

# Two-Hadron Physics on the Lattice<sup>a</sup>

Merritt S Cook and HRF, LHPCollaboration

SYSTEMS WITH HEAVY (STATIC) QUARKS

*Heavy-light  $K-\Lambda$  like system*

SYSTEMS WITH LIGHT QUARKS

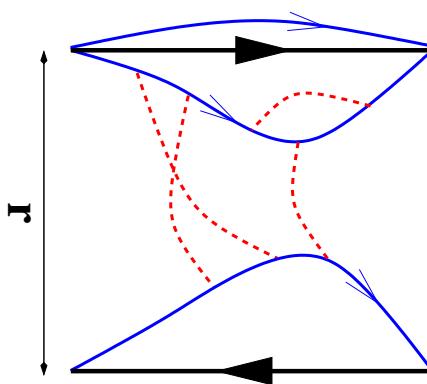
*See Lattice 2005*

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<sup>a</sup>This material is based upon work supported by the National Science Foundation under Grant No. 0300065 and upon resources provided by the Lattice Hadron Physics Collaboration LHPC through the SciDac program of the US Department of Energy.

Goal: Effective hadron-hadron interaction  
possible strategies

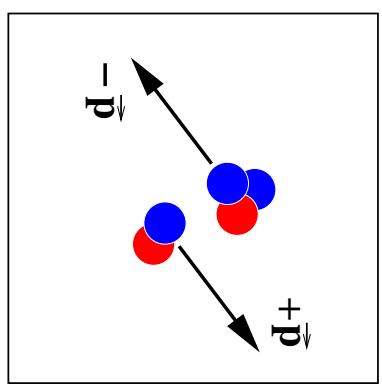
1. Heavy (static) quarks



$r$ =relative distance

total energy  $E(r) = V(r) + \text{const}$   
 $\Rightarrow$  adiabatic potential  $V(r)$

2. Light quarks



finite box mass spectrum [Lüscher]  
scattering phase shifts  
 $\Rightarrow \delta(k_n^2)$

# Search for hadronic molecules in $K-\Lambda$ like systems <sup>a</sup>

Merritt S Cook and HRF, LHPCollaboration

*Motivation*

*Operators*

*Correlator functions*

*Fuzzing and smearing*

*Lattice simulation*

*Analysis issues*

*Current results*

*Assessment*

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## Motivation

$\Theta(1539)$

$N(1647)$

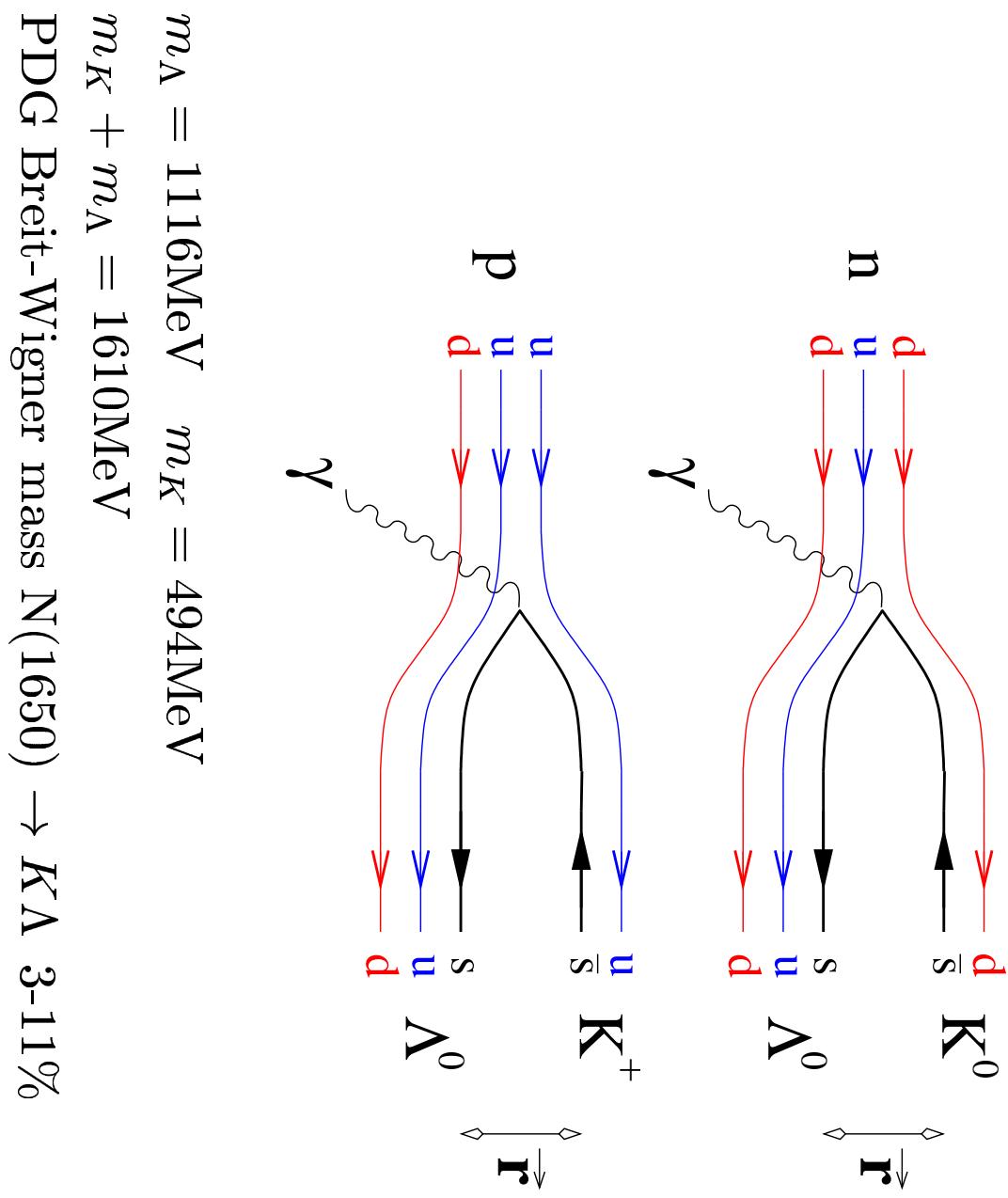
$\Sigma(1755)$

$\Xi(1862)$

$\Theta^+ \sim uudd\bar{s}$



5-quark(?) decouplet [Daikov, Petrov, 2004]



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$$m_\Lambda = 1116 \text{ MeV} \quad m_K = 494 \text{ MeV}$$

$$m_K + m_\Lambda = 1610 \text{ MeV}$$

PDG Breit-Wigner mass  $N(1650) \rightarrow K\Lambda$  3-11%

## Operators

local,  $A, B, C = \text{color}, \alpha, \beta, \gamma = \text{spin, Dirac}$

$$\begin{aligned}
 K^+(x) &= \bar{s}_{C\alpha}(x)\gamma_{5,\alpha\beta}u_{C\beta}(x) \\
 \Lambda_\alpha^0(x) &= \epsilon_{CDE}(C\gamma_5)_{\beta\gamma} \\
 &\quad [ u_{C\alpha}(x)(d_{D\beta}(x)s_{E\gamma}(x) - s_{D\beta}(x)d_{E\gamma}(x)) \\
 &\quad + d_{C\alpha}(x)(s_{D\beta}(x)u_{E\gamma}(x) - u_{D\beta}(x)s_{E\gamma}(x)) \\
 &\quad + 2s_{C\alpha}(x)(u_{D\beta}(x)d_{E\gamma}(x) - d_{D\beta}(x)u_{E\gamma}(x))]
 \end{aligned}$$

relative distance =  $\vec{r}$

$$\mathcal{O}_\alpha(\vec{r}; t) = V^{-1/2} \sum_{\vec{x}} \sum_{\vec{y}} \delta_{\vec{x}-\vec{y}, \vec{r}} K^+(\vec{x}t) \Lambda_\alpha^0(\vec{y}t)$$

spin index  $\alpha$  belongs to baryon, so def

$$\overline{\mathcal{O}}_\mu(\vec{r}; t) = \mathcal{O}_\alpha^\dagger(\vec{r}; t) \gamma_{4,\alpha\mu}$$

