The Greek Letters

Chapter 17
Example (Page 365)

- A bank has sold for $300,000 a European call option on 100,000 shares of a non-dividend-paying stock (the price of the option per share, “textbook style”, is therefore $3.00).

- $S_0 = 49$, $K = 50$, $r = 5\%$, $\sigma = 20\%$, $T = 20$ weeks, $\mu = 13\%$

- (Note that even though the price of the option does not depend on $\mu$, we list it here because it can impact the effectiveness of the hedge)

- The Black-Scholes-Merton value of the option is $240,000$ (or $2.40 for an option on one share of the underlying stock).

- How does the bank hedge its risk?
Naked & Covered Positions

- Naked position:
  - One strategy is to simply do nothing: taking no action and remaining exposed to the option risk.
  - In this example, since the firm wrote/sold the call option, it hopes that the stock remains below the strike price ($50) so that the option doesn’t get exercised by the other party and the firm keeps the premium (price of the call it received originally) in full.
  - But if the stock rises to, say, $60, the firm loses (60-50)x100,000: a loss of $1,000,000 is much more than the $300,000 received before.

- Covered position:
  - The firm can instead buy 100,000 shares today in anticipation of having to deliver them in the future if the stock rises.
  - If the stock declines, however, the option is not exercised but the stock position is hurt: if the stock goes down to $40, the loss is (49-40)x100,000 = $900,000.
Stop-Loss Strategy

- The stop-loss strategy is designed to ensure that the firm owns the stock if the option closes in-the-money and does not own the stock if the option closes out-of-the-money.

- The stop-loss strategy involves:
  - Buying 100,000 shares as soon as the stock price reaches $50.
  - Selling 100,000 shares as soon as the stock price falls below $50.
However, repeated transactions (costly) and the fact that you will always buy at $K+\epsilon$ and sell at $K-\epsilon$ makes this strategy not a very viable one.
Delta (See Figure 17.2, page 369)

- Delta ($\Delta$) is the rate of change of the option price with respect to the underlying.

![Diagram showing delta and its relationship with option price and stock price]
Hedge

- Trader would be hedged with the position:
  - Short/wrote 1000 options (each on one share)
  - buy 600 shares

- Gain/loss on the option position is offset by loss/gain on stock position

- Delta changes as stock price changes and time passes.

- Hedge position must therefore be rebalanced often, an example of *dynamic hedging*.
Delta Hedging

- Delta hedging involves maintaining a delta neutral portfolio: a position with a delta of zero.

- The delta of a European call on a non-dividend-paying stock is $N(d_1)$

- The delta of a European put on the stock is $[N(d_1) - 1]$ or equivalently: $-N(-d_1)$.
First Scenario for the Example:
Table 17.2 page 372

<table>
<thead>
<tr>
<th>Week</th>
<th>Stock price</th>
<th>Delta</th>
<th>Shares purchased</th>
<th>Cost ('$000)</th>
<th>Cumulative Cost ($000)</th>
<th>Interest ($000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>49.00</td>
<td>0.522</td>
<td>52,200</td>
<td>2,557.8</td>
<td>2,557.8</td>
<td>2.5</td>
</tr>
<tr>
<td>1</td>
<td>48.12</td>
<td>0.458</td>
<td>(6,400)</td>
<td>(308.0)</td>
<td>2,252.3</td>
<td>2.2</td>
</tr>
<tr>
<td>2</td>
<td>47.37</td>
<td>0.400</td>
<td>(5,800)</td>
<td>(274.7)</td>
<td>1,979.8</td>
<td>1.9</td>
</tr>
<tr>
<td>......</td>
<td>......</td>
<td>......</td>
<td>......</td>
<td>......</td>
<td>......</td>
<td>......</td>
</tr>
<tr>
<td>19</td>
<td>55.87</td>
<td>1.000</td>
<td>1,000</td>
<td>55.9</td>
<td>5,258.2</td>
<td>5.1</td>
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<tr>
<td>20</td>
<td>57.25</td>
<td>1.000</td>
<td>0</td>
<td>0</td>
<td>5,263.3</td>
<td></td>
</tr>
</tbody>
</table>
First Scenario results

- At the end, the option is in-the-money.

- The buyer of the call option exercises it and pays the strike of 50 per share on the 100,000 shares held by the firm.

