Internal null-controllability for a structurally damped beam equation, Version 4.0

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Abstract In this paper we study the null-controllability of a beam equation with hinged ends and structural damping, the damping depending on a positive parameter. We prove that this system is exactly null controllable in arbitrarily small time. This result is proven using a combination of Ingham-type inequalities, adapted for complex frequencies, and exponential decay on various frequency bands. We then let the damping parameter tend to zero and we recover an earlier null-controllability result for the undamped beam equation.

1 Introduction and statement of main results

Let $T > 0$, and let ω be an open subset of $(0, 1)$. Let $\rho > 0$. Consider the controllability problem: given

$$u(x, 0) = u_0(x) \in H^2(0, 1) \cap H_0^1(0, 1),\ u_t(x, 0) = u_1(x) \in L^2(0, 1),$$

can we find a control function $f \in L^2(\omega \times (0, T))$ such that the solution u of the structurally damped beam equation

$$\begin{cases} u_{tt} + u_{xxxx} - \rho u_{xxt} = \chi_\omega f & \text{in } (0, 1) \times (0, T) \\ u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0 & \text{in } (0, T) \\ u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x), & \text{in } (0, 1), \end{cases} \tag{0}$$

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satisfies

$$u(x,T) = u_t(x,T) = 0, \ a.e. x \in (0,1)? \tag{1}$$

The system (0) with $f = 0$ is the one-dimensional version of the mathematical model for linear elastic systems with structural damping introduced by Chen and Russell in [2]. Null-controllability in time for arbitrarily small T for the undamped case ($\rho = 0$)was proven in [14]. Various results for boundary control in the undamped case can be found in Zuazua's survey in [8]. For a study of optimal boundary controllability of the damped plate equation, see [12]. Recently Lasiecka and Triggiani [6] studied the null-controllability of the abstract equation:

$$w_{tt} + Sw + \rho S^{\alpha} w_t = u, \ w(0) = w_0, w_t(0) = w_1, \rho > 0, \alpha \in [1/2, 1].$$

Here S is a strictly positive, self-adjoint unbounded operator with compact resolvent, and the control u is assumed to be distributed throughout $(0,1)$. Although the operator S is more general than operator Δ^2 appearing in our case, the authors did not assume that the control u is confined to a proper subset $\omega \subset (0,1)$ as we do here; this makes our problem harder because we cannot use Bessel's inequality as they do. Also, their controllability is not uniform in ρ. Hansen [3] proved an estimate on functions biorthogonal to a certain family of exponential functions on $[0,T]$. As an application, he studied the null-controllability of a damped vibrating rectangular plate equation subject to boundary control on one side, with $\rho < 2$, but his methods will also apply to the vibrating beam equation with internal damping to prove boundary null-controllability in time T for any $\rho < 2$. However, his proof, which uses a compactness argument, will not yield estimates on the control that are uniform in ρ. The main purpose of this paper is to prove null-controllability of the system in Eq.(0) uniformly with respect to ρ for small ρ. This permits us to show that as $\rho \to 0$, we recover existing controllability results for the undamped beam equation with hinged ends [14]. We are now in a position to state our main results.

Theorem 1 *Fix $T > 0, \rho > 0$. Assume $\rho \neq 2$. For any pair of initial conditions (u_0, u_1), there exists a function $f_\rho \in L^2(\omega \times (0,T))$ such that $u(x,T) = u_t(x,T) = 0$. Also,*

$$\int_0^T \int_\omega |f_\rho|^2 dx dt \leq C(\|u_0\|^2_{H^2(0,1)} + \|u_1\|^2_{L^2(0,1)}),$$

with C independent of u_0, u_1. Furthermore, for $\rho < 2$, the constant C can be chosen independent of ρ.

We remark that the restriction $\rho \neq 2$ is probably an artifact of the framework of [6] that we adopt here.

The following theorem follows from Theorem 1 by a standard argument.

Theorem 2 *Fix $T > 0$ and fix initial conditions u_0, u_1. Let $f_\rho \in L^2(\omega \times (0,T))$ be the control function associated with Eq.0. Then f_ρ converges in $L^2(\omega \times (0,T))$ to f_0, where f_0 is a control function for the unperturbed beam equation.*

Convergence results analogous to Theorem 2 have been proven in various physical settings [10],[11], and [9].

We now give a brief outline of the proof of Theorem 1. For ρ bounded away from zero, we use a generalization of Bessel's inequality due to Lebeau-Zuazua [7] together with exponential decay. For ρ close to zero, the exponential decay is not uniform as $\rho \to 0$, and therefore we need a more complicated argument to achieve a bound on $\|f_\rho\|$ which is uniform in ρ. For high frequencies, we still use the Lebeau-Zuazua estimate together with exponential decay, while for low frequencies we use Ingham-type inequalities. Our adaptation of Ingham's argument to complex frequencies is, to the best of our knowledge, new, and might be of independent interest. For related lower bounds on trigonometric polynomials, see [13],[5] and references therein.

The remainder of this paper is organized as follows. In the next section, we will recast the problem in the notation of [6]. In Section 3, we will prove Theorem 1 for $\rho > 2$. In Section 4, we will prove the theorem for $0 < \rho < 2$. Theorem 2 then follows from an essentially well known argument which is given in the appendix for the reader's convenience.

2 Preliminaries

In what follows we use the framework of [6], which applies for all $\rho \neq 2$.. Thus let $H = L^2(0,1) \times L^2(0,1)$, with Hilbert space norm $\|\mathbf{y}\|_H^2 = \int_0^1 |y_1|^2 + |y_2|^2 dx$. Solving Eq.(0) is equivalent to solving the system

$$\tilde{\mathbf{y}}_t = \tilde{A}\tilde{\mathbf{y}} + \chi_\omega \tilde{B} f, \;\; \tilde{\mathbf{y}}(0) = \mathbf{y_0}, \tag{2}$$

with

$$\tilde{\mathbf{y}} = \frac{1}{r_1 - r_2} \begin{pmatrix} -r_2 & -1 \\ r_1 & 1 \end{pmatrix} \begin{pmatrix} -\Delta u \\ u_t \end{pmatrix},$$

$$\tilde{A} = \begin{pmatrix} r_1\Delta & 0 \\ 0 & r_2\Delta \end{pmatrix},\ \tilde{A}^* = \begin{pmatrix} \overline{r_1}\Delta & 0 \\ 0 & \overline{r_2}\Delta \end{pmatrix},$$

$$\tilde{B} = \frac{1}{r_2 - r_1}(I, -I)^T,\ \tilde{B}^*((y_1, y_2)^T) = \frac{1}{\overline{r_1 - r_2}}(y_2 - y_1),$$

and

$$r_{1,2} = (\rho \pm \sqrt{\rho^2 - 4})/2. \tag{3}$$

The exact null-controllability of Eq.(0) in the original dynamics is equivalent to the exact null-controllability in H within the class of $L^2((0,T), L^2(\omega))$ controls of Eq. 2 (see [LT], p.46-48), provided $\rho \neq 2$. The associated adjoint equation is:

$$-\mathbf{w}_t = \tilde{A}^*\mathbf{w},\ \mathbf{w}(T) = \mathbf{w_T}. \tag{4}$$

Also set $\|v\|_\omega^2 = \int_\omega |v|^2 dx$. Note that the spectrum of the matrix $\tilde{A}$ is given by $\{-\pi^2 n^2 r_1\}, \{-\pi^2 n^2 r_2\}$, $n = 1, 2, \ldots$.

We now cite a well known result on the equivalence of null-controllability and observability.

Lemma 1 *The following assertions are equivalent:*

i) for every $\mathbf{w} \in H$*,*

$$\int_{t=0}^{T} \|\tilde{B}^* exp(t\tilde{A}^*)\mathbf{w}\|_\omega^2 dt \geq C \|exp(T\tilde{A}^*)\mathbf{w}\|_H^2, \tag{5}$$

ii) for every $y_0 \in H$ *and every* $T > 0$*, system(2) is null-controllable in time* T*, with control* f *satisfying*

$$\int_0^T \|f\|_\omega^2 dt \leq \frac{1}{C}\|y_0\|_H^2. \tag{6}$$

The proof of this result will appear in the appendix.

We note that Eq. 5 is equivalent to

$$\int_{t=0}^{T} \|\tilde{B}^* \exp((T-t)\tilde{A}^*)\mathbf{w}\|_\omega^2 dt \geq C \| \exp(T\tilde{A}^*)\mathbf{w}\|_H^2, \forall \mathbf{w} \in H. \tag{7}$$

3 Proof of Theorem 1 for $\rho > 2$.

Note that the assumption $\rho > 2$ implies that the "spectrum" of the problem is real, with

$$r_1 > r_2.$$

In what follows we prove a sequence of observability results, formulated as in Lemma 1.

Fix initial conditions

$$\mathbf{w_j} = (\sum_{2^{j-1}}^{2^j-1} a_n \sin(n\pi x), \sum_{2^{j-1}}^{2^j-1} b_n \sin(n\pi x))^T.$$

Clearly

$$\tilde{B}^* exp(t\tilde{A}^*)\mathbf{w_j} = \frac{1}{r_2 - r_1} \sum_{2^{j-1}}^{2^j-1} (a_n e^{-t\pi^2 n^2 r_1} - b_n e^{-t\pi^2 n^2 r_2}) \sin(\pi n x).$$

We apply a generalized Bessel's inequality proven in [7]: there exists C independent of j, such that

$$\| \sum_{n=2^{j-1}}^{2^j-1} (a_n e^{-t\pi^2 n^2 r_1} - b_n e^{-t\pi^2 n^2 r_2}) \sin(\pi n x)\|_\omega^2 \geq e^{-C2^j} \sum_{n=2^{j-1}}^{2^j-1} |a_n e^{-t\pi^2 n^2 r_1} - b_n e^{-t\pi^2 n^2 r_2}|^2. \tag{8}$$

Next, we apply an inequality found in ([6], Steps. 5-6). For each j, let T_j be a positive constant to be specified later. Then

$$e^{-C2^j} \int_0^{T_j} \sum_{n=2^{j-1}}^{2^j-1} |a_n e^{-t\pi^2 n^2 r_1} - b_n e^{-t\pi^2 n^2 r_2}|^2 dt$$

$$\geq e^{-C2^j} \sum_{n=2^{j-1}}^{2^j-1} \left(\frac{1}{2a}((1-2k) - (1+2k)e^{-2aT_j})|a_n|^2 + \frac{1}{2b}((1-2k) - (1+2k)e^{-2bT_j})|b_n|^2 \right) \tag{9}$$