## Correlator functions

$$C = \langle \mathcal{O}_\mu(\vec{r}; t) \bar{\mathcal{O}}_\nu(\vec{s}; t_0) \rangle - \langle \mathcal{O}_\mu(\vec{r}; t) \rangle \langle \bar{\mathcal{O}}_\nu(\vec{s}; t_0) \rangle$$

separable term vanishes

$$\langle \mathcal{O}_\mu \rangle \sim \langle K^+ \Lambda^0 \rangle \sim \langle \bar{s} u \bar{u} d s \rangle = 0$$

quark propagators (contractions)

$$\frac{s_{A\alpha}(\vec{x}t)\bar{s}_{B\beta}(\vec{y}t_0)}{d_{A\alpha}(\vec{x}t)\bar{d}_{B\beta}(\vec{y}t_0)} = H_{A\alpha, B\beta}(\vec{x}t, \vec{y}t_0) \quad \text{heavy } (\textcolor{blue}{s})$$

Wick's theorem  $\implies C(H, G)$

Which sources are needed for propagator generation?

look at site-sum structure

$$C = \left\langle \frac{1}{V^{1/2}} \sum_{\vec{x}} \sum_{\vec{y}} \delta_{\vec{x} - \vec{y}, \vec{r}} \frac{1}{V^{1/2}} \sum_{\vec{u}} \sum_{\vec{v}} \delta_{\vec{u} - \vec{v}, \vec{s}} \dots \right\rangle$$

$$= \left\langle \sum_{\vec{y}} \sum_{\vec{v}} \frac{1}{V} \right.$$

$$[H(\vec{y}, \vec{y} + \vec{r}) H(\vec{v} + \vec{s}, \vec{v}) - H(\vec{y}, \vec{v}) H(\vec{v} + \vec{s}, \vec{y} + \vec{r})]$$

$$G(\vec{y}, \vec{v}) [G(\vec{y}, \vec{v}) G(\vec{y} + \vec{r}, \vec{v} + \vec{s}) - G(\vec{y}, \vec{v} + \vec{s}) G(\vec{y} + \vec{r}, \vec{v})] \rangle$$

- option for computation of  $G$

random-source estimator  $\Rightarrow$  all-to-all  $G$

- rather, use translational invariance of  $\langle \dots \rangle$

add  $\vec{r}_1 - \vec{v}$  subst  $\vec{y} \rightarrow \vec{y} - \vec{r}_1 + \vec{v}$  then

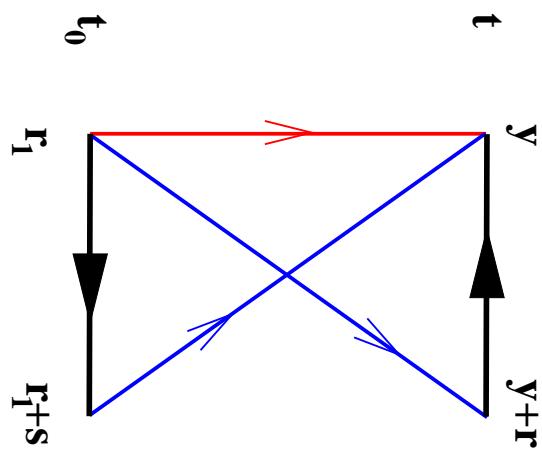
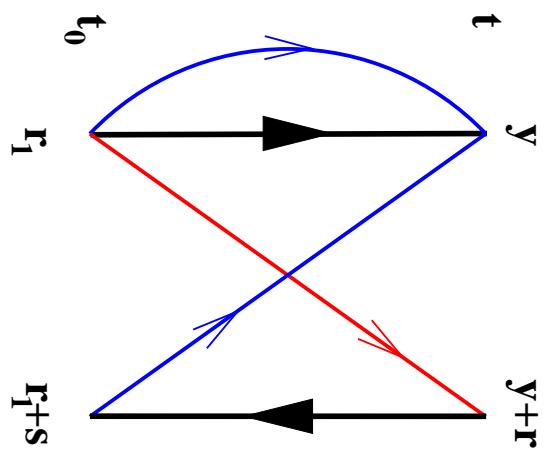
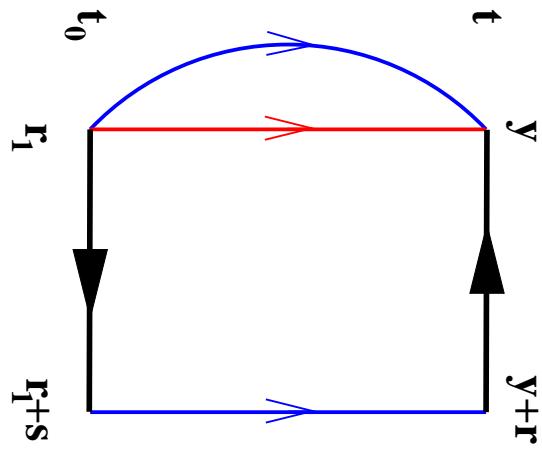
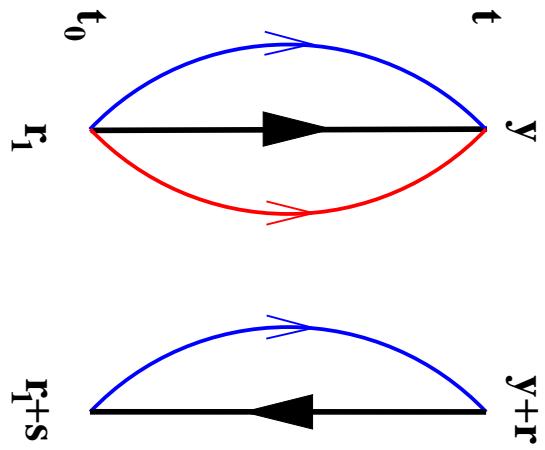
e.g.  $H(\vec{y}, \vec{v}) \rightarrow H(\vec{y} + \vec{r}_1 - \vec{v}, \vec{r}_1) \rightarrow H(\vec{y}, \vec{r}_1)$  etc

$$C = \langle \sum_{\vec{y}} \sum_{\vec{v}} \frac{1}{V}$$

$$\begin{aligned} & [H(\vec{y}, \vec{y} + \vec{r}) H(\vec{r}_1 + \vec{s}, \vec{r}_1) - H(\vec{y}, \vec{r}_1) H(\vec{r}_1 + \vec{s}, \vec{y} + \vec{r})] \\ & G(\vec{y}, \vec{r}_1) [+G(\vec{y}, \vec{r}_1) G(\vec{y} + \vec{r}_1, \vec{r}_1 + \vec{s}) \\ & - G(\vec{y}, \vec{r}_1 + \vec{s}) G(\vec{y} + \vec{r}_1, \vec{r}_1)] \rangle \end{aligned}$$

indep of  $\vec{v} \rightarrow \sum_{\vec{v}} = V$  spatial volume

- fixed set of spatial source locations  $\vec{r}_1 + \vec{s}$  for  $G$
- all-to-all spatial source locations  $\vec{y} + \vec{r}$  for  $H$