- This revenue of $5,000,000 partially offsets the cumulative cost of $5,263,300 and leaves a net cost of $263,300.
Second Scenario for the Example

Table 17.3 page 373

<table>
<thead>
<tr>
<th>Week</th>
<th>Stock price</th>
<th>Delta</th>
<th>Shares purchased</th>
<th>Cost (‘$000)</th>
<th>Cumulative Cost ($000)</th>
<th>Interest ($000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>49.00</td>
<td>0.522</td>
<td>52,200</td>
<td>2,557.8</td>
<td>2,557.8</td>
<td>2.5</td>
</tr>
<tr>
<td>1</td>
<td>49.75</td>
<td>0.568</td>
<td>4,600</td>
<td>228.9</td>
<td>2,789.2</td>
<td>2.7</td>
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<tr>
<td>2</td>
<td>52.00</td>
<td>0.705</td>
<td>13,700</td>
<td>712.4</td>
<td>3,504.3</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>....</td>
<td>....</td>
<td>....</td>
<td>....</td>
<td>....</td>
<td>....</td>
</tr>
<tr>
<td>19</td>
<td>46.63</td>
<td>0.007</td>
<td>(17,600)</td>
<td>(820.7)</td>
<td>290.0</td>
<td>0.3</td>
</tr>
<tr>
<td>20</td>
<td>48.12</td>
<td>0.000</td>
<td>(700)</td>
<td>(33.7)</td>
<td>256.6</td>
<td></td>
</tr>
</tbody>
</table>
Second Scenario results

- At the end, the option is out-of-the-money.

- The buyer of the call option does not exercise it and nothing happens on that front.

- The delta having gone to zero, the firm has no shares left.

- The net cumulative cost of hedging is $256,600.
Comments on dynamic hedging

- Since by hedging the short/written call the firm was essentially *synthetically replicating* a long position in the option, the (PV of the) cost of the replication should be close to the Black-Scholes price of the call: $240,000.

- They differ a little in reality because the hedge was rebalanced once a week only, instead of continuously.

- Also, in reality, volatility may not be constant, and there are some transaction costs.
The concept of delta is not limited to one security.

The delta of a portfolio of options or derivatives dependent on a single asset with price $S$ is given by $\Delta_\Pi = \Delta\Pi/\Delta S$.

If a portfolio of $n$ options consists of a quantity $q_i$ of option $i$, the delta of the portfolio is given by:

$$\Delta_\Pi = \sum_{i=1}^{n} q_i \Delta_i$$
Delta of a Portfolio: example

Suppose a bank has the following 3 option positions:
- A long position in 100,000 call options with a delta of 0.533 for each.
- A short position in 200,000 call options with a delta of 0.468 for each.
- A short position in 50,000 put options with a delta of -0.508 for each.

The delta of the whole portfolio is:
- \( \Delta_\Pi = 100,000 \times 0.533 - 200,000 \times 0.468 - 50,000 \times (-0.508) \)
- \( \Delta_\Pi = -14,900 \)

The portfolio can thus be made delta neutral by buying 14,900 shares of the underlying stock.

It also means that when the stock goes up by $1, the portfolio goes down by $14,900.
Theta

- The Theta ($\Theta$) of a derivative (or portfolio of derivatives) is the rate of change of its value with respect to the passage of time.

- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of a long call or put option declines.

- It is sometimes referred to as the *time decay* of the option.
Theta

- For European options on a non-dividend-paying stock, it can be shown from the Black-Scholes formulas that:

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} - rK e^{-rT} N(d_2)$$

with $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$$\Theta(\text{put}) = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} + rK e^{-rT} N(-d_2)$$
Theta for Call Option:
\[ K=50, \quad \sigma = 25\%, \quad r = 5\%, \quad T = 1 \]

Note that theta is here expressed per calendar day. The theta obtained from the formula is annual. For example, for \( S=50, \theta=-3.6252 \) and so we have \( \theta = -3.6252/365 = -0.01 \) per calendar day.
Gamma

- Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset, so it is the “change of the change”.

- Gamma is greatest for options that are close to being at-the-money since this is where the slope, delta, changes the most.

- If Gamma is high, then delta changes rapidly.

- If Gamma is low, then delta changes slowly.