Here $a = \pi^2 n^2 r_1$, $b = \pi^2 n^2 r_2$, and $k = 1/\rho$. Let $\tilde{\delta} = \frac{1-2k}{4}$. Let K be the minimal positive number such that $2bT_j \geq K$ implies

$$[(1-2k) - (1+2k)e^{-2bT_j}]/2 \geq \tilde{\delta}.$$

Set

$$T_j = \frac{KC_0}{2\pi^2 r_2 2^{j-1}}, \tag{10}$$

with $C_0 > 1$ to be chosen later. Then $n \in [2^{j-1}, 2^j - 1]$ implies $2bT_j \geq K$ and hence the RHS of Eq. 9 is greater than or equal to (letting C be various constants independent of j)

$$\begin{aligned}
\tilde{\delta} e^{-C2^j} \sum_{n=2^{j-1}}^{2^j-1} \frac{|a_n|^2}{\pi^2 n^2 r_1} + \frac{|b_n|^2}{\pi^2 n^2 r_2} &\geq \tilde{\delta} \frac{e^{-C2^j}}{\pi^2 r_1 (2^j)^2} \sum_{n=2^{j-1}}^{2^j-1} |a_n|^2 + |b_n|^2 \\
&\geq \tilde{\delta} e^{-C2^j} \sum_{n=2^{j-1}}^{2^j-1} |a_n|^2 + |b_n|^2. \qquad (11)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\| \exp(T_j \tilde{A}^* \mathbf{w_j}) \|_H^2 &= \| \sum_{2^{j-1}}^{2^j-1} a_n e^{-T_j \pi^2 n^2 r_1} \sin(n\pi x) \|_{L^2(0,1)}^2 + \| \sum_{2^{j-1}}^{2^j-1} b_n e^{-T_j \pi^2 n^2 r_2} \sin(n\pi x) \|_{L^2(0,1)}^2 \\
&= \sum_{2^{j-1}}^{2^j-1} |a_n e^{-T_j \pi^2 n^2 r_1}|^2 + |b_n e^{-T_j \pi^2 n^2 r_2}|^2 \\
&\leq e^{-2T_j \pi^2 (2^{j-1})^2 r_2} \sum_{2^{j-1}}^{2^j-1} |a_n|^2 + |b_n|^2 \qquad (12)
\end{aligned}$$

Combining Eqs. 12, 11, 9, and 8 we get for any j

$$\begin{aligned}
\int_0^{T_j} \| \tilde{B}^* \exp(t\tilde{A}^*) \mathbf{w_j} \|_\omega^2 &\geq \frac{\tilde{\delta}}{(r_2 - r_1)^2} e^{-C2^j + 2T_j \pi^2 (2^{j-1})^2 r_2} \| e^{T_j \tilde{A}^*} \mathbf{w_j} \|_H^2 \\
&\geq \delta e^{-C2^j + 2T_j \pi^2 (2^{j-1})^2 r_2} \| e^{T_j \tilde{A}^*} \mathbf{w_j} \|_H^2, \qquad (13)
\end{aligned}$$

where

$$\delta = \frac{1}{4\rho(\rho + 2)}$$

is bounded away from zero for $\rho \in (2, N)$ for any $N > 2$.

Now by Eq. 10

$$-C2^j + 2T_j \pi^2 (2^{(j-1)})^2 r_2 = 2^j (C_0 K/2 - C).$$

We choose C_0 such that $C_0\frac{K}{2} - C \equiv \alpha > 0$. Thus by Eq. 13,

$$\int_{t=0}^{T_j} \|\tilde{B}^* exp(t\tilde{A}^*)\mathbf{w_j}\|_\omega^2 dt \geq \delta e^{\alpha 2^j} \| \exp(T_j\tilde{A}^*)\mathbf{w_j}\|_H^2, \ \forall j. \tag{14}$$

We choose $P > 0$ such that $j \geq P$ implies

$$T_j = \frac{KC_0}{2\pi^2 r_2 2^{j-1}} < T.$$

In what follows it will be convenient to define an orthogonal projection $\Pi_N^M : H \rightarrow H$ as follows:

$$\Pi_N^M (\sum_{n=1}^\infty c_n \sin(n\pi x), \sum_{n=1}^\infty d_n \sin(n\pi x))^T = (\sum_{n=N}^M c_n \sin(n\pi x), \sum_{n=N}^M d_n \sin(n\pi x))^T. \tag{15}$$

Suppose now that the initial conditions are $\mathbf{y}(0) = (\sum_1^\infty c_n \sin(n\pi x), \sum_1^\infty d_n \sin(n\pi x))^T$. We define a sequence of controls as follows. First we apply a standard compactness argument to argue that there exists $C_T > 0$ such that

$$\int_{t=0}^{T} \|\tilde{B}^* exp(t\tilde{A}^*)\mathbf{w_0}\|_\omega^2 dt \geq C_T \| \exp(T\tilde{A})^*\mathbf{w_0}\|_H^2,$$

for $\mathbf{w_0} = (\sum_1^{2^P-1} a_n \sin(n\pi x), \sum_1^{2^P-1} b_n \sin(n\pi x))^T$, for arbitrary $\{(a_n, b_n)\}$. It then follows that there exists a control $f_0 \in L^2(0, T, L^2(\omega))$ such that the system in Eq. 2 with initial conditions $\Pi_1^{2^P-1}\mathbf{y}(0)$ can be brought to rest in time T.

For $j \geq P$, we consider the initial conditions

$$(\sum_{2^j}^{2^{j+1}-1} c_n \sin(n\pi x), \sum_{2^j}^{2^{j+1}-1} d_n \sin(n\pi x))^T.$$

Applying Eq. 14, there exists a control f_j that will bring these initial conditions to rest in time $T_j < T$. Note that by Lemma 1 along with Eq. 14,

$$\begin{aligned} \int_0^{T_j} \|f_j\|_\omega^2 dt &\leq \frac{1}{\delta} e^{-\alpha 2^{(j+1)}} \|\Pi_{2^j}^{2^{(j+1)}-1}\mathbf{y}(0)\|_H \\ &\leq \frac{1}{\delta} e^{-\alpha 2^{(j+1)}} \|\mathbf{y}(0)\|_H^2. \end{aligned}$$

Thus we obtain a sequence of controls $f_P, f_{P+1}, \ldots$. We extend each of these controls to $(0,1)\times(0,T)$ by setting $f_j(x,t)=0$ for $t>T_j$. Define f by $f\equiv f_0+\sum_P^\infty f_j$ on $[0,T]$, $f=0$ for $t>T$. Then f will bring the system given by Eq. 2 to rest in time T, with

$$\begin{aligned}(\int_0^T \|f\|_\omega^2 dt)^{1/2} &\leq \|f_0\|_\omega + (\sum_{j=P}^\infty \frac{e^{-\alpha 2^{(j+1)}}}{\delta}\|\mathbf{y}(0)\|_H^2)^{1/2} \\ &\leq C\|\mathbf{y}(0)\|_H.\end{aligned}$$

Remark In the case $\rho=2$, we have $r_1=r_2=\rho/2$, and hence Lasiecka-Trigianni's vector framework no longer applies (see, for instance, the formula for $\tilde{B}$). However, the uniform exponential decay still applies for $\rho=2$, and this together with the Lebeau-Zuazua estimate should be adaptable to the scalar framework (ie. Eq.0) to prove appropriate observability inequalities, thus yielding null-controllability for any $T>0$, and any pair of initial conditions (u_0,u_1).

Remark Note that $\delta\to 0$ as $\rho\to\infty$, so the controllability is not uniform for in ρ for $\rho >> 0$. This reflects the fact that $r_2\to 0$ as $\rho\to\infty$, so that the exponential decay is not uniform.

4 Proof of Theorem 1 for $\rho\in(0,2)$

We first note that for $\rho<2$, the spectrum of the matrix $\tilde{A}$ is complex and is given by $\{-\pi^2n^2r_1\},\{-\pi^2n^2r_2\}$, with $r_{1,2}=(\rho\pm i\sqrt{4-\rho^2})/2$.

Let

$$\rho_0\in(0,2)$$

be a number to be chosen below. We will employ different arguments depending on whether $\rho<\rho_0$ or $\rho\in[\rho_0,2)$.

4.1 Proof for $\rho\geq\rho_0$

Since $\rho\geq\rho_0$, then the exponential decay of the homogeneous system is uniform, and the argument of the previous section can easily be adapted. In place of Eq. 9, we use ([6]-Lemma 2.5): for all j,

$$\int_0^{T_j}\sum_{n=2^{j-1}}^{2^j-1}|a_ne^{-t\pi^2n^2r_1}-b_ne^{-t\pi^2n^2r_2}|^2dt \geq \sum_{n=2^{j-1}}^{2^j-1}\left(\frac{1}{2a}((1-2h)-(1+2h)e^{-2aT_j})\right)$$

$$\times \quad (|a_n|^2 + |b_n|^2)\Big) . \tag{16}$$

Here $a = \pi^2 n^2 \rho/2$ and $h = \rho/4$. The rest of the argument from the previous section now carries over word for word, except that $\tilde{\delta} = (1-2h)/4$ and $\delta = 1/8(2+\rho)$. Inspection of this argument makes it clear that for $\rho \in [\rho_0, 2)$, all observability estimates are uniform with respect to ρ.

4.2 Strategy for proving uniform control for $\rho < \rho_0$

We wish to prove null-controllability that is uniform in ρ; this uniformity is necessary in the proof of Theorem 2. The method used in Sections 3 and 4.1 to prove null-controllability for $\rho > \rho_0$ may also be applied in the present case. However, for ρ close to zero, the exponential decay of the system in Eq.(0) is no longer uniform in ρ, and hence the resulting estimates on f would not be uniform in ρ.

This section will be organized as follows. We will give different arguments to prove controllability results for different "frequency bands". Fix initial condition

$$\mathbf{y}(0) = (\sum_{n=1}^{\infty} c_n \sin(n\pi x), \sum_{n=1}^{\infty} d_n \sin(n\pi x)),$$

with $c_n, d_n \in \ell^2$. We will determine constants Q, V, P, each independent of ρ, such that null-controllability, uniform in ρ, can be proven for each of the following initial conditions:

a)$(\sum_{n=\lfloor Q/\rho \rfloor}^{\infty} c_n \sin(n\pi x), \sum_{n=\lfloor Q/\rho \rfloor}^{\infty} d_n \sin(n\pi x))$, to be proven in 4.3,

b)$(\sum_{n=(\lfloor V/\sqrt{\rho} \rfloor+1)}^{(\lfloor Q/\rho \rfloor-1)} c_n \sin(n\pi x), \sum_{n=(\lfloor V/\sqrt{\rho} \rfloor+1)}^{(\lfloor Q/\rho \rfloor-1)} d_n \sin(n\pi x))$, to be proven in 4.4,

c) $(\sum_{n=1}^{\lfloor V/\sqrt{\rho} \rfloor} c_n \sin(n\pi x), \sum_{n=1}^{\lfloor V/\sqrt{\rho} \rfloor} d_n \sin(n\pi x))$, to be proven in 4.5.

More precisely, for $\rho < \rho_0$ and for each of the initial conditions listed in a,b,c above, the system is null-controllable in time T, with the corresponding controls f_a, f_b, f_c satisfying

$$\int_\omega \int_0^T |f_i|^2 \leq C \|\mathbf{y}(0)\|_H^2, \ i = a, b, c,$$

and with constant C independent of ρ. By superposition, we obtain null-controllability for $\mathbf{y}(0)$ with control $f = f_\rho$ satisfying $\int_\omega \int_0^T |f|^2 \leq C\|\mathbf{y}(0)\|_H^2$, with C independent of ρ. From this follows the uniform observability inequality: for $\rho < \rho_0$,

$$\int_{t=0}^{T} \|\tilde{B}^* exp(-t\tilde{A}^*)\mathbf{w}\|_\omega^2 dt \geq \frac{1}{C}\|\mathbf{w}\|_H^2, \forall \mathbf{w} \in H. \tag{17}$$

Theorem 2 is an easy corollary of this inequality. For completeness, the proof that Eq. 17 implies Theorem 2 is included in the appendix.