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- use  $H$  in (static) limit  $\kappa \rightarrow 0$  (hopping parameter expansion)

$$H(\vec{x}t, \vec{y}t_0) = \delta_{\vec{x}, \vec{y}} (2\kappa)^{t-t_0} \frac{1}{2}(1 + \gamma_4) U^\dagger(\vec{x}; t_0 t)$$

$$U^\dagger(\vec{x}; t_0 t) = U_4(\vec{x}t_0) U_4(\vec{x}t_0 + 1) \dots U_4(\vec{x}t - 1)$$

→ equal-time diagrams

$$H(\vec{y}t, \vec{y} + \vec{r}t) H(\vec{r}_1 + \vec{s}t_0, \vec{r}_1 t_0) \propto \delta_{\vec{r}, \vec{0}} \delta_{\vec{s}, \vec{0}}$$

→ mixed-time diagrams

$$H(\vec{y}t, \vec{r}_1 t_0) H(\vec{r}_1 + \vec{s}t_0, \vec{y} + \vec{r}t) \propto \delta_{\vec{y}, \vec{r}_1} \delta_{\vec{s}, \vec{r}}$$

$\Rightarrow$  correlator, site-sum structure only

$$C \simeq \delta_{\vec{r}, \vec{0}} \delta_{\vec{s}, \vec{0}} \langle \sum_{\vec{y}} H(\vec{y}, \vec{y}) H(\vec{r}_1, \vec{r}_1) G(\vec{y}, \vec{r}_1) [G()G() - G()G()] \rangle$$

$$- \delta_{\vec{r}, \vec{s}} \langle H(\vec{r}_1, \vec{r}_1) H(\vec{r}_1 + \vec{s}, \vec{r}_1 + \vec{r})$$

$$G(\vec{r}_1, \vec{r}_1) [G(\vec{r}_1, \vec{r}_1) G(\vec{r}_1 + \vec{r}, \vec{r}_1 + \vec{s}) - G(\vec{r}_1, \vec{r}_1 + \vec{s}) G(\vec{r}_1 + \vec{r}, \vec{r}_1)] \rangle$$

- distance  $\vec{r} = \vec{0}$  is special, color  $\mathcal{O} \neq K\Lambda$  possible, ignore
- write  $\vec{r}_2 = \vec{r}_1 + \vec{r}$  then, for  $\vec{r} \neq \vec{0}$

$$C \simeq \delta_{\vec{r}, \vec{r}_2 - \vec{r}_1} \langle H(\vec{r}_1, \vec{r}_1) H(\vec{r}_2, \vec{r}_2)$$

$$G(\vec{r}_1, \vec{r}_1) [-G(\vec{r}_1, \vec{r}_1) G(\vec{r}_2, \vec{r}_2) + G(\vec{r}_1, \vec{r}_2) G(\vec{r}_2, \vec{r}_1)] \rangle$$

- may choose a fixed set of source sites
- note loss of  $\sum_{\vec{y}}$  space-site sum  $\rightarrow$  bad for statistics

## Fuzzing and smearing

correlation functions built from operators

$$\mathcal{O} = \mathcal{O}[U, \psi, \bar{\psi}]$$

construct spatially “spread out” fields

$$\begin{aligned} U &\longrightarrow U^{\{k\}} \\ \psi &\longrightarrow \psi^{\{k\}} \\ \bar{\psi} &\longrightarrow \bar{\psi}^{\{k\}} \end{aligned}$$

enhance (ground-state signal), replace

$$\mathcal{O} \longrightarrow \mathcal{O}^{\{k\}} = \mathcal{O}[U^{\{k\}}, \psi^{\{k\}}, \bar{\psi}^{\{k\}}]$$

correlator

$$C = \sum_{\alpha} \langle \mathcal{O}_{\alpha}^{\{k\}}(\vec{r}; t) \overline{\mathcal{O}}_{\alpha}^{\{\ell\}}(\vec{r}; t_0) \rangle - \langle \dots \rangle \langle \dots \rangle, \quad k, \ell = 1 \dots K$$

→ is  $K \times K$  matrix

## Gauge field fuzzing (APE)

$$U_i^{\{0\}}(x) = U(x)$$

$$U_i^{\{k\}}(x) = [U_i^{\{k-1\}}(x) + \sum_j \alpha_{ij} (UU^\dagger U^\dagger + U^\dagger U^\dagger U)] Z_i(x)$$

$$x \rightarrow x+i = \longrightarrow + \sum \alpha \left( \begin{array}{c} \nearrow \\ \searrow \\ + \\ \nearrow \\ \searrow \end{array} \right)$$

spatial directions only  $i, j = 1, 2, 3$     $\alpha_{4j} = 0$     $\alpha_{ii} = 0$   
 not unitary, after each iteration step choose  $Z_i(x)$  such

$$\text{Tr } U_i^{\{k\}\dagger}(x) U_i^{\{k\}}(x) = 3 \quad (\text{gauge invariant})$$

→ numerically well behaved, during iteration  $k = 0, 1 \dots K$

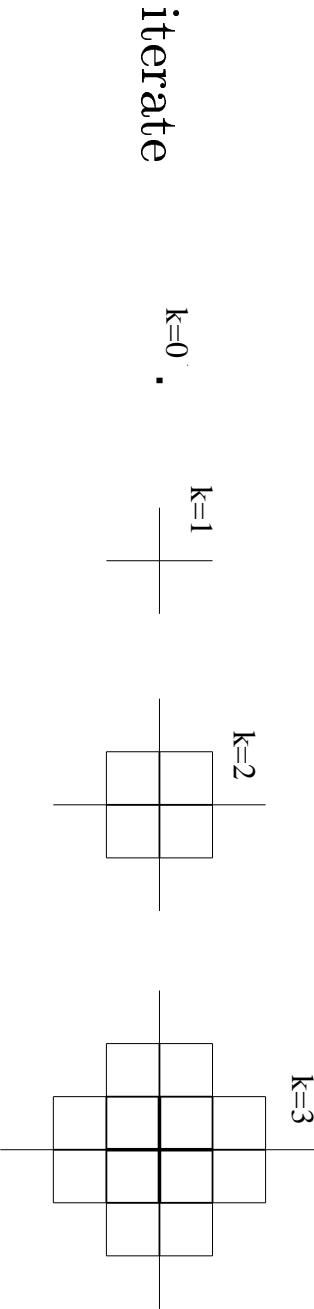
## Quark field smearing (WUP)

$$\psi_{A\mu}^{\{0\}}(\vec{x}t) = \psi_{A\mu}(x)$$

$$\psi_{A\mu}^{\{k\}}(\vec{x}t) = \sum_{\vec{y}} \sum_B K_{AB}(t; \vec{x}, \vec{y}) \psi_{B\mu}^{\{k-1\}}(\vec{y}t) Z$$

$$K_{AB}(t; \vec{x}, \vec{y}) = \delta_{AB} \delta_{\vec{x}, \vec{y}} + \sum_i \alpha_i [U_{i,AB}(\vec{x}t) \delta_{\vec{x}, \vec{y} - \hat{i}} + U_{i,AB}^\dagger(\vec{y}t) \delta_{\vec{x}, \vec{y} + \hat{i}}]$$

spatial directions only  $i = 1, 2, 3$



normalize, after each iteration step choose  $Z$  such that

$$\sum_A \sum_\mu \{\psi_{A\mu}^{\{k\}}(\vec{x}t), \bar{\psi}_{A\mu}^{\{k\}}(\vec{x}t)\} = 3 \cdot 4 \quad \text{gauge inv}$$

$$\rightarrow Z^{-2} = 1 + 2 \sum_i \alpha_i^2 \quad \text{numerically well behaved}$$