- Gamma is the second derivative of the option price with respect to the stock: it can be stated as $\Gamma = \frac{d^2C}{dS^2}$ or $\Gamma = \frac{d^2P}{dS^2}$.
Gamma

- For European options on a non-dividend-paying stock, it can be shown from the Black-Scholes formulas that:

\[
\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}
\]

with \( N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \)

- The formula is valid for both call and put options.
Gamma for Call or Put Option:
$K=50$, $\sigma = 25\%$, $r = 5\%$, $T = 1$
Gamma Addresses Delta Hedging Errors Caused By Curvature
(Figure 17.7, page 377)
Interpretation of Gamma

- For a delta neutral portfolio,

\[ \Delta \Pi \approx \Theta \Delta t + \frac{1}{2} \Gamma \Delta S^2 \]

Positive Gamma

Negative Gamma
Relationship Between Delta, Gamma, and Theta

For a portfolio $\Pi$ of derivatives on a non-dividend-paying stock, we have:

$$\Theta + rS_0 \Delta + \frac{1}{2} \sigma^2 S_0^2 \Gamma = r\Pi$$

or

$$\frac{\partial C}{\partial t} + rS_0 \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 C}{\partial S^2} = rC$$

if the portfolio consists of one Call only

Solving this partial differential equation for $C$ (or $P$ if a Put) yields the Black-Scholes pricing equation, by the way.
How to make a portfolio “Gamma Neutral”

- Recall that, just like for any other Greek (Delta,…), the Gamma of a portfolio $\Pi$ is given by: $\Gamma_\Pi = n_1\Gamma_1 + n_2\Gamma_2 + n_3\Gamma_3 \ldots$
- One can thus make the total Gamma equal to zero with the appropriate addition of a certain number of options with a certain Gamma.

- As an example, assume your position is currently Delta neutral ($\Delta=0$) but that $\Gamma=-3,000$. You however find a call option with $\Delta_C=0.62$ and $\Gamma_C=1.50$
- You need $\Gamma_\Pi = (1)\Gamma_{\text{existing position}} + n_C\Gamma_C = 0$ i.e. need $-3,000 + n_C(1.50) = 0$.
- Buy $n_C = 3,000/1.5 = 2,000$ Call options and now have a Gamma of zero.

- The only issue is that the addition of the call options shifted your delta.
- Your new portfolio delta is $\Delta_\Pi = (1)0 + 2,000\Delta_C = 2,000(0.62) = 1,240$.
- Therefore 1,240 shares of the underlying asset must be sold from the portfolio in order to keep it delta neutral. It is now Delta-Gamma neutral.
Vega (not an actual Greek letter, but it sounds like one: good enough)

- Vega (\( \nu \)) is the rate of change of the value of a derivative (or a derivatives portfolio) with respect to the volatility of the underlying asset.

- Vega is an important measure because in practice, the volatility \( \sigma \) is not constant and changes over time.

- If Vega is highly positive or negative, the portfolio’s value is very sensitive to small changes in volatility.

- If Vega is close to zero, volatility changes have almost no impact on the value of the portfolio.
Vega

- For European options on a non-dividend-paying stock, it can be shown from the Black-Scholes formulas that:

\[
\nu = S_0 \sqrt{T} N'(d_1)
\]

with \( N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \)

- The formula is valid for both call and put options.
Vega for a Call or Put Option:
$K=50, \sigma = 25\%, \ r = 5\%, \ T = 1$
Managing Delta, Gamma, & Vega risk *all at once*

- We know that Delta can be changed by taking a position in the underlying asset.

- We also know that to adjust Gamma and Vega, it is necessary to take a position in an option or other derivative.

- However, if only one other derivative is added, either the Gamma risk or the Vega risk will be canceled, but not both at the same time (except by some coincidence).

- So we need to add *two new derivatives* to hedge all risks.
What position in option 1 and the underlying asset will make the portfolio delta and gamma neutral?

**Answer:** Long 10,000 options, short 6,000 of the asset.

What position in option 1 and the underlying asset will make the portfolio delta and vega neutral?

**Answer:** Long 4,000 options, short 2,400 of the asset.
Example if we do

<table>
<thead>
<tr>
<th></th>
<th>Delta</th>
<th>Gamma</th>
<th>Vega</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio</td>
<td>0</td>
<td>-5000</td>
<td>-8000</td>
</tr>
<tr>
<td>Option 1</td>
<td>0.6</td>
<td>0.5</td>
<td>2.0</td>
</tr>
<tr>
<td>Option 2</td>
<td>0.5</td>
<td>0.8</td>
<td>1.2</td>
</tr>
</tbody>
</table>

What position in option 1, option 2, and the asset will make the portfolio delta, gamma, and vega neutral \textit{all at once}?