4.3 Controllability of high frequencies

Proposition 1 *There exists a constant Q, with Q depending only on ω and T, such that if*

$$N \geq \frac{Q}{\rho}, \tag{18}$$

then for any initial conditions of the form

$$\mathbf{y} \equiv (\sum_N^\infty c_n \sin(n\pi x), \sum_N^\infty d_n \sin(n\pi x))^T \in H,$$

the system in Eq. 2 is null-controllable in time T with $\int_\omega \int_0^T |f|^2 \leq C\|\mathbf{y}(0)\|^2$ for some constant C independent of ρ.

Proof: We mimic the arguments of the Sections 3 and 4.1. As in that section,

$$\tilde{B}^* exp(t\tilde{A}^*)\mathbf{w}_j = \frac{1}{r_2 - r_1} \sum_{2^{j-1}}^{2^j - 1} (a_n e^{-t\pi^2 n^2 r_1} - b_n e^{-t\pi^2 n^2 r_2}) \sin(\pi n x),$$

and

$$\| \sum_{n=2^{j-1}}^{2^j-1} (a_n e^{-t\pi^2 n^2 r_1} - b_n e^{-t\pi^2 n^2 r_2}) \sin(\pi n x)\|_\omega^2 \geq e^{-C2^j} \sum_{n=2^{j-1}}^{2^j-1} |a_n e^{-t\pi^2 n^2 r_1} - b_n e^{-t\pi^2 n^2 r_2}|^2. \tag{19}$$

Next, we apply the argument of [[6]-Lemma 2.5]. For all j,

$$\begin{aligned} e^{-C2^j} \int_0^{T_j} \sum_{n=2^{j-1}}^{2^j-1} |a_n e^{-t\pi^2 n^2 r_1} - b_n e^{-t\pi^2 n^2 r_2}|^2 dt &\geq e^{-C2^j} \sum_{n=2^{j-1}}^{2^j-1} \left(\frac{1}{2a}((1-2h) - (1+2h)e^{-2aT_j}) \right) \\ &\times \left(|a_n|^2 + |b_n|^2 \right) . \end{aligned} \tag{20}$$

Here $a = \pi^2 n^2 \rho/2$ and $h = \rho/4$. Set $\tilde{\delta} = \frac{1-2h}{4}$. Let K be the minimal positive number such that $2aT_j \geq K$ implies

$$[(1-2h) - (1+2h)e^{-2aT_j}] \geq \tilde{\delta}.$$

Set

$$T_j = \frac{KC_0}{\pi^2 \rho 2^{j-1}}, \tag{21}$$

with $C_0 > 1$ to be chosen later. Then $n \in [2^{j-1}, 2^j - 1]$ implies $2aT_j \geq K$ and hence the RHS of Eq. 20 is greater than or equal to (letting C be various constants independent of j and ρ)

$$\begin{aligned}
\tilde{\delta} e^{-C2^j} \sum_{n=2^{j-1}}^{2^j-1} \frac{|a_n|^2}{\pi^2 n^2 \rho} + \frac{|b_n|^2}{\pi^2 n^2 \rho} &\geq \tilde{\delta} \frac{e^{-C2^j}}{\pi^2 \rho (2^j)^2} \sum_{n=2^{j-1}}^{2^j-1} |a_n|^2 + |b_n|^2 \\
&\geq \frac{\tilde{\delta} e^{-C2^j}}{\rho} \sum_{n=2^{j-1}}^{2^j-1} |a_n|^2 + |b_n|^2. \qquad (22)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|exp(T_j \tilde{A}^* \mathbf{w_j})\|_H &= \| \sum_{n=2^{j-1}}^{2^j-1} a_n e^{-T_j \pi^2 n^2 r_1} \sin(n\pi x)\|^2_{L^2(0,1)} + \| \sum_{2^{j-1}}^{2^j-1} b_n e^{-T_j \pi^2 n^2 r_2} \sin(n\pi x)\|^2_{L^2(0,1)} \\
&= \sum_{2^{j-1}}^{2^j-1} |a_n e^{-T_j \pi^2 n^2 r_1}|^2 + |b_n e^{-T_j \pi^2 n^2 r_2}|^2 \\
&\leq e^{-T_j \pi^2 (2^{j-1})^2 \rho} \sum_{2^{j-1}}^{2^j-1} |a_n|^2 + |b_n|^2 \qquad (23)
\end{aligned}$$

Combining Eqs. 23, 22, 16, and 19 and using $\rho < 1$, we get for any j

$$\begin{aligned}
\int_0^{T_j} \|\tilde{B}^* exp(t\tilde{A}^*) \mathbf{w_j}\|^2_\omega &\geq \frac{\tilde{\delta}}{|r_1 - r_2|^2} e^{-C2^j + T_j \pi^2 (2^{j-1})^2 \rho - \ln \rho} \|e^{T_j \tilde{A}^*} \mathbf{w_j}\|^2_H \\
&\geq \delta e^{-C2^j + T_j \pi^2 (2^{j-1})^2 \rho} \|e^{T_j \tilde{A}^*} \mathbf{w_j}\|^2_H, \qquad (24)
\end{aligned}$$

with $\delta = \tilde{\delta}/|r_1 - r_2|^2$ bounded away from zero independently of ρ. Now by Eq. 21

$$-C2^j + T_j \pi^2 (2^{(j-1)})^2 \rho = 2^j (C_0 K/2 - C).$$

We choose C_0 such that $C_0\frac{K}{2} - C \equiv \alpha > 0$. Thus by Eq. 24,

$$\int_{t=0}^{T_j} \|\tilde{B}^* exp(t\tilde{A}^*)\mathbf{w_j}\|_\omega^2 dt \geq \delta e^{\alpha 2^j} \| \exp(T_j\tilde{A}^*)\mathbf{w_j}\|_H^2. \tag{25}$$

We choose $P > 0$ such that $\rho 2^P > \frac{KC_0}{\pi^2 T}$. Then for $j \geq P+1$,

$$\begin{aligned} T_j &= \frac{KC_0}{\pi^2 \rho 2^{j-1}} \\ &< T. \end{aligned}$$

The proof of the proposition now follows by mimicking the argument appearing in the Section 3 for $\rho > 2$, setting $Q = \lceil \frac{KC_0}{T\pi^2} \rceil$ and $N = 2^P$.

4.4 Ingham-type argument for intermediate frequencies

In this section we treat the case

$$\frac{V}{\sqrt{\rho}} \leq n < \frac{Q}{\rho},$$

with V a constant to be determined below. We begin with a technical lemma:

Lemma 2 *Let*

$$g_n(m) \equiv \frac{4-\rho^2}{\rho^2 n^2}(1+\frac{m}{n})^2(n-m)^2 + (1+\frac{m^2}{n^2})^2.$$

Then, assuming $n\rho < Q$, there exists a constant P_0 such that if $m, n \geq P_0$, with $m \neq n$, then there exists a constant $\gamma > 0$ such that

$$g_n(m) > 4 + \gamma. \tag{26}$$

Proof: Noting that

$$g_n(m) = \frac{4-\rho^2}{\rho^2 n^4}(n+m)^2(n-m)^2 + (1+\frac{m^2}{n^2})^2,$$

we have that

$$g_n'(m) = m^3(\frac{4(4-\rho^2)}{n^4\rho^2} + \frac{4}{n^4}) - m(\frac{4(4-\rho^2)}{\rho^2 n^2} - \frac{4}{n^2}).$$

It is thus easy to verify that on the positive reals, $g_n' < 0$ on $(0, n\sqrt{1-\rho^2/2})$ and $g_n' > 0$ on $(n\sqrt{1-\rho^2/2}, \infty)$. Since $\rho n \leq Q$, there exists P_1 such that $n > P_1$ implies that $n\sqrt{1-\rho^2/2} \in (n-1, n)$. We conclude that when m is an integer,$m \neq n$, then g_n will be minimized at $m = n \pm 1$. One calculates

$$g_n(n \pm 1) \geq 4 + \frac{4(4-\rho^2)}{\rho^2 n^2}(1 - \frac{1}{n}) - \frac{8}{n} + O(\frac{1}{n^2}). \tag{27}$$

Since $n\rho \leq Q$, it follows that we can choose P_2 and $\gamma \in (0, 4(4-\rho^2)/Q^2)$ such that $n \geq P_2$ implies

$$\frac{4(4-\rho^2)}{\rho^2 n^2}(1 - \frac{1}{n}) - \frac{8}{n} + O(\frac{1}{n^2}) > \gamma.$$

The proof is completed by setting $P_0 = \max(P_1, P_2)$.

The main result of this section is then:

Proposition 2 *Let Q be as in Proposition 1, and P_0 as in Lemma 2. Then there exist positive constants ρ_1 and V, with V independent of ρ, such that if $\rho < \rho_1$, and*

$$N = \lfloor \frac{Q}{\rho} \rfloor - 1, \ P = \max(P_0, \lfloor \frac{V}{\sqrt{\rho}} \rfloor + 1),$$

then for initial conditions

$$\mathbf{y}(0) \equiv (\sum_P^N c_n \sin(n\pi x), \sum_P^N d_n \sin(n\pi x))^T,$$

we have null-controllability in time T with $\int_\omega \int_0^T |f|^2 \leq C\|\mathbf{y}(0)\|^2$ for some constant C independent of ρ.

The first step in the proof of this proposition is the following Ingham-type inequality:

Lemma 3 *Let $d \in \mathbf{R}^+$. Assume N satisfies $N \leq Q/\rho$. Then there exist $\delta > 0$, $\rho_1 > 0$ and positive integers R and $\tilde{P}$ such that if $\rho < \rho_1$, if*

$$P \geq \max(P_0, \sqrt{\frac{\tilde{P}}{\rho d \pi^2}}),$$

and if

$$f(t)=\sum_{n=P}^{N} a_n e^{-tdn^2\pi^2 r_1}+b_n e^{-tdn^2\pi^2 r_2},$$

then

$$\int_{-1}^{1}|f(t)|^2dt \geq \delta \sum_{n=P}^{N} \frac{e^{\pi^2 d\rho n^2}}{(\pi^2 d^2\rho^2 n^4)^{R+2}}(|a_n|^2+|b_n|^2).$$

Here $\delta, R, \tilde{P}$ are independent of f, d, ρ.

Proof: We define $C_1, C_2 \in \mathbf{R}$ by

$$\begin{aligned} d\pi^2 r_1 &= d\pi^2\rho/2 + id\pi^2\sqrt{4-\rho^2}/2 \\ &= C_1 + iC_2, \end{aligned} \tag{28}$$

and hence

$$d\pi^2 r_2 = C_1 - iC_2.$$

Thus, since $f(t)=\sum a_n e^{-tn^2(C_1+iC_2)}+b_n e^{-tn^2(C_1-iC_2)}$, we have

$$\begin{aligned} |f(t)|^2 &= \sum_{m=P}^{N}\sum_{n=P}^{N} a_n\overline{a_m}\exp(-C_1(n^2+m^2)t+C_2(m^2-n^2)it) \\ &+ \sum_{m=P}^{N}\sum_{n=P}^{N} b_n\bar{b}_m\exp(-C_1(n^2+m^2)t+C_2(-m^2+n^2)it) \\ &+ 2\Re(\sum_{m=P}^{N}\sum_{n=P}^{N} a_n\bar{b}_m\exp(-C_1(n^2+m^2)t+C_2(-m^2-n^2)it)) \\ &\equiv I(t)+II(t)+III(t). \end{aligned} \tag{29}$$

Set

$$K(u)=\int_{-\infty}^{\infty} k(t)e^{-uti}dt.$$

The idea of the proof is to choose k so that $K(u)$ has certain desirable growth properties as $u{\rightarrow}\infty$ along various sectors in the complex plane.