## Quark propagators

smearing of  $G$  and  $H$ ?  
iterated smearing kernel

$$\begin{aligned}\mathcal{K}^{\{0\}}(t; \vec{x}, \vec{y}) &= \mathbf{1} \delta_{\vec{x}, \vec{y}} \\ \mathcal{K}^{\{k\}}(t; \vec{x}, \vec{y}) &= \sum_{\vec{z}} K(t; \vec{x}, \vec{z}) \mathcal{K}^{\{k-1\}}(t; \vec{z}, \vec{y})\end{aligned}$$

$$\mathcal{K}^{\{k\}}(t; \vec{x}, \vec{y}) = 0 \quad \text{if} \quad |\vec{x} - \vec{y}| > k$$

- light-quark propagator ( $\psi = u, d$ )

$$\begin{aligned}G^{\{k\ell\}}(\vec{x}t, \vec{y}t_0) &= \overline{\psi^{\{k\}}(\vec{x}t)} \bar{\psi}^{\{\ell\}}(\vec{y}t_0) = \\ &= \sum_{\vec{z}, \vec{z}'} \mathcal{K}^{\{k\}}(t; \vec{x}, \vec{z}) \overline{\psi(\vec{z}t)} \bar{\psi}(\vec{z}'t_0) \mathcal{K}^{\{\ell\}\dagger}(t_0; \vec{y}, \vec{z}')\end{aligned}$$

in practice:  $\rightarrow$  smear source $^{\{\ell\}}$   $\rightarrow$  call (cG)inverter  $G^{\{\ell\}}$   $\rightarrow$   
 $\rightarrow$  smear sink $^{\{k\}}$  at operator level

- heavy-quark propagator ( $\psi = s$  static)

$$\text{recall } \overline{\psi(\vec{z}t)} \bar{\psi}(\vec{z}'t_0) \propto \delta_{\vec{z}, \vec{z}'}$$

thus

$$H^{\{k, \ell\}}(\vec{x}t, \vec{y}t_0) = 0 \quad \text{if} \quad |\vec{x} - \vec{y}| > k + \ell$$

loss of localization, ill-defined relative distance  $\vec{r}$ , unless  $k + \ell = 0$

$\Rightarrow$  choose smearing levels  $k = \ell = 0$  for  $H$   
 $\Rightarrow$  gauge links  $U_\mu(x)$  used in  $H$  *not* fuzzed

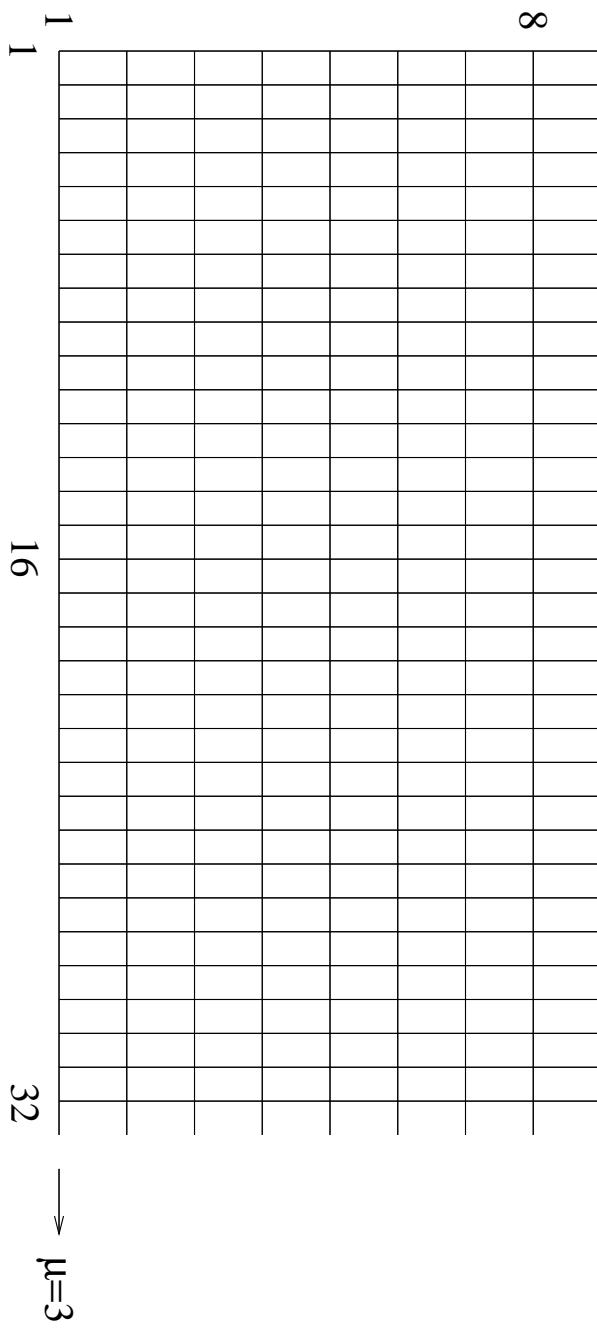
## Lattice simulation

geometry: anisotropic and asymmetric

$$L_1 \times L_2 \times L_3 \times L_4 = 8 \times 8 \times 32 \times 16$$

$$a_1 = a_2 = 2a_3 = 2a_4 \quad (\text{bare})$$

$\mu=1,2$



- gauge field action, anisotropic Wilson

$$S_G = \sum_x \sum_{\mu < \nu} \beta_{\mu\nu} \left( 1 - \frac{1}{3} \text{ReTr} P_{\mu\nu} \right)$$

$$\beta_{\mu\nu} = \beta \frac{a_1 a_2 a_3 a_4}{(a_\mu a_\nu)^2} \quad \text{used} \quad \beta = 6.2$$