We solve

\[-5000 + 0.5w_1 + 0.8w_2 = 0\]
\[-8000 + 2.0w_1 + 1.2w_2 = 0\]

to get \(w_1 = 400\) and \(w_2 = 6,000\).

We therefore require long positions of 400 and 6,000 in option 1 and option 2.

However, because these additions result in an incremental positive delta of \(400(0.6) + 6,000(0.5) = 3,240\), we also need to take a \textit{short position} of 3,240 in the asset in order to also make the portfolio delta neutral.
Rho

- Rho is the rate of change of the value of a derivative with respect to the interest rate.

- It is usually small and not a big issue in practice, unless the option is deep in-the-money and has a long horizon (discounting a larger cash flow over a longer horizon is more relevant then).
Hedging in Practice

- Traders usually ensure that their portfolios are delta-neutral at least once a day.

- Whenever the opportunity arises, they improve gamma and vega.

- As the portfolio becomes larger, hedging becomes less expensive since the trading cost per option goes down.
Scenario Analysis

- In addition to monitoring risks such as delta, gamma, and vega, option traders often also conduct a scenario analysis.

- A scenario analysis involves computing the gains and losses on the portfolio over a specified period of time under a variety of different scenarios.

- Often the two main sources of risk looked as variables (the scenarios) are the underlying asset price and volatility.
Greek Letters for European Options on an Asset that Provides a (dividend) Yield at Rate $q$ (Table 17.6, page 386)

<table>
<thead>
<tr>
<th>Greek Letter</th>
<th>Call Option</th>
<th>Put Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delta</td>
<td>$e^{-qT} N(d_1)$</td>
<td>$e^{-qT}[N(d_1) - 1]$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$</td>
<td>$\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$</td>
</tr>
<tr>
<td>Theta</td>
<td>$-S_0N'(d_1)\sigma e^{-qT} / (2\sqrt{T}) + qS_0N(d_1)e^{-qT} - rKe^{-rT}N(d_2)$</td>
<td>$-S_0N'(d_1)\sigma e^{-qT} / (2\sqrt{T}) - qS_0N(-d_1)e^{-qT} + rKe^{-rT}N(-d_2)$</td>
</tr>
<tr>
<td>Vega</td>
<td>$S_0\sqrt{T}N'(d_1)e^{-qT}$</td>
<td>$S_0\sqrt{T}N'(d_1)e^{-qT}$</td>
</tr>
<tr>
<td>Rho</td>
<td>$KTe^{-rT}N(d_2)$</td>
<td>$-KTe^{-rT}N(-d_2)$</td>
</tr>
</tbody>
</table>
Using Futures for Delta Hedging

- The delta of a futures contract on an asset paying a yield at rate $q$ is $e^{(r-q)T}$, since we know that $F = Se^{(r-q)T}$.

- The position required in futures for delta hedging (instead of using the spot asset) is therefore $e^{- (r-q)T}$ times the position required in the corresponding spot asset.
Example of Using Futures for Delta Hedging (instead of the spot)

- A portfolio of currency options held by a US bank can be made delta neutral with a short position of 458,000 pounds sterling (of the spot (currency) asset, if it were used).

- If the US riskless rate is 4% and the UK rate 7%, hedging for 9 months using a short position in futures contracts instead of shorting the spot currency would require shorting:

  \[ e^{-(0.04-0.07) \times 9/12} \times 458,000 \text{ or } £468,422 \text{ in futures contracts.} \]

- Since each futures contract is for £62,500 the number of contracts to be shorted is \( 468,422/62,500 = 7.49 \) contracts therefore rounded to 7 contracts.
Hedging vs. Creation of an Option Synthetically

- When we are hedging we take positions that offset delta, gamma, vega, etc...

- When we create an option synthetically we take positions that match delta, gamma, vega, etc...
In October of 1987, many portfolio managers attempted to create a put option on a portfolio synthetically. This involves initially selling enough of the portfolio (or of index futures) to match the $\Delta$ of the put option.
Portfolio Insurance (continued)

- As the value of the portfolio increases, the $\Delta$ of the put becomes less negative and some of the original portfolio is repurchased.

- As the value of the portfolio decreases, the $\Delta$ of the put becomes more negative and more of the portfolio must be sold.

- The strategy did not work well on October 19, 1987 because since everyone was doing the same thing, liquidity became an issue.

- Additionally, investors that anticipated the portfolio insurers reaction sold their positions as well, exacerbating the problem and precipitating the price decrease, making the crash worse.