Let R be an even positive integer whose exact value will be determined later. Define $\tilde{k}(t)$ to be a polynomial on $[0,1]$ satisfying:

$$\begin{aligned} \tilde{k}^{(j)}(1) &= 0,\ j=0,\ldots,R \\ \tilde{k}(0) &= 1, \\ \tilde{k}^{(j)}(0) &= 0,\ 0<j\leq R,\ j\ \textit{odd}. \end{aligned} \tag{30}$$

Proving that $\tilde{k}$ exists is an amusing exercise in linear algebra. Let M be the order of $\tilde{k}$.

We then extend $\tilde{k}$ to the real line as follows:

$$k(t) = \begin{cases} 0, & |t| > 1 \\ \tilde{k}(-t), & t \in [-1, 0] \\ \tilde{k}(t), & t \in [0, 1]. \end{cases}$$

Thus $k \in C^R(-\infty, \infty)$, with the support being the interval $[-1, 1]$. Furthermore, $\|k\|_\infty = k(0) = 1$.

Thus, using integration by parts and Eq. 30,

$$\begin{aligned} K(u) &\equiv \int_{\mathbf{R}} e^{-itu} k(t) dt \\ &= 2 \int_0^1 k(t) \cos(tu) dt \\ &= 2[\sum_{j=0}^{M} u^{-j-1} k^{(j)}(t) (\frac{d^j}{ds^j} \sin s)|_{s=tu}]_0^1 \\ &= 2 \left(\sum_{j=R+1}^{M} u^{-j-1} k^{(j)}(1) (\frac{d^j}{ds^j} \sin s)|_{s=u} \right. \\ &\quad \left. - \sum_{j=R/2+1}^{M} (-1)^{j+1} u^{-2j} k^{(2j-1)}(0) \right). \end{aligned}$$

It follows that, for $u, v \in \mathbf{R}$ and $u >> 0$,

$$|K(v - iu)| = \frac{|k^{(R+1)}(1)| e^u}{|v - iu|^{R+2}} + O(|v - iu|^{-R-3} e^u). \tag{31}$$

We begin by analysing

$$\begin{aligned} \int_{\mathbf{R}} I(t) k(t) dt &= \sum_{m=P}^{N} \sum_{n=P}^{N} a_n \overline{a_m} K(-iC_1(n^2 + m^2) + C_2(n^2 - m^2)) \\ &= \sum_{n=P}^{N} |a_n|^2 K(-2iC_1 n^2) + \sum_{m,n; m \neq n} a_n \overline{a_m} K(-iC_1(n^2 + m^2) + C_2(n^2 - m^2)). \end{aligned}$$

Let $\epsilon > 0$. We choose $\tilde{P}_1 = \tilde{P}_1(R, \epsilon)$ sufficiently large that the following holds for $2C_1 n^2 = \pi^2 d\rho n^2 \geq \tilde{P}_1$,

$$K(-2iC_1 n^2) \;\; \geq \;\; (1 - \epsilon) \frac{|k^{(R+1)}(1)| e^{2C_1 n^2}}{(2C_1 n^2)^{R+2}},$$

and hence, setting $P_1 = \lceil \sqrt{\frac{\tilde{P}_1}{\pi^2 d\rho}} \rceil$,

$$\sum_{n=P_1}^{N} |a_n|^2 |K(-2iC_1 n^2)| \geq (1-\epsilon)|k^{(R+1)}(1)| \sum_{n=P_1}^{N} \frac{|a_n|^2 e^{2C_1 n^2}}{(2C_1 n^2)^{R+2}}. \tag{32}$$

Next, by Eq. 28

$$\begin{aligned} |C_2(n^2-m^2)+iC_1(n^2+m^2)|^2 &= C_2^2(n^2-m^2)^2 + C_1^2(n^2+m^2)^2 \\ &= C_1^2 n^4 \left(\frac{4-\rho^2}{\rho^2 n^2}(1+\frac{m}{n})^2(n-m)^2 + (1+\frac{m^2}{n^2})^2 \right) \\ &= C_1^2 n^4 g_n(m), \end{aligned} \tag{33}$$

with $g_n(m)$ as in Lemma 2.

Hence, there exists $\tilde{P}_2 = \tilde{P}_2(\epsilon, R)$ such that for for $\pi^2 d\rho m^2, \pi^2 d\rho n^2 \geq \tilde{P}_2$, we have by Eq. 31

$$|K(C_2(n^2-m^2)+iC_1(n^2+m^2))| \leq (1+\epsilon)\frac{|k^{(R+1)}(1)|e^{C_1(n^2+m^2)}}{|C_2(n^2-m^2)+iC_1(n^2+m^2)|^{R+2}}$$

Hence, setting $P_2 = \lceil \sqrt{\frac{\tilde{P}_2}{\pi^2 d\rho}} \rceil$ and using Eq. 33,

$$\begin{aligned} &\sum_{m,n=P_2, m\neq n}^{N} |a_n \overline{a_m} K(C_2(n^2-m^2)+iC_1(n^2+m^2))| \\ &\leq \frac{1}{2}\sum_{m,n\geq P_2, m\neq n}^{N} (1+\epsilon)\frac{|k^{(R+1)}(1)|(|a_n|^2 e^{2C_1 n^2} + |a_m|^2 e^{2C_1 m^2})}{|C_2(n^2-m^2)+iC_1(n^2+m^2)|^{R+2}} \\ &\leq \sum_{n=P_2}^{N} |a_n|^2 e^{2C_1 n^2} \sum_{m=P_2, m\neq n}^{N} \frac{|k^{(R+1)}(1)|(1+\epsilon)}{|C_2(n^2-m^2)+iC_1(n^2+m^2)|^{R+2}} \\ &\leq (1+\epsilon)|k^{(R+1)}(1)| \sum_{n=P_2}^{N} \frac{|a_n|^2 e^{2C_1 n^2}}{(C_1 n^2)^{R+2}} \sum_{m=P_2, m\neq n}^{N} \frac{1}{(g_n(m))^{\frac{R+2}{2}}} \end{aligned} \tag{34}$$

Set $P = \max(P_0, P_1, P_2)$.We analyze

$$\sum_{m=P, m\neq n}^{N} g_n(m)^{-\frac{R+2}{2}} \leq \sum_{1\leq |m-n| \leq Q^2+1}^{N} g_n(m)^{-\frac{R+2}{2}} + \sum_{|m-n|\geq Q^2+2}^{N} g_n(m)^{-\frac{R+2}{2}}. \tag{35}$$

It follows from Lemma 2 that for $R = R(Q, \gamma)$ sufficiently large,

$$\begin{aligned}
\sum_{|m-n| \le Q^2+1} g_n(m)^{-(R+2)/2} &\le \sum_{|m-n| \le Q^2+1} (4+\gamma)^{-(R+2)/2} \\
&\le (2Q^2+2)(4+\gamma)^{-(R+2)/2} \\
&\le \frac{1}{4} 4^{-(R+2)/2} \\
&= \frac{1}{4} 2^{-(R+2)}.
\end{aligned} \tag{36}$$

Fixing R as above, we then estimate

$$\sum_{|m-n| \ge Q^2+2} g_n(m)^{-(R+2)/2} \le \sum_{|m-n| \ge Q^2+2} \left(\frac{4-\rho^2}{\rho^2 n^4}\right)^{\frac{-R-2}{2}} |n^2 - m^2|^{-(R+2)}$$

Assume for the moment that $n - Q^2 > P$ and that $n + Q^2 < N$. Then

$$\sum_{|m-n| \ge Q^2+2} \left(\frac{4-\rho^2}{\rho^2 n^4}\right)^{\frac{-R-2}{2}} |n^2 - m^2|^{-(R+2)} \le \left(\int_{m=P}^{n-Q^2} + \int_{m=Q^2+n}^{\infty} \right) \left(\frac{4-\rho^2}{\rho^2 n^4}\right)^{\frac{-R-2}{2}} |m^2 - n^2)|^{-(R+2)} dm.$$

We estimate the first of these terms. Recalling that $n\rho \le Q$,

$$\begin{aligned}
\int_{m=P}^{n-Q^2} \left(\frac{4-\rho^2}{n^4\rho^2}\right)^{\frac{-R-2}{2}} (n^2-m^2)^{-(R+2)} dm &\le \left(\frac{\rho^2 n^4}{4-\rho^2}\right)^{(R+2)/2} \int_{m=P}^{n-Q^2} \left(\frac{1/(2n)}{n-m} + \frac{1/(2n)}{n+m}\right)^{R+2} \\
&\le \left(\frac{n^2\rho^2}{4(4-\rho^2)}\right)^{(R+2)/2} \int_{m=P}^{n-Q^2} \left(\frac{1}{n-m} + \frac{1}{n+m}\right)^{R+2} \\
&\le \left(\frac{n^2\rho^2}{4-\rho^2}\right)^{(R+2)/2} \int_{m=P}^{n-Q^2} \left(\frac{1}{n-m}\right)^{R+2} \\
&\le \left(\frac{n^2\rho^2}{4-\rho^2}\right)^{(R+2)/2} \cdot \frac{Q^{-2R-2} - (n-P)^{-R+1}}{R+1} \\
&\le \left(\frac{1}{4-\rho^2}\right)^{(R+2)/2} \cdot \frac{Q^{-R}}{R+1} \\
&\le \frac{1}{8 \cdot 2^{R+2}},
\end{aligned} \tag{37}$$

provided $Q > 2$ and $R \ge 1$. The other integral, and the cases where $n - Q^2 < P$ or $n + Q^2 > N$, can be easily shown to satisfy the same estimate. Thus, combining Eqs. 37, 36, 35 and 34, we see that

$$\sum_{m,n=P_2,m\neq n}^{N} |a_n\overline{a_m}K(C_2(n^2-m^2)+iC_1(n^2+m^2))| \leq \frac{1+\epsilon}{2}|k^{(R+1)}(1)| \sum_{n=P_2}^{N} \frac{|a_n|^2 e^{2C_1n^2}}{(2C_1n^2)^{R+2}}.$$

Comparing this with Eq. 32, we get (assuming ϵ is sufficiently small)

$$\int_{\mathbf{R}} I(t)k(t)dt \geq \delta \sum_{n=P}^{N} |a_n|^2 \frac{e^{2C_1n^2}}{(2C_1n^2)^{R+2}},$$

for some $\delta > 0$.