- fermion action, anisotropic Wilson (light quarks)

$$S_F = \sum_{x,y} \bar{\psi}(x) Q(x,y) \psi(y)$$

$$Q(x,y) = \mathbf{1} \delta_{x,y} - \sum_\mu \kappa_\mu [(\mathbf{1} - \gamma_\mu) U_\mu(x) \delta_{x+\mu,y} + (\mathbf{1} + \gamma_\mu) U_\mu^\dagger(y) \delta_{x,y+\mu}]$$

hopping parameters

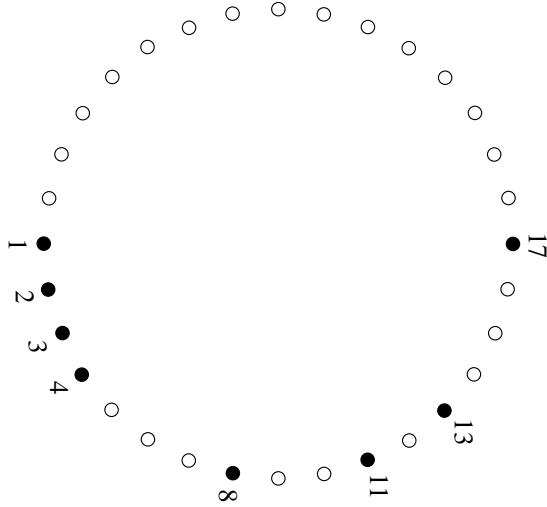
$$\kappa_\mu = \frac{\kappa}{a_\mu \frac{1}{4} \sum_{\nu=1}^4 \frac{1}{a_\nu}} \quad \text{used} \quad \kappa = 0.140, 0.136, 0.132, 0.128$$

multiple mass solver [Frommer et al, 1995]

## Placement of sources

spatial sites  $\vec{r}_1$  and  $\vec{r}_2$  selected from  $x = (5, 5, n, 3)$  with

$$n = 1, 2, 3, 4, 8, 11, 13, 17$$



available relative distances  $r$ , multiplicity  $m$  (averaged over)

$r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$m$	3	3	2	2	2	2	2	1	3	2	1	1	1	1	1	1

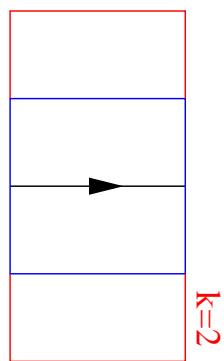
## Asymmetric fuzzing (gauge field)

fuzzing dir	$i =$	1	2	3
iterations	$k =$	1	2	3
	2	2	4	
	3	3	6	

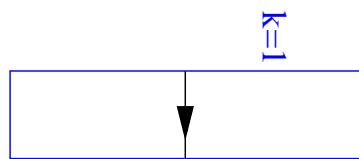
Fuzzing direction  $i$

$i=2$

$\uparrow$



$\Rightarrow i=3$



$k=1$

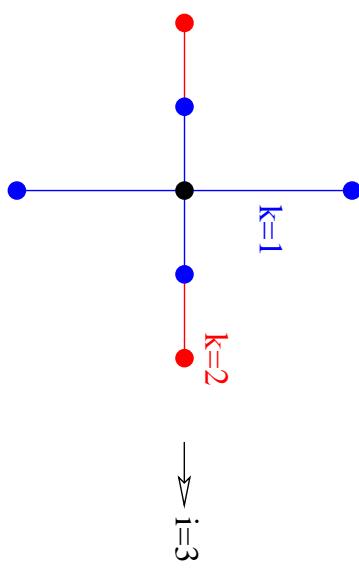
- $K = 3$  sets of fuzzing levels
- $\alpha = 2.5$  for all directions  $i$

## Asymmetric smearing (fermion field)

smearing dir     $i = \begin{matrix} 1 & 2 & 3 \end{matrix}$

iterations     $k = \begin{matrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{matrix}$

Smeating direction     $i=2$   
 $\uparrow$



- $K = 3$  sets of smearing levels
- $\alpha = 2.5$  for all directions  $i$

## Analysis issues

- used: matching sets of fuzzing–smearing levels size  $K \times K = 3 \times 3$  correlation matrix  $C(t, t_0)$  at each  $\vec{r}$
- noise: long-path operators (distances  $r$ )
  - static approx  $\rightarrow$  lack of space-site  $\sum_{\vec{x}}$
  - notation: omit source time slice  $t_0 \dots$
- generalized eigenvalue problem [Lüscher, Wolff, 1990]

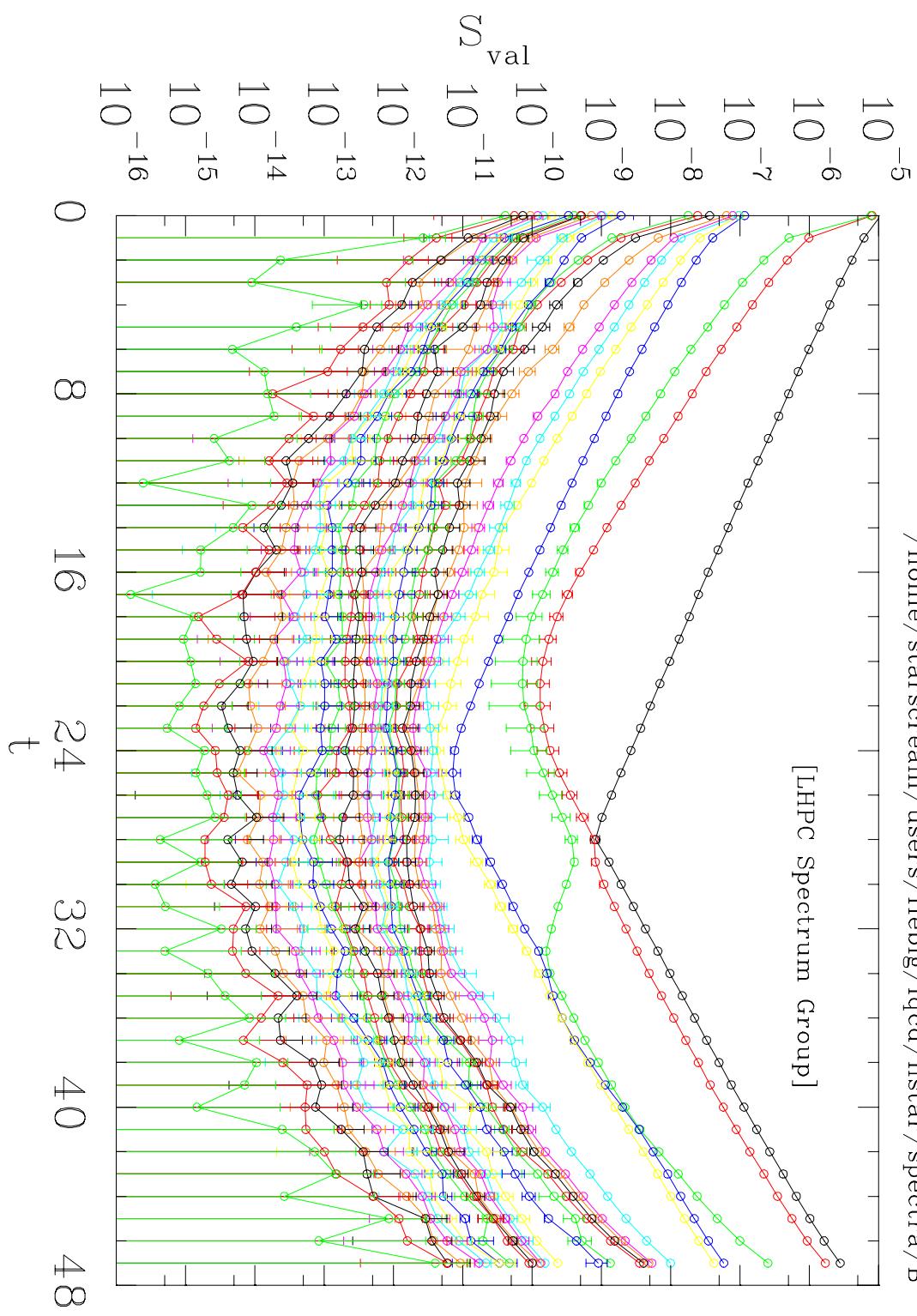
$$C(t)v = C(t_1)v \lambda(t, t_1) \quad t_1 \gtrsim t_0$$

