^{2n}, J) be a compact complex manifold and assume that ω is a Kähler form and β is a holomorphic $(2,0)$ form on (M^{2n}, J). Then for any $t \in \mathbb{R}$, the form $\omega_t = \omega + t\,\mathrm{Re}(\beta)$ is a symplectic form on M^{2n}. Furthermore, if we assume that $c_1 = -[\omega]$, then the following hold:*

- (i) *If $n = 2m$ and β^m is not identically 0, then for $|t|$ large enough, (M^{4m}, ω_t) does not satisfy inequalities (14) and (1), hence it does not admit compatible Kähler metrics, nor compatible Einstein metrics.*
- (ii) *If $n = 2m + 1$ and β^m is not identically 0, then for $|t|$ large enough, (M^{4m+2}, ω_t) does not satisfy inequality (2), hence it does not admit compatible Einstein metrics.*
- (iii) *If $n = 2m$ or $n = 2m + 1$ and the highest non-zero power of β is $k < m$, with $(25/9)(n - 2k) < n$, then, for $|t|$ large enough, (M^{2n}, ω_t) does not satisfy inequality (2), hence it does not admit compatible Einstein metrics.*

Proof. It is well known that on a Kähler manifold any holomorphic form is closed. Thus, ω_t is closed for any t. To check the non-degeneracy, observe that the only non-vanishing terms from the binomial expansion of ω_t^n are those of the form $\omega^{n-2l} \wedge \beta^l \wedge \overline{\beta}^l$. But for any form α of type $(2l, 0)$, we have the (pointwise) Hodge–Riemann bilinear relation (see [12, pp. 123 and 110])

$$\omega^{n-2l} \wedge \alpha \wedge \bar{\alpha} = (n-2l)!\,|\alpha|^2 \frac{\omega^n}{n!}, \tag{27}$$

where the norm is the one induced by the Kähler metric corresponding to (J, ω). Thus ω_t is a symplectic form for any t.

Assuming now that $c_1 = -[\omega]$, the statements from (i), (ii) and (iii) follow by computing

$$L = \lim_{t \to \pm\infty} \frac{(c_1^2 \vee [\omega_t]^{n-2}(M)) \cdot ([\omega_t]^n(M))}{((c_1 \vee [\omega_t]^{n-1})(M))^2},$$

in each case. This is easily accomplished identifying the top powers of t in the following binomial expansions

$$\omega_t^n = \sum_{l=0}^{[\frac{n}{2}]} C_n^{2l} C_{2l}^l (t/2)^{2l} \omega^{n-2l} \wedge \beta^l \wedge \overline{\beta}^l,$$

$$\omega \wedge \omega_t^{n-1} = \sum_{l=0}^{[\frac{n-1}{2}]} C_{n-1}^{2l} C_{2l}^l (t/2)^{2l} \omega^{n-2l} \wedge \beta^l \wedge \overline{\beta}^l,$$

$$\omega^2 \wedge \omega_t^{n-2} = \sum_{l=0}^{[\frac{n-2}{2}]} C_{n-2}^{2l} C_{2l}^l (t/2)^{2l} \omega^{n-2l} \wedge \beta^l \wedge \overline{\beta}^l.$$

It follows that $L = +\infty$ in case (i), $L = 0$ in case (ii) and $L = \frac{n(n-2k-1)}{(n-1)(n-2k)}$ in case (iii). Now the statements are clear, noting in case (iii) that $\frac{n(n-2k-1)}{(n-1)(n-2k)} < \frac{n-(25/9)}{n-1} \Leftrightarrow (25/9)(n-2k) < n$. □

Remarks. (a) Certainly the condition $c_1 = -[\omega]$ cannot be replaced by $c_1 = [\omega]$, as in that case there are no non-trivial holomorphic forms by Kodaira's vanishing theorem. One would like to understand better the condition that β^m is not identically 0, for $n = 2m$, or $n = 2m + 1$. This is trivially satisfied if $n = 2$, or 3, by any non-trivial holomorphic $(2, 0)$ form. Further, the condition is stable under products: if β_1 has this property on M_1 and β_2 on M_2, then so does $\beta_1 + \beta_2$ on $M_1 \times M_2$. However, for product manifolds (or holomorphic fiber bundles), case (iii) does occur when β is a holomorphic $(2, 0)$ form coming from one of the factors (or from the base).

(b) With the notations from the above proposition, we showed in [9] that if (M^4, J, ω) is a compact Kähler surface with $c_1 = -[\omega]$, then for *all* values of $t \neq 0$, the symplectic forms ω_t violate inequality (14), hence they do not admit compatible Kähler metrics. The same was shown to be true in all higher dimensions for *small* non-zero values of t. Now we obtain the same conclusion when $|t|$ is sufficiently large and n is even. It is perhaps tempting to conjecture that in any dimension and for any holomorphic $(2, 0)$ form β, the symplectic 2-form ω_t does not admit compatible Kähler metrics, for any $t \neq 0$.

The next source of examples is the following proposition, suggested to me by Claude LeBrun.

Proposition 5. *Let* $(M_1^{2n_1}, \eta)$, $(M_2^{2n_2}, \mu)$ *be symplectic manifolds such that* $c_1(M_1) = -[\eta]$, $c_1(M_2) = -[\mu]$. *On* $M^{2n} = M_1^{2n_1} \times M_2^{2n_2}$ $(n = n_1 + n_2)$, *consider the symplectic forms* $\omega_t = \eta + t\mu$, *for* $t > 0$. *Then* *the manifold* (M^{2n}, ω_t) *does not satisfy inequality* (2) *and, hence, does not admit compatible Einstein metrics, in any of the following cases*:

(i) *if* $2n_1 = 4$, *and* t *is sufficiently large*;
(ii) *if* $2n_2 = 4$, *and* t *is sufficiently small*;
(iii) *if* $2n_1 \geqslant 6$, $2n_2 \geqslant 6$, $(25/9)n_1 < n$, *and* t *is sufficiently large*;
(iv) *if* $2n_1 \geqslant 6$, $2n_2 \geqslant 6$, $(25/9)n_2 < n$, *and* t *is sufficiently small.*

Proof. First note that cases (ii) and (iv) can be obtained from (i), respectively (iii), by substituting t with $1/t$. For (i) and (iii), we compute as in the previous proposition

$$L = \lim_{t\to\infty} \frac{(c_1^2 \vee [\omega_t]^{n-2}(M)) \cdot ([\omega_t]^n(M))}{((c_1 \vee [\omega_t]^{n-1})(M))^2}.$$

Note that $c_1(M) = -([\eta] + [\mu])$. We easily obtain

$$\omega_t^n = C_n^{n_1} t^{n_2} \eta^{n_1} \wedge \mu^{n_2},$$
$$(\eta + \mu)^2 \wedge \omega_t^{n-2} = (C_{n-2}^{n_1-2} t^{n_2} + 2C_{n-2}^{n_1-1} t^{n_2-1} + C_{n-2}^{n_1} t^{n_2-2}) \eta^{n_1} \wedge \mu^{n_2},$$
$$(\eta + \mu) \wedge \omega_t^{n-1} = (C_{n-1}^{n_1-1} t^{n_2} + C_{n-1}^{n_1} t^{n_2-1}) \eta^{n_1} \wedge \mu^{n_2},$$

with the convention that a binomial coefficient C_a^b is 0 if $a \leqslant 0$, or $b < 0$, or $a < b$. It follows that $L = 0$ in case (i) and $L = \frac{n(n_1-1)}{n_1(n-1)}$ in case (iii) and the statements are now clear. □

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