Similarly,

$$\int_{\mathbf{R}} II(t)k(t)dt \geq \delta \sum_{n=P}^{N} |b_n|^2 \frac{e^{2C_1n^2}}{(2C_1n^2)^{R+2}}.$$

It remains to estimate the term $\int III(t)k(t)dt$. Arguing as we did for $\int I(t)k(t)dt$, we have, for some $\epsilon > 0$,

$$\begin{aligned} |\int III(t)k(t)dt| &= 2|\sum_{m,n=P}^{N} a_n\overline{b_m}K(-iC_1(n^2+m^2)+C_2(m^2+n^2))| \\ &\leq (1+\epsilon)|k^{(R+1)}(1)| \sum_{m,n=P}^{N} \frac{|a_n|^2 e^{2C_1n^2} + |b_m|^2 e^{2C_1m^2}}{(C_1^2+C_2^2)^{(R+2)/2}(n^2+m^2)^{(R+2)}}. \end{aligned} \tag{38}$$

Consider the term involving $|a_n|^2$. We have

$$\begin{aligned} \sum_{m,n=P}^{N} \frac{|a_n|^2 e^{2C_1n^2}}{(C_1^2+C_2^2)^{(R+2)/2}(n^2+m^2)^{(R+2)}} &= \sum_{n=P}^{N} \frac{|a_n|^2 e^{2C_1n^2}}{(d^2\pi^4)^{(R+2)/2}} \sum_{m=P}^{N} \frac{1}{(m^2+n^2)^{(R+2)}} \\ &\leq \sum_{n=P}^{N} \frac{|a_n|^2 e^{2C_1n^2}}{(d\pi^2)^{(R+2)}} \frac{N-P}{n^{(2R+4)}} \\ &\leq \sum_{n=P}^{N} \frac{|a_n|^2 e^{2C_1n^2} \rho^{R+2} N}{(2C_1n^2)^{(R+2)}} \\ &\leq Q\rho^{R+1} \sum_{n=P}^{N} \frac{|a_n|^2 e^{2C_1n^2}}{(2C_1n^2)^{(R+2)}}. \end{aligned} \tag{39}$$

A similar estimate holds for the terms involving b_n, so the lemma follows if we choose ρ_1 to satisfy $(1+\epsilon)Q\rho_1^{R+1} = \delta/2$.

Remark. The most common choice for k in Ingham type arguments has been the cut off cosine function found in [4]. But this choice, which implicitly sets $R = 2$, is not sufficient to give Eq. 36.

Remark. Our choice for k do not seem to improve on the original Ingham inequality appearing in [4].

Remark. It is of interest to try to extend this lemma to the case $N = \infty$. However, because $g_n(n\sqrt{1-\rho^2/2}) = 4-\rho^2 < 4$, it follows that the hypothesis $\rho n \le Q$ for some Q is necessary in the proof of Eq. 26. Thus we cannot extend this lemma to $N = \infty$, at least with the current methods.

Corollary 1 *Assume $T > 0$ and let $\rho < \rho_1$. Set $d = T/2$. Let $N \le Q/\rho$, $P \ge \max(\sqrt{\frac{\tilde{P}}{\pi^2\rho d}}, P_0)$, with $\tilde{P}$ as in previous lemma. Then if*

$$f(t) = \sum_{n=P}^{N} a_n e^{-tn^2\pi^2 r_1} + b_n e^{-tn^2\pi^2 r_2},$$

then

$$\int_0^T |f(t)|^2 dt \ge CT \sum_P^N \frac{|a_n|^2 + |b_n|^2}{(\rho n^2 T)^{R+2}},$$

where $C = C(R)$ is a positive constant independent of $\rho, \{a_n\}\{b_n\}, T$.

Proof: We have, using the substitution $s = 2t/T - 1$ and then the previous lemma (with $d = T/2$),

$$\begin{aligned}
\int_0^T |f(t)|^2 dt &= \frac{T}{2}\int_{-1}^{1} |\sum_{n=P}^{N} (a_n e^{-n^2\pi^2 r_1 T/2}) e^{-sdn^2\pi^2 r_1} + (b_n e^{-n^2\pi^2 r_2 T/2}) e^{-sdn^2\pi^2 r_2}|^2 ds \\
&\ge \frac{T\delta}{2} \sum_{n=P}^{N} (|a_n e^{-\rho\pi^2 n^2 T/4}|^2 + |b_n e^{-\rho\pi^2 n^2 T/4}|^2) \frac{e^{\rho\pi^2 n^2 T/2}}{(\rho\pi^2 n^2 T/2)^{R+2}} \\
&\ge TC \sum_{n=P}^{N} \frac{|a_n|^2 + |b_n|^2}{(\rho n^2 T)^{R+2}},
\end{aligned}$$

Proposition 3 *Let ρ, N, T, R be as above. There exists V such that if $P \geq \max(P_0, V/\sqrt{\rho})$, then, given initial conditions $(\sum_P^N c_n \sin(n\pi x), \sum_P^N d_n \sin(n\pi x))$, there exists a control f bringing the system in Eq. 2 to rest in time T. Furthermore f satisfies*

$$\int_0^T \int_\omega |f|^2 \leq \tilde{C} \sum_{n=P}^N |c_n|^2 + |d_n|^2,$$

where $\tilde{C}$ is independent of ρ.

Proof: It suffices to prove the observability inequality

$$\int_0^T \|\tilde{B} \exp(t\tilde{A}^*)\mathbf{w}\|_\omega^2 dt \geq \frac{1}{\tilde{C}} \| \exp(T\tilde{A}^*)\mathbf{w}\|_H^2, , \tag{40}$$

for $\mathbf{w} = (\sum_P^N a_n \sin(n\pi x), \sum_P^N b_n \sin(n\pi x))$ and for $\tilde{C}$ some constant independent of ρ.

Recall

$$\tilde{B}^* exp(t\tilde{A}^*)\mathbf{w} = \frac{1}{r_2 - r_1} \sum_P^N (a_n e^{-t\pi^2 n^2 r_1} - b_n e^{-t\pi^2 n^2 r_2}) \sin(\pi n x).$$

Hence, applying Cor. 1, there exists C such that,

$$\begin{aligned}
\int_{t=0}^T \|\tilde{B}^* \exp(t\tilde{A}^*)\mathbf{w}\|_\omega^2 &= \frac{1}{|r_1 - r_2|^2} \int_\omega \int_{t=0}^T |\sum_{n=P}^N (a_n e^{-t\pi^2 n^2 r_1} - b_n e^{-t\pi^2 n^2 r_2}) \sin(\pi n x)|^2 dt dx \\
&\geq TC \sum_P^N \frac{(|a_n|^2 + |b_n|^2)}{(\rho n^2 T)^{R+2}} \int_\omega | \sin(n\pi x)|^2 \, dx \\
&\geq TC\kappa \sum_P^N \frac{|a_n|^2 + |b_n|^2}{(\rho n^2 T)^{R+2}},
\end{aligned}$$

with $\kappa = \kappa(\omega)$ a positive constant which is independent of N, P, ρ.

On the other hand, by Eq. 4

$$\begin{aligned}
\| \exp(T\tilde{A}^*)\mathbf{w}\|_H^2 &= \|(\sum_P^N a_n e^{-T\pi^2 n^2 r_1} \sin(n\pi x), \sum_P^N b_n e^{-T\pi^2 n^2 r_2} \sin(n\pi x))\|_H^2 \\
&= \sum_P^N |a_n e^{-T\pi^2 n^2 r_1}|^2 + |b_n e^{-T\pi^2 n^2 r_2}|^2 \\
&= \sum_P^N e^{-T\pi^2 n^2 \rho} (|a_n|^2 + |b_n|^2)
\end{aligned} \tag{41}$$

Choose W such that $\pi^2\rho n^2 \geq W$ implies

$$e^{-T\pi^2 n^2 \rho} \leq (Tn^2\rho)^{-(R+2)}.$$

Then, setting

$$V = \max(\sqrt{\frac{2\tilde{P}}{T\pi^2}}, \frac{\sqrt{W}}{\pi}),$$

Eq. 40 clearly follows by setting $1/\tilde{C} = TC\kappa$.

It follows from Eq. 40 and Lemma 1 that there exists a control f which will bring the system to rest in time T, with

$$\int_0^T \int_\omega |f|^2 dxdt \leq \tilde{C} \sum_{n=P}^{N} |a_n|^2 + |b_n|^2.$$

Proposition 2 is proven.

4.5 Low frequencies

In this section we prove the following:

Proposition 4 *Let V, ρ_1 be as in the previous subsection. There exists $\rho_0 \leq \rho_1$ such that if $N \leq V/\sqrt{\rho}$, then for initial conditions $(\sum_1^N c_n \sin(n\pi x), \sum_1^N d_n \sin(n\pi x))$, the system in Eq. 2 is null-controllable in time T with a control f satisfying $\int_\omega \int_0^T |f|^2 \leq C$ for some constant C independent of ρ.*

Remark Note that for frequencies considered in this section, we that $\Re(\pi^2 n^2 r_i) < 2V$ for $i = 1, 2$, ie. the real parts of the frequencies are bounded with bound independent of ρ. The Ingham-type inequalities proven in this section can be compared with those surveyed in Young's book [13] (also see [1]), where (when their language is adapted to the framework of this paper) complex frequencies with an upper bound on the real parts are also considered. The hypotheses in [13] are stronger than those found here, though, and the resulting family of exponential functions actually forms a Riesz basis on some interval.

To prove Prop. 4, we will need to prove observability estimates on three different frequency bands whose union makes up $\{n^2 r_{1,2}\}_{n=1}^{\lceil V/\sqrt{\rho}\rceil}$. The first two of these observability estimates, proven in Lemmas 4 and 5, are again proven using Ingham type inequalities, each one adapted to the frequency band in question. Why these two lemmas cannot be combined among themselves or with Lemma 3 will be pointed out during the proofs. The third observability estimate in this section, Lemma 6, will correspond to a set of frequencies $\{n^2 r_{1,2}\}_{n=1}^{P}$, with P a constant independent of ρ. Here a standard compactness argument yields an observability estimate that is uniform in ρ.

In what follows we set

$$\rho_0 = \min(\rho_1, 1/10, 1/V^2).$$

Lemma 4 *Let $d \in \mathbf{R}^+$. Assume $\rho < \rho_0$, and that N, P satisfy:*

$$\begin{aligned} P &> \max(16, \lceil 1/d \rceil, \sqrt{\frac{2}{\pi^2 d\rho}}) \\ V^2 \ln(P)/P &< 1/17 \\ N &\leq V/\sqrt{\rho}. \end{aligned}$$

There exists $\delta > 0$ such that if

$$f(t) = \sum_{n=P}^{N} a_n e^{-tn^2 d\pi^2 r_1} + b_n e^{-tn^2 d\pi^2 r_2},$$

then

$$\int_{-\pi}^{\pi} |f(t)|^2 dt \geq \delta \sum_{n=P}^{N} \frac{e^{\pi^3 d\rho n^2}}{d^2 \rho^2 n^4} (|a_n|^2 + |b_n|^2).$$

Here δ is independent of N, P, f, d, ρ.

Proof: Using the notation of the proof of Lemma 3, we have

$$\int_{\mathbf{R}} I(t)k(t)dt = \sum_{n=P}^{N} |a_n|^2 K(-2iC_1 n^2) + \sum_{m,n;m\neq n} a_n \overline{a_m} K(-iC_1(n^2+m^2) + C_2(n^2-m^2)). \tag{42}$$

For this lemma we use a test function used by Ingham in [4]:

$$k(t) = \begin{cases} \cos(t/2), & |t| \leq \pi \\ 0, & |t| > \pi. \end{cases} \tag{43}$$

Remark Although we could choose our test function to be as in Lemma 3, our choice here has the advantage of giving a simpler formula for K. In fact, here $K(u) = \frac{4\cos(\pi u)}{1-4u^2}$. On the other hand, Ingham's test function could not be used in Lemma 3, where it is vital for the denominator of K to be of order $R >> 0$ in the argument leading to Eq. 36.