$C(t_1)$  hermitean, non-singular, pos-definite

$$\lambda(t, t_1) \rightarrow e^{-W(t-t_1)} [1 + \mathcal{O}(e^{-\Delta W(t-t_1)})]$$

- if size of  $C(t)$  is large (e.g.  $27 \times 27$ ,  $N^*$  project)  
many very small eigenvalues  $\rightarrow$  numerical problems

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## Singular value decomposition (SVD)

- superior numerical algorithm if small eigenvalues are present

let  $C_{ij}(t)$  with  $i = 1 \dots m, j = 1 \dots n$

$m \times n$  correlation matrix

for simplicity, assume  $m = n$

the SVD is

$$C(t) = U(t)\Sigma(t)V^\dagger(t)$$

with

$$U^\dagger(t)U(t) = \mathbf{1}$$

$$V^\dagger(t)V(t) = \mathbf{1}$$

$$\Sigma(t) = \text{diag}(\sigma_1(t), \dots, \sigma_n(t))$$

$$\sigma_1(t) \geq \dots \geq \sigma_n(t) \geq 0$$

## Relation to diagonalization

let  $C$  be

1. hermitean
2. non-degenerate
3. pos-definite

consider its SVD  $C = U\Sigma V^\dagger$

$$\text{square } S = C^\dagger C = C C^\dagger = V \Sigma^2 V^\dagger = U \Sigma^2 U^\dagger \quad \curvearrowright$$

$$SV = V\Sigma^2 \text{ and } SU = U\Sigma^2$$

columns of  $V$  and  $U$  are both eigenvectors of  $S$ , to same  $\sigma_k^2$

$$\begin{bmatrix} S_{11} & \dots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & \dots & S_{nn} \end{bmatrix} \begin{bmatrix} V_{1k} \\ \vdots \\ V_{nk} \end{bmatrix} = \begin{bmatrix} V_{1k} \\ \vdots \\ V_{nk} \end{bmatrix} \sigma_k^2, \quad "V \rightarrow U"$$

all eigenspaces 1-dim  $\Rightarrow$

$$\begin{bmatrix} U_{1k} \\ \vdots \\ U_{nk} \end{bmatrix} = \begin{bmatrix} V_{1k} \\ \vdots \\ V_{nk} \end{bmatrix} \eta_k \quad \text{with} \quad |\eta_k| = 1$$

or  $U = VE$ ,  $E = \text{diag}(\eta_1, \dots, \eta_n)$

use this in

$$C = U\Sigma V^\dagger = VE\Sigma V^\dagger = V\Lambda V^\dagger$$

$$E\Sigma = \text{diag}(\eta_1\sigma_1 \dots \eta_n\sigma_n) = \text{diag}(\lambda_1 \dots \lambda_n) = \Lambda$$

$$\eta_k\sigma_k = \lambda_k > 0 \Rightarrow \eta_k = +1 \Rightarrow \sigma_k = \lambda_k$$

• Above assumptions: SVD  $\Rightarrow$  diagonalization

$\Leftarrow$  (trivial)

- o if  $C^\dagger = C$  only: usual eigvec ambiguities, eigval same
- o available: generalized SVD

Def: SVD-projected operators, correlation matrix

$$\mathcal{U}_{k\alpha}(\vec{r}; t) = \sum_{i=1}^K U_{ki}^\dagger \mathcal{O}_\alpha^{\{i\}}(\vec{r}; t), \quad \mathcal{V}_{k\alpha}(\vec{r}; t) = \dots V^\dagger \dots$$

$$\Sigma_{k\ell}(\vec{r}; t; t_0) = \langle \mathcal{U}_{k\alpha}(\vec{r}; t) \bar{\mathcal{V}}_{\ell\alpha}(\vec{r}; t_0) \rangle = U_{ki}^\dagger C_{ij}(\vec{r}; t; t_0) V_{j\ell}$$

- SVD-diag done at fixed  $t = t_1 \Rightarrow U, V = \text{const}$

$$\Sigma_{k\ell}(\vec{r}; t_1; t_0) \propto \delta_{k\ell} \quad \text{at } t_1$$

$$\Sigma_{k\ell}(\vec{r}; t; t_0) = \sum_{n \neq 0} \langle 0 | \mathcal{U}_{k\alpha}(\vec{r}; t_0) | n \rangle \langle n | \bar{\mathcal{V}}_{\ell\alpha}(\vec{r}; t_0) | 0 \rangle e^{-W_n(t-t_0)}$$