Note that the hypothesis $P \geq \sqrt{2/(\pi^2 \rho d)}$ implies that if $n \geq P$, then $C_1 n^2 \geq 1$. Hence for $n \geq P$,

$$\begin{aligned} K(-2iC_1n^2) &= \frac{4\cosh(2\pi C_1 n^2)}{1+16C_1^2n^4} \\ &\geq \frac{2e^{2\pi C_1 n^2}}{17C_1^2n^4}. \end{aligned} \tag{44}$$

Also, there exists $\gamma > 0$ such that

$$\begin{aligned} |K(-iC_1(n^2+m^2)+C_2(n^2-m^2))| &= 4|\frac{\cos(\pi(-iC_1(n^2+m^2)+C_2(n^2-m^2)))}{1-4(-iC_1(n^2+m^2)+C_2(m^2-n^2))^2}| \\ &= 2|\frac{e^{\pi i C_2(n^2-m^2)+\pi C_1(n^2+m^2)} + e^{-\pi i C_2(n^2-m^2)-\pi C_1(n^2+m^2)}}{1-4(C_2(n^2-m^2)+iC_1(n^2+m^2))^2}| \\ &\leq \frac{1}{2-2\gamma}\frac{e^{\pi C_1(n^2+m^2)}}{C_2^2(n^2-m^2)^2+C_1^2(n^2+m^2)^2}. \end{aligned} \tag{45}$$

This last inequality was derived as follows. Assume first that $n > m$. Note that $\rho < 1/10$ implies $C_2 > 3d\pi^2/2$. Also, the hypothesis $P > 1/d$ implies $n^2 - m^2 \geq 2/d$. Hence $C_2(n^2-m^2) - \frac{1}{2} \geq (1-\gamma)C_2(n^2-m^2)$ with $\gamma = 1/3$, and hence

$$\begin{aligned} |1-4(C_2(n^2-m^2)+iC_1(n^2+m^2))^2| &= 4|\frac{1}{2} - C_2(n^2-m^2) - iC_1(n^2+m^2)| \\ &\quad \cdot|\frac{1}{2} + C_2(n^2-m^2) + iC_1(n^2+m^2)| \\ &\geq 4|-(1-\gamma)C_2(n^2-m^2) - iC_1(n^2+m^2)| \\ &\quad \cdot|\frac{1}{2} + C_2(n^2-m^2) + iC_1(n^2+m^2)| \\ &\geq 4(1-\gamma)|-C_2(n^2-m^2) - iC_1(n^2+m^2)| \\ &\quad \cdot|\frac{1}{2} + C_2(n^2-m^2) + iC_1(n^2+m^2)| \\ &\geq 4(1-\gamma)(C_2^2(n^2-m^2)^2 + C_1^2(n^2+m^2)^2). \end{aligned}$$

The proof for $m > n$ is similar. Hence, for any P, N satisfying the hypotheses of the lemma,

$$\sum_{m,n=P,m\neq n}^{N} |a_n \overline{a_m} K(-iC_1(n^2+m^2)+C_2(m^2-n^2))|$$
$$\leq \frac{1/2}{2-2\gamma} \sum_{m,n\geq P,m\neq n}^{N} \frac{|a_n|^2 e^{2\pi C_1 n^2} + |a_m|^2 e^{2\pi C_1 m^2}}{C_2^2(n^2-m^2)^2 + C_1^2(n^2+m^2)^2}$$
$$= \frac{1}{2-2\gamma} \sum_{n=P}^{N} |a_n|^2 e^{2\pi C_1 n^2} \sum_{m=P,m\neq n}^{N} \frac{1}{C_2^2(n^2-m^2)^2 + C_1^2(n^2+m^2)^2}. \qquad (46)$$

We analyze the term

$$\sum_{m=P,m\neq n}^{N} \frac{1}{C_2^2(n^2-m^2)^2 + C_1^2(n^2+m^2)^2} = \frac{1}{C_1^2+C_2^2} \sum_{m=P,m\neq n}^{N} \frac{1}{n^4+2\beta n^2 m^2 + m^4}, \qquad (47)$$

with

$$\beta \equiv \frac{C_1^2 - C_2^2}{C_1^2 + C_2^2} = -1 + \rho^2/2.$$

One easily verifies that

$$n^4 + 2\beta n^2 m^2 + m^4 = (m+nx)(m-nx)(m+ny)(m-ny)$$

with $x = \sqrt{-\beta - i\sqrt{1-\beta^2}}$, $y = \sqrt{-\beta + i\sqrt{1-\beta^2}}$, and that

$$\frac{1}{n^4+2\beta n^2m^2+m^4} = \frac{1}{-4in^3x\sqrt{1-\beta^2}}\left(\frac{1}{m-nx} - \frac{1}{m+nx}\right) + \frac{1}{4in^3y\sqrt{1-\beta^2}}\left(\frac{1}{m-ny} - \frac{1}{m+ny}\right).$$

Suppose, for the moment, that $n = P$. Thus, since $x = \overline{y} \sim 1$ and $P - Px \sim 0$,

$$\begin{aligned}
\sum_{m=P,m\neq n}^{N} \frac{1}{n^4+2\beta n^2m^2+m^4} &= \sum_{m=P+1}^{N} \frac{1}{n^4+2\beta n^2m^2+m^4} \\
&\sim \int_{m=P+1}^{N} \frac{dm}{n^4+2\beta n^2m^2+m^4} \\
&\leq \frac{1}{|-4in^3x\sqrt{1-\beta^2}|} \ln\left|\frac{(N-Px)(P+1+Px)}{(P+1-Px)(N+Px)}\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{|-4in^3y\sqrt{1-\beta^2}|}\ln|\frac{(N-Py)(P+1+Py)}{(P+1-Py)(N+Py)}| \\
&\leq \quad \frac{\ln(N)}{n^3\sqrt{1-\beta^2}} \\
&\leq \quad \frac{\ln(N)(C_1^2+C_2^2)}{2n^3C_1C_2} \qquad (48)
\end{aligned}
$$

It is easy to verify that the same estimate holds for $n > P$. Thus, using $C_2 > 3\pi^2 d/2$, along with Eqs. 46, 47, 48 and the hypotheses $\rho N^2 \leq V^2$, $V^2\ln(N)/N < 1/17$,

$$
\begin{aligned}
\sum_{m,n=P,m\neq n}^{N} |a_n\overline{a_m}K(-iC_1(n^2+m^2)+C_2(n^2-m^2))| &\leq \frac{1}{2-2\gamma}\sum_{n=P}^{N}\frac{\ln(N)}{2n^3C_1C_2}|a_n|^2e^{2\pi C_1n^2} \\
&\leq \sum_{n=P}^{N}\frac{1/17}{C_1^2n^4}|a_n|^2e^{2\pi C_1n^2}. \qquad (49)
\end{aligned}
$$

It follows from Eqs. 42, 44 and 49 that

$$
\int_{\mathbf{R}} I(t)k \ \geq \ \sum_{n=P}^{N}|a_n|^2\frac{e^{2\pi C_1n^2}}{17C_1^2n^4} \qquad (50)
$$

A similar argument yields

$$
\int_{\mathbf{R}} II(t)k \ \geq \ \sum_{n=P}^{N}|b_n|^2\frac{e^{2\pi C_1n^2}}{17C_1^2n^4} \qquad (51)
$$

Finally, we consider $\int III(t)kdt$. We have, using the arguments leading to Eq. 45, that there exists $\gamma > 0$ such that

$$
\begin{aligned}
\int III(t)k(t)dt &= 2\sum_{m,n=P}^{N}|a_n\overline{b_m}|K(-iC_1(n^2+m^2)+C_2(m^2+n^2))| \\
&= 2\sum_{m,n=P}^{N}|a_n\overline{b_m}|\frac{4\cos(-i\pi C_1(n^2+m^2)+C_2\pi(m^2+n^2))}{1-4(-iC_1(n^2+m^2)+C_2(-m^2-n^2))^2} \\
&\leq 2\sum_{m,n=P}^{N}|a_n\overline{b_m}|\frac{e^{C_1\pi(n^2+m^2)}}{(2-\gamma)(C_1^2+C_2^2)(m^2+n^2)^2}
\end{aligned}
$$

$$\leq \sum_{m,n=P}^{N} |a_n \overline{b_m}| \frac{2e^{\pi C_1(n^2+m^2)}}{(2-\gamma)\pi^4 d^2(m^2+n^2)^2}$$
$$\leq \sum_{m,n=P}^{N} \frac{|a_n|^2 e^{2\pi C_1 n^2} + |b_m|^2 e^{2\pi C_1 m^2}}{(2-\gamma)\pi^4 d^2(m^2+n^2)^2}. \tag{52}$$

Consider the term involving $|a_n|^2$. We have, using Eq. 28, the hypotheses on P, and the hypotheses $\rho \leq 1/V^2$ and $P > 17$:

$$\begin{aligned}
\sum_{m,n=P}^{N} \frac{|a_n|^2 e^{2\pi C_1 n^2}}{(2-\gamma)\pi^4 d^2(m^2+n^2)^2} &= \sum_{n=P}^{N} \frac{|a_n|^2 e^{2\pi C_1 n^2} \rho^2}{2(2-\gamma)C_1^2} \sum_{m=P}^{N} \frac{1}{(m^2+n^2)^2} \\
&\leq \sum_{n=P}^{N} \frac{|a_n|^2 e^{2\pi C_1 n^2} \rho^2}{2(2-\gamma)C_1^2 n^3} \\
&= \sum_{n=P}^{N} \frac{|a_n|^2 e^{2\pi C_1 n^2} \rho^2 n}{2(2-\gamma)C_1^2 n^4} \\
&\leq \sum_{n=P}^{N} \frac{|a_n|^2 e^{2\pi C_1 n^2} \rho V^2/N}{2(2-\gamma)C_1^2 n^4} \\
&\leq \sum_{n=P}^{N} \frac{|a_n|^2 e^{2\pi C_1 n^2}}{34(2-\gamma)C_1^2 n^4} \\
&\leq \sum_{n=P}^{N} \frac{|a_n|^2 e^{2\pi C_1 n^2}}{34 C_1^2 n^4},
\end{aligned} \tag{53}$$

provided γ is small enough. The same argument shows that

$$\sum_{m,n=P}^{N} \frac{|b_m|^2 e^{2\pi C_1 m^2}}{(2-\gamma)\pi^4 d^2(m^2+n^2)^2} \leq \sum_{m=P}^{N} \frac{|b_m|^2 e^{2\pi C_1 m^2}}{34 C_1^2 m^4}. \tag{54}$$

The lemma is completed by using Eqs. 54, 53, 51, 50.

The lemma follows. We now state the corresponding result for integrals over the interval $[0, T]$.

Corollary 2 *Set $d = \frac{T}{2\pi}$. Assume the hypotheses of the previous lemma. If*

$$f(t) = \sum_{n=P}^{N} a_n e^{-tn^2\pi^2 r_1} + b_n e^{-tn^2\pi^2 r_2},$$

then
$$\int_0^T |f(t)|^2 dt \geq C \sum_P^N |a_n|^2 + |b_n|^2,$$
where C is a positive constant independent of $\rho, \{a_n\}, \{b_n\}$.

Proof: We have, using the substitution $s = t/d - \pi$ and then the previous lemma (with $d = T/2\pi$),

$$\begin{aligned}
\int_0^T |f(t)|^2 dt &= \int_0^T \sum_{n=P}^N |a_n e^{-tn^2\pi^2 r_1} + b_n e^{-tn^2\pi^2 r_2}|^2 dt \\
&= \frac{T}{2\pi} \int_{-\pi}^{\pi} |\sum_{n=P}^N (a_n e^{-n^2\pi^2 r_1 T/2}) e^{-sn^2\pi^2 r_1 d} + (b_n e^{-n^2\pi^2 r_2 T/2}) e^{-sn^2\pi^2 r_2 d}|^2 ds \\
&\geq \frac{T}{68\pi} \sum_{n=P}^N (|a_n e^{-n^2\pi^2\rho T/4}|^2 + |b_n e^{-n^2\pi^2\rho T/4}|^2) \frac{e^{\pi^2 n^2 \rho T/2}}{\pi^4 d^2 \rho^2 n^4} \\
&\geq C \sum_{n=P}^N |a_n|^2 + |b_n|^2.
\end{aligned}$$

Next, we prove observability for $n \leq \sqrt{2/(\pi^2 \rho d)}$.

Lemma 5 *Let T, V, d be as in the previous corollary, and assume $\rho < \rho_0$. Assume $\pi^2 d\rho N^2 \leq 2$ Then there exists a positive constant δ, independent of ρ, such that if $P > \max(\lceil 1/d \rceil, 16, P_0)$ and $V \ln(P)/P < 1/17$, then*
$$f(t) = \sum_{n=P}^N a_n e^{-tn^2 d\pi^2 r_1} + b_n e^{-tn^2 d\pi^2 r_2}$$
satisfies the inequality
$$\int_{-\pi}^{\pi} |f(t)|^2 dt \geq \delta \sum_{n=P}^N (|a_n|^2 + |b_n|^2).$$

Proof: The proof follows along the lines of the proof of the previous lemma, whose notation we use. As in the previous lemma, $K(u) = \frac{4\cos(\pi u)}{1-4u^2}$. Consider first the term
$$\int_{\mathbf{R}} I(t)k(t)dt = \sum_{n=P}^N |a_n|^2 K(-2iC_1 n^2) + \sum_{m,n;m\neq n} a_n \overline{a_m} K(-iC_1(n^2+m^2) + C_2(n^2-m^2)).$$

Note that

$$\begin{aligned} K(-2iC_1n^2) &= \frac{4\cosh(2\pi C_1 n^2)}{1+16C_1^2n^4} \\ &\geq 4\frac{1+(2\pi C_1n^2)^2/2}{1+16C_1^2n^4} \\ &\geq 4. \end{aligned} \quad (55)$$

Also, $\pi^2 d\rho N^2 \leq 2$ implies that for $n \leq N$, $n^2C_1 \leq 1$.

Remark Since n^2C_1 is not bounded below in this case, Eq. 44 no longer holds. Hence, the proof of Lemma 4 no longer applies. Instead, we have

$$\begin{aligned} |K(-iC_1(n^2+m^2)+C_2(n^2-m^2))| &= 2\left|\frac{e^{\pi iC_2(n^2-m^2)+\pi C_1(n^2+m^2)}+e^{-\pi iC_2(n^2-m^2)-\pi C_1(n^2+m^2)}}{1-4(C_2(n^2-m^2)+iC_1(n^2+m^2))^2}\right| \\ &\leq \frac{2(e^{2\pi}+e^{-2\pi})}{4C_2^2(n^2-m^2)^2-17}. \end{aligned} \quad (56)$$

Note that $\rho < 1/10$ implies $C_2 > (.99)d\pi^2$, and hence $\frac{e^{2\pi}+e^{-2\pi}}{2C_2^2/d^2} < 3-\delta$, for some $\delta > 0$. Also, since $n, m \geq P$, we have by hypothesis $n^2-m^2 \geq (n-m)(2/d)$. Hence

$$\begin{aligned} \sum_{m=P,m\neq n}^{N} |K(-iC_1(n^2+m^2)+C_2(n^2-m^2))| &\leq 2(e^{2\pi}+e^{-2\pi}) \sum_{m=P,m\neq n}^{N} \frac{1}{4C_2^2(n^2-m^2)^2-17} \\ &\leq \frac{(e^{2\pi}+e^{-2\pi})}{2C_2^2/d^2} \sum_{m=P,m\neq n}^{N} \frac{1}{4(n-m)^2-\frac{17d^2}{4C_2^2}} \\ &\leq (3-\delta) \sum_{m=P,m\neq n}^{N} \frac{1}{4(n-m)^2-1} \\ &\leq (3-\delta)\cdot 2\sum_{r=1}^{\infty} \frac{1/2}{2r-1} - \frac{1/2}{2r+1} \\ &< 3-2\delta. \end{aligned} \quad (57)$$

Similarly,

$$\sum_{n=P,m\neq n}^{N} |K(-iC_1(n^2+m^2)+C_2(n^2-m^2))| < 3-2\delta. \quad (58)$$

Thus, by Eqs. 57, 58 and 55, and the inequality $|a_m a_n| \leq (|a_m|^2+|a_n|^2)/2$,

$$\begin{aligned}\int_{\mathbf{R}} I(t)k(t)dt &\geq \sum_{n=P}^{N} 4|a_n|^2 - \sum_{m,n\geq P; m\neq n}^{N} \frac{|a_n|^2+|a_m|^2}{2}|K(-iC_1(n^2+m^2)+C_2(n^2-m^2))| \\ &\geq \sum_{n=P}^{N} 4|a_n|^2 - \frac{1}{2}(3-\delta)\sum_{m=P}^{N}|a_m|^2 - \frac{1}{2}(3-\delta)\sum_{n=P}^{N}|a_n|^2 \\ &\geq (1+\delta)\sum_{n=P}^{N}|a_n|^2\end{aligned}$$

Similarly,

$$\int_{\mathbf{R}} II(t)k(t)dt \geq (1+\delta)\sum_{n=P}^{N}|b_n|^2.$$

To complete the proof, it suffices to show that

$$|\int III(t)k(t)dt| \leq \sum_{P}^{N}|a_n|^2+|b_n|^2. \tag{59}$$

In fact, by Eq. 52, we have, for some $\gamma > 0$,

$$|\int III(t)k(t)dt| \leq \sum_{m,n=P}^{N} \frac{|a_n|^2 e^{2\pi C_1 n^2}+|b_m|^2 e^{2\pi C_1 m^2}}{(2-\gamma)\pi^4 d^2(m^2+n^2)^2}.$$

Consider the term involving $|a_n|^2$. We have

$$\begin{aligned}\sum_{m,n=P}^{N}\frac{|a_n|^2 e^{2\pi C_1 n^2}}{(2-\gamma)\pi^4 d^2(m^2+n^2)^2} &\leq \sum_{n=P}^{N}\frac{|a_n|^2 e^{2\pi}}{d^2\pi^4(2-\gamma)}\sum_{m=P}^{N}\frac{1}{(m^2+n^2)^2} \\ &\leq \sum_{n=P}^{N}\frac{2|a_n|^2 e^{2\pi}}{d^2\pi^4(2-\gamma)n^3} \\ &\leq \sum_{n=P}^{N}\frac{2|a_n|^2 e^{2\pi}}{\pi^4(2-\gamma)n} \\ &\leq \sum_{n=P}^{N}|a_n|^2,\end{aligned}$$

because $P > 16$ implies $\frac{2e^{2\pi}}{\pi^4(2-\gamma)P} \leq 1$ provided γ is small enough. The same argument then shows that

$$\sum_{m,n=P}^{N}\frac{|b_m|^2 e^{2\pi C_1 m^2}}{2(2-\gamma)\pi^4 d^2(m^2+n^2)^2} \leq \sum_{m=P}^{N}|b_m|^2.$$

The lemma now follows.

Corollary 3 *Assume the hypotheses of the previous lemma. If*

$$f(t) = \sum_{n=P}^{N} a_n e^{-tn^2\pi^2 r_1} + b_n e^{-tn^2\pi^2 r_2}$$

then there exists $C > 0$ *such that*

$$\int_0^T |f(t)|^2 dt \geq C \sum_{n=P}^{N} |a_n|^2 + |b_n|^2.$$

Proof: As in the proof of the previous corollary,

$$\begin{aligned}
\int_0^T |f(t)|^2 dt &= \frac{T}{2\pi} \int_{-\pi}^{\pi} |\sum_{n=P}^{N} (a_n e^{-n^2\pi^2 r_1 T/2}) e^{-tn^2\pi^2 r_1 d} + (b_n e^{-n^2\pi^2 r_2 T/2}) e^{-tn^2\pi^2 r_2 d}|^2 dt \\
&\geq \frac{T\delta}{2\pi} \sum_{n=P}^{N} (|a_n e^{-n^2\pi^2\rho T/4}|^2 + |b_n e^{-n^2\pi^2\rho T/4}|^2) \\
&\geq \frac{T\delta e^{-\pi/2}}{2\pi} \sum_{n=P}^{N} |a_n|^2 + |b_n|^2.
\end{aligned}$$

Finally, by a standard compactness argument, we have

Lemma 6 *Let* T, P *be as in the previous lemma. If*

$$f(t) = \sum_{n=1}^{P} a_n e^{-tn^2\pi^2 r_1} + b_n e^{-tn^2\pi^2 r_2}$$

then there exists $C > 0$ *such that*

$$\int_0^T |f(t)|^2 dt \geq C \sum_{n=1}^{P} |a_n|^2 + |b_n|^2.$$

5 Appendix

5.1 Proof of Lemma 1

Assume Eq. 7. Define

$$J(w_T) = \int_\omega \int_{t=0}^{T} \frac{1}{2} |\tilde{B}^* w(t)|^2 dt dx + \Re \int_0^1 \overline{w(0)} \cdot \tilde{y}(0) dx.$$

Since J is coercive and strictly convex and continuous, there exists a unique minimum z_T. Setting $z(t) = e^{(T-t)\tilde{A}^*} z_T$, the associated Euler equation is :

$$0 = \delta_{w_T} J(z_T) = \int_\omega \int_{t=0}^{T} \Re(\overline{\tilde{B}^* w(t)} \tilde{B}^* z(t)) dt dx + \Re \int_0^1 \overline{w(0)} \cdot \tilde{y}(0)\ dx. \quad (60)$$

We now set $\tilde{f} = \tilde{B}^* z$. Then by Eq. 2,

$$\int_0^T \int_0^1 \tilde{y}_t \cdot \overline{w} = \int_0^T \int_0^1 \tilde{A}\tilde{y} \cdot \overline{w} + \chi_\omega \tilde{B}\tilde{B}^* z \cdot \overline{w}.$$

Integrating by parts, we get

$$\int_0^1 \tilde{y}(T) \cdot \overline{w(T)} - \tilde{y}(0) \cdot \overline{w(0)}\ dx = \int_\omega \int_0^T \tilde{B}^* z \overline{\tilde{B}^* w}\ dt\ dx.$$

Taking the real part of this equation and using Eq. 60, we get

$$\Re \int_0^1 \tilde{y}(T) \cdot \overline{w(T)}\ dx = 0,\ \forall w(T) \in H.$$

It follows that $\tilde{y}(T) = 0$. Thus $\tilde{f} = \tilde{B}^* z$ is the control that sends $\tilde{y}(t)$ to rest.