expect  $\Sigma(\vec{r}; t; t_0)$  diag domination

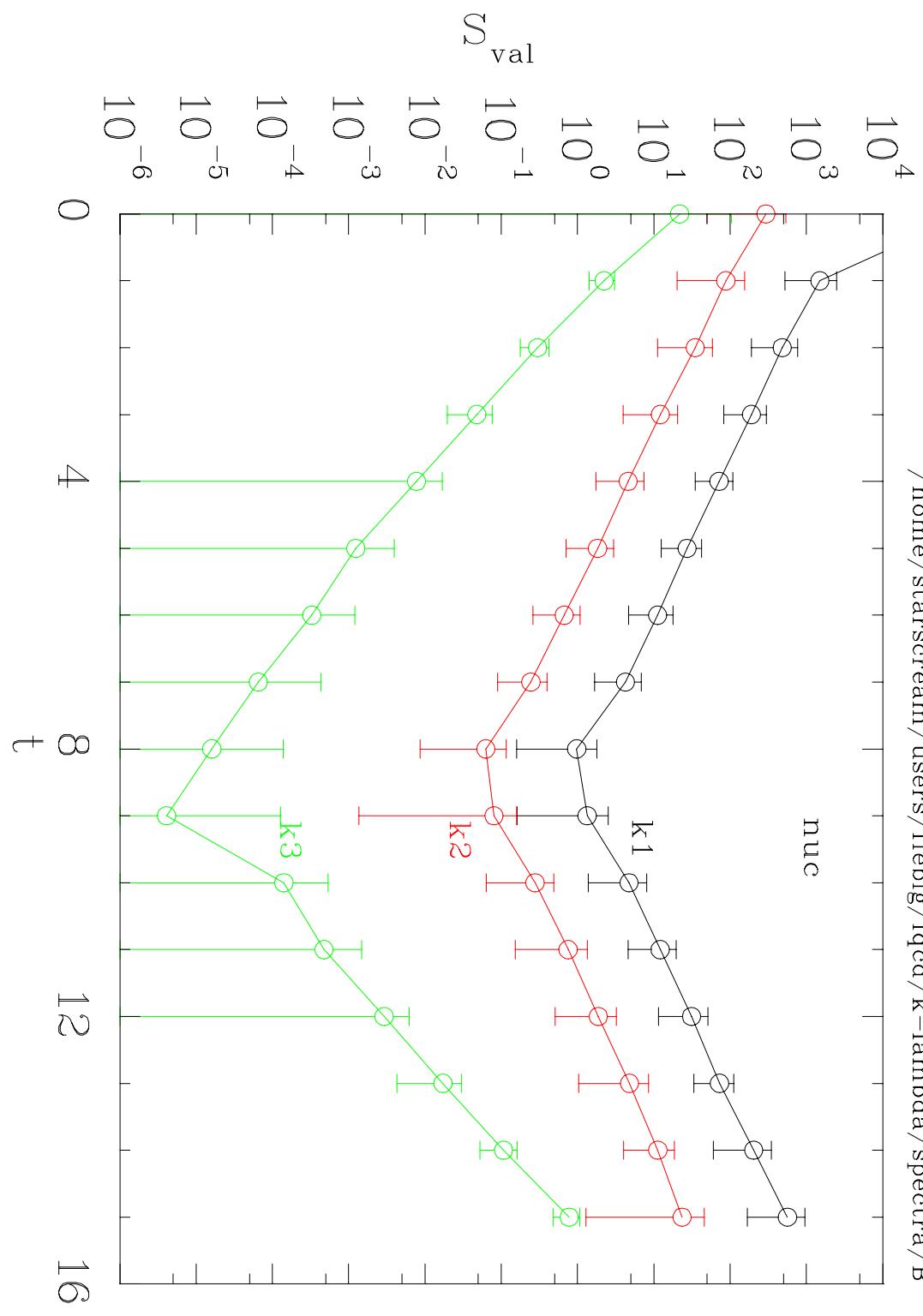
- SVD-diag done at all  $t$

$$\Sigma_{k\ell}(\vec{r}; t; t_0) \propto \delta_{k\ell} \sigma_k(t) \quad \text{all } t$$

$$\text{expect } \sigma_k(t) \longrightarrow s_k e^{-W_k(t-t_0)} [1 + \mathcal{O}(e^{-\Delta W_k(t-t_0)})]$$

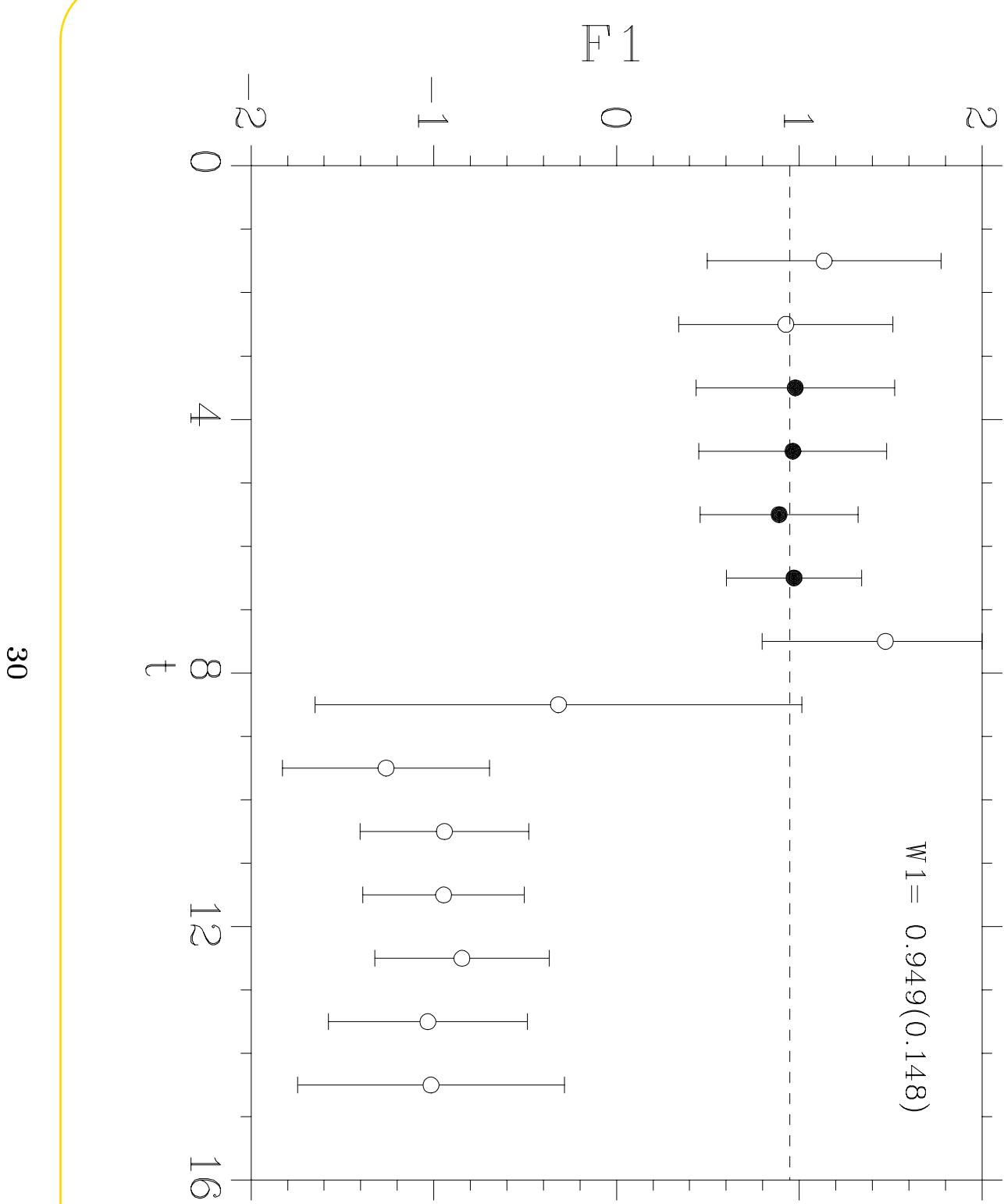
$$\Delta W_k = \min\{|W_k - W_\ell| : \ell \neq k\}$$

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W1 = 0.949(0.148)



## Spectrum analysis

- Assume correlators have ‘diagonal’ form<sup>a</sup>

$$D_k(t, t_0) = \langle \hat{\Psi}_k^\dagger(t) \hat{\Psi}_k(t_0) \rangle = \sum_{n \neq 0} |\langle n | \hat{\Psi}_k(t_0) | 0 \rangle|^2 e^{-W_n(t-t_0)}$$

- Spectral model

$$F(\rho|t, t_0) = \int_{-\infty}^{+\infty} d\omega \rho(\omega) e^{-\omega(t-t_0)}$$

if exact  $F(\rho|t, t_0) = D_k(t, t_0)$  then

$$\rho(\omega) = \sum_{n \neq 0} \delta(\omega - W_n) |\langle n | \hat{\Psi}(t_0) | 0 \rangle|^2$$

---

a.e.g.  $D \sim \lambda; \sigma$  and  $\Psi \sim \mathcal{O}; \sim \mathcal{U}, \mathcal{V}$  with  $\hat{\Psi} = \Psi - \langle \Psi \rangle$  possibly  $t \rightarrow \infty$

- Discretization  $\omega_i = \Delta\omega i$ ,  $i = I_1 \dots I_2$

$$P(\rho|t, t_0) \simeq \sum_{i=I_1}^{I_2} \rho_i e^{-\omega_i(t-t_0)} \quad \rho_i = \text{parameters}$$

# of parameters ( $\sim 40$ )  $\gg$  # of lattice data ( $\sim 10$ )

- Bayesian inference: Interpret  $\rho_i$  as stochastic variables!

given set of data  $D$ , fixed

→ find conditional pdf of parameters  $\rho$

$$\mathcal{P}(\rho \leftarrow D) \propto \mathcal{P}(D \leftarrow \rho) \times \mathcal{P}(\rho)$$

posterior  $\propto$  likelihood  $\times$  prior

→ maximize posterior probability  $\Rightarrow$  best “fit”

→ more parameters than data? ... no problem!



## Likelihood

lattice data  
↓  
spectral model

$$\chi^2(D, \rho) = \sum_{t_1, t_2} [D(t_1, t_0) - F(\rho|t_1, t_0)] \Gamma^{-1}(t_1, t_2) [.. t_1 \rightarrow t_2 ..]$$

↑ covar matrix

given  $[\rho]$  make (large number of) measurements of  $[D]$ ,  
then (central limit theorem)  $\Rightarrow$  Gaussian pdf

$$\mathcal{P}(D \leftarrow \rho) = e^{-\chi^2(D, \rho)/2}$$

## Prior

Shannon-Jaynes Entropy

$$S(\rho) = \sum_k \left( \rho_k - m_k - \rho_k \ln\left(\frac{\rho_k}{m_k}\right) \right) \leq 0 = S(m)$$

measures (lack of) information relative to default model = [m]

$$\mathcal{P}(\rho) = e^{\alpha S} \frac{\text{entropy weight}}{\alpha} = \alpha$$

## Posterior

$$\mathcal{P}(\rho \leftarrow C) \propto \mathcal{P}(C \leftarrow \rho) \mathcal{P}[\rho] \propto e^{-W[\rho]} \quad \text{with} \quad W[\rho] = \chi^2/2 - \alpha S$$

- Maximum Entropy Method (MEM)

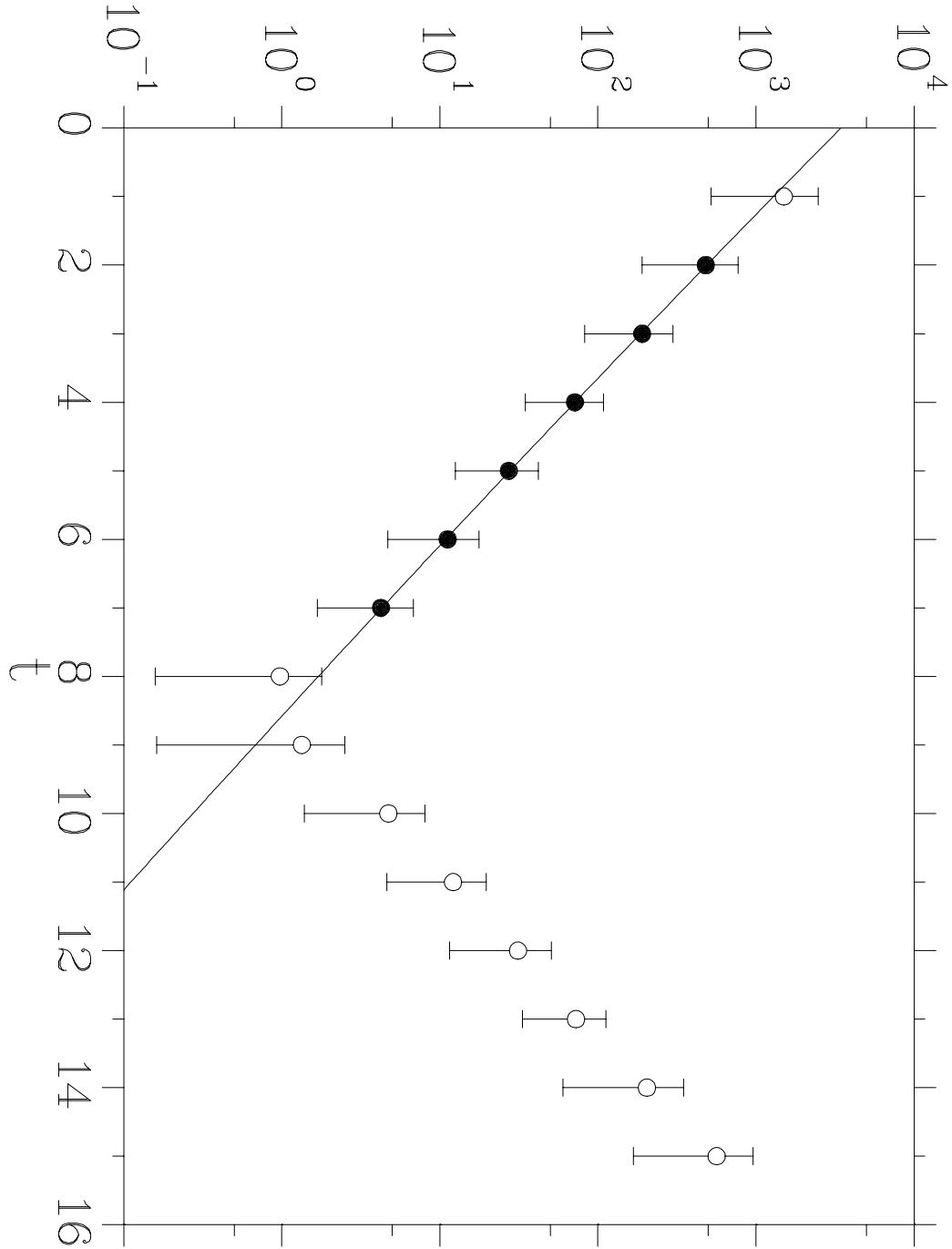
given data  $[D] \Rightarrow$  solve  $\mathcal{P}(\rho \leftarrow D) = \max \equiv W[\rho] = \min$

- Simulated Annealing (cooling)

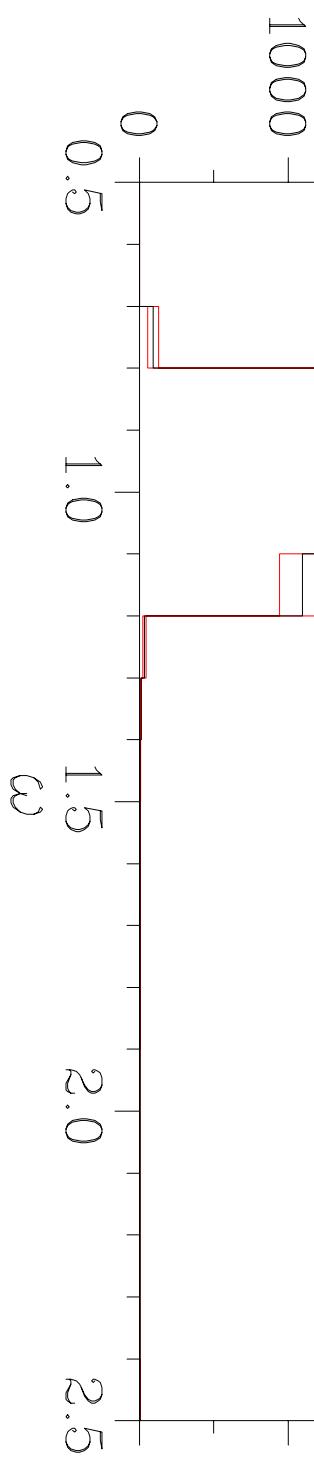