Next, we estimate the norm of the control. By Eqs. 60 and 7

$$\begin{aligned} \int_0^T \int_0^1 |\tilde{B}^* z(t)|^2 dx dt &= -\Re \int_0^1 z(0) \cdot \tilde{y}(0) dx \\ &\leq \|z(0)\|_H \|\tilde{y}(0)\|_H \\ &\leq (\frac{1}{C} \int_0^T \int_0^1 |\tilde{B}^* z(t)|^2 dx dt)^{1/2} \|\tilde{y}(0)\|_H, \end{aligned}$$

and hence

$$\sqrt{C}(\int_0^T \int_0^1 |\tilde{B}^* z(t)|^2 dx dt)^{1/2} \leq \|\tilde{y}(0)\|_H;$$

this is just a phrasing for Eq. 6

We now prove the converse. To this end, dot Eq. 2 by $\overline{w}$, and integrate by parts in t and x to obtain

$$\begin{aligned} \int_0^1 \tilde{y}(0) \cdot \overline{w(0)} &= -\int_0^1 \int_0^T \chi_\omega \tilde{B} f \cdot \overline{w} \\ &= -\int_\omega \int_0^T f \overline{\tilde{B}^* w} \end{aligned}$$

Thanks to Eq. 6, we conclude

$$
\begin{aligned}
|\int_0^1 \tilde{y}(0)\cdot\overline{w(0)}| &\leq \int_\omega\int_0^T |f\overline{\tilde{B}^*w}| \\
&\leq \frac{1}{\sqrt{C}}(\int_\omega\int_0^T |\tilde{B}^*w|^2 dtdx)^{1/2}\|\tilde{y}(0)\|_H, \forall \tilde{y}(0)\in H.
\end{aligned}
$$

Setting $\tilde{y}(0) = w(0)$, Eq. 7 follows. This completes the proof of the lemma.

5.2 Proof of Theorem 2

We now prove that Eq. 17 implies Theorem 2.

The proof follows the argument appearing in [11], except that in their case the limit problem remains parabolic, whereas in our case the limit problem is hyperbolic.

In what follows, we will highlight the ρ-dependence of various objects using the notation $f_\rho, \tilde{A}^*_\rho$, etc.

Step 1 By Theorem, there exists $C > 0$ independent of ρ such that

$$\int_0^T \|f_\rho\|^2_\omega dt < C,\ \forall\rho > 0.$$

It follows that there exists a subsequence, which we again label f_ρ, such that $f_\rho \rightarrow f_0$ weakly in $L^2(\omega\times(0,T))$. The Euler equation implies that for any fixed $w_T \in H$,

$$0 = \int_\omega\int_{t=0}^T \Re(\overline{\tilde{B}^*_\rho w_\rho(t)}\tilde{B}^*_\rho z_\rho(t))dtdx + \Re\int_0^1 \overline{w_\rho(0)}\cdot\tilde{y}(0)\ dx. \qquad (61)$$

Here w_ρ solves Eq. 4. Now it is easy to verify, by studying the Fourier coefficients, that for any fixed $w_T \in L^2(0,1)$, $\tilde{B}^*_\rho w_\rho \rightarrow \tilde{B}^*_0 w_0$ in $L^2((0,1)\times(0,T))$ and $w_\rho(x,0)\rightarrow w_0(x,0)$ in H. Here w_0 is a weak solution to $(w_0)_t + \tilde{A}^*_0 w_0 = 0,\ w_0(T) = w_T$ in the sense that

$$\int_0^1\int_0^T w_0\cdot\overline{(-(\phi)_t + \tilde{A}\phi)} = 0,$$

$$
\begin{aligned}
&\forall\phi\in\left(L^2((0,T),H^2(0,1)\cap H^1_0(0,1))\cap H^1((0,T),L^2(0,1))\right)\\
&\times\left(L^2((0,T),H^2(0,1)\cap H^1_0(0,1))\cap H^1((0,T),L^2(0,1))\right),
\end{aligned}
$$

$$\text{such that } \phi(0) = \phi(T) = 0. \tag{62}$$

Thus, letting $\rho \to 0$ in Eq. 61,

$$0 = \int_\omega \int_{t=0}^T \Re(\overline{\tilde{B}_0^* w_0(t)} f_0) dt dx + \Re \int_0^1 \overline{w_0(0)} \cdot \tilde{y}(0) \ dx. \tag{63}$$

It follows that the limit function f_0 is a null control for the unperturbed ($\rho = 0$) beam equation. To complete the proof of the Corollary, it suffices to prove the following:

a) the limit f_0 is uniquely determined, ie. independent of choice of subsequence,

b) the convergence $f_\rho \to f_0$ is strong.

Step 2 We begin by proving a convergence result for z_ρ on $(0,1) \times (0,T)$. Note that uniform observability inequality holds for any $T > 0$. In fact, examination of the argument in Section 4 shows that for any $\tau < T$, we have that for $t \leq \tau$ and for $\rho < \rho_0(\tau)$,

$$\|exp((\tau - t)\tilde{A}_\rho{}^*)\mathbf{w}\|_H^2 < C(T-\tau) \int_\tau^T \int_\omega |\tilde{B}_\rho^* exp(-t\tilde{A}_\rho{}^*)\mathbf{w}|^2 dx dt, \forall \mathbf{w} \in H, \tag{64}$$

with $C(T - \tau) \to \infty$ and $\rho_0 \to 0$ as $\tau \to T$. Fixing τ for the moment, we deduce that for $t \leq \tau$, $\|z_\rho(t)\|_H^2 < C(T - \tau)$ and hence the two components of z_ρ are each bounded in $L^\infty((0,\tau), L^2(0,1))$. Thus, passing to subsequences, there exists ζ such that $z_\rho \to \zeta$ weakly in $L^\infty((0,\tau), L^2(0,1)) \times L^\infty((0,\tau), L^2(0,1))$. Since $\tau < T$ is arbitrary, it follows that ζ can be defined to be in $L^\infty_{loc}((0,T), L^2(0,1)) \times L^\infty_{loc}((0,T), L^2(0,1))$. Furthermore, ζ satisfies

$$\int_0^1 \int_0^\tau \zeta \cdot \overline{(\tilde{A}_0(\phi) - \phi_t)} dt dx = 0,$$

$$\forall \phi \in \Big(L^2((0,\tau), H^2(0,1) \cap H_0^1(0,1)) \cap H^1((0,\tau), L^2(0,1))\Big)$$
$$\times \Big(L^2((0,\tau), H^2(0,1) \cap H_0^1(0,1)) \cap H^1((0,\tau), L^2(0,1))\Big),$$
$$\text{such that } \phi(0) = \phi(\tau) = 0. \tag{65}$$

It follows that $\zeta = \exp(-t\tilde{A}_0^*)\zeta(0)$ for some $\zeta(0) \in H$. Since $\exp(-t\tilde{A}_0^*)$ is unitary, it follows that $\zeta \in L^\infty((0,T), L^2(0,1))$. It then follows that $\zeta = \exp((T-t)\tilde{A}_0^*)\zeta_T$ for some $\zeta_T \in H$. Also, note that $\tilde{B}^* z_\rho$ converges weakly to $\tilde{B}^*\zeta$, hence

$$\tilde{B}^*\zeta|_{\omega \times (0,T)} = f_0. \tag{66}$$

Thus Eq. 63 can be rewritten

$$0 = \int_\omega \int_{t=0}^T \Re \left(\overline{\tilde{B}_0^* w_0(t)} \tilde{B}_0^* \zeta \right) dt dx + \Re \int_0^1 \overline{w_0(0)} \cdot \tilde{y}(0) \; dx. \tag{67}$$

Step 3
Define the functional $J_0 : H \rightarrow \mathbf{R}$ by

$$J_0(w_T) = \int_\omega \int_{t=0}^T \frac{1}{2} |\tilde{B}_0^* w(t)|^2 dt dx + \Re \int_0^1 \overline{w(0)} \cdot \tilde{y}(0) dx,$$

with w a weak solution to the adjoint equation as in Eq. 62,

We now recall the well known observability inequality for the unperturbed beam equation:

$$\| \exp(T\tilde{A}_0^*) w\|_{L^2(0,1)}^2 \leq C \int_0^T \int_\omega |\tilde{B}_0^* \exp((T-t)\tilde{A}_0^*) w|^2 dx dt,$$

with C a positive constant independent of w. It follows that J_0 achieves a unique minimum $w = \psi$ which is characterized by the Euler Equation

$$0 = \int_\omega \int_{t=0}^T \Re \left(\overline{\tilde{B}_0^* w(t)} \tilde{B}_0^* \psi(t) \right) dt dx + \Re \int_0^1 \overline{w(0)} \cdot \tilde{y}(0) \; dx, \tag{68}$$

this holding for all w solution of Eq. 62. Comparing with Eq. 67, it follows that $\psi = \zeta$. It now follows that the limit functions ζ and f_0 are independent of the choice of subsequence.

Step 3 It remains to show that the convergence of f_ρ to f_0 is strong. For this, first we show that $z_\rho(x,0)$ converges weakly in H to $\zeta(x,0)$. To this end, note first that by Eq. 7, $z_\rho(x,0)$ is bounded in H. Hence there exists $\mu(x) \in H$ such that $z_\rho(x,0) \rightarrow \mu$ weakly in H. Now let $\tau \in (0,T)$ and let $\phi \in C^\infty((0,1) \times (0,\tau)) \times C^\infty((0,1) \times (0,\tau))$ vanish on $\{x=0\}, \{x=1\}$, and $\{t=\tau\}$. Dotting the equation $(z_\rho)_t + \tilde{A}_\rho^* z_\rho = 0$ by $\overline{\phi}$ and integrating by parts, we obtain

$$\int_0^\tau \int_0^1 z_\rho \cdot \overline{(\phi_t - \tilde{A}_\rho \phi)} dx dt + \int_0^1 z_\rho(x,0) \overline{\phi(x,0)} dx = 0.$$

Letting $\rho \rightarrow 0$, we obtain

$$\int_0^\tau \int_0^1 \zeta \cdot \overline{(\phi_t - \tilde{A}_0 \phi)} dx dt + \int_0^1 \mu(x) \overline{\phi(x,0)} dx = 0. \tag{69}$$

Also, replacing ϕ and ζ by their Fourier expansions and then integrating by parts shows that

$$\int_0^\tau \int_0^1 \zeta \cdot \overline{(\phi_t - \tilde{A}_0\phi)} dxdt + \int_0^1 \zeta(x,0)\overline{\phi(x,0)} dx = 0. \tag{70}$$

Comparing Eqs. 69 and 70, we get that $\mu(x) = \zeta(x,0)$.

Step 4 Finally, by Eq. 63, we have

$$0 = \int_\omega \int_{t=0}^T |f_\rho|^2 dtdx + \Re \int_0^1 \overline{z_\rho(0)} \cdot \tilde{y}(0)\ dx.$$

Thus we have

$$\begin{aligned}\lim_{\rho\longrightarrow 0} \|f_\rho\|^2_{L^2(\omega\times(0,T))} &= -\Re \int_0^1 \mu(x) \cdot \tilde{y}(0) dx \\ &= -\Re \int_0^1 \zeta(x,0) \cdot \tilde{y}(0) dx \qquad (71)\end{aligned}$$

Also, by Eqs. 67, 66, we have

$$\int_\omega \int_{t=0}^T |f_0|^2 dtdx = -\Re \int_0^1 \overline{\zeta(x,0)} \cdot \tilde{y}(0)\ dx \tag{72}$$

It follows immediately from Eqs. 71, 72 that $\lim_{\rho\longrightarrow 0} \|f_\rho\|^2_{L^2(\omega\times(0,T))} = \|f_0\|^2_{L^2(\omega\times(0,T))}$. Since $f_\rho \longrightarrow f_0$ weakly in $L^2(\omega\times(0,T))$, this implies that $f_\rho\longrightarrow f_0$ strongly in the same topology.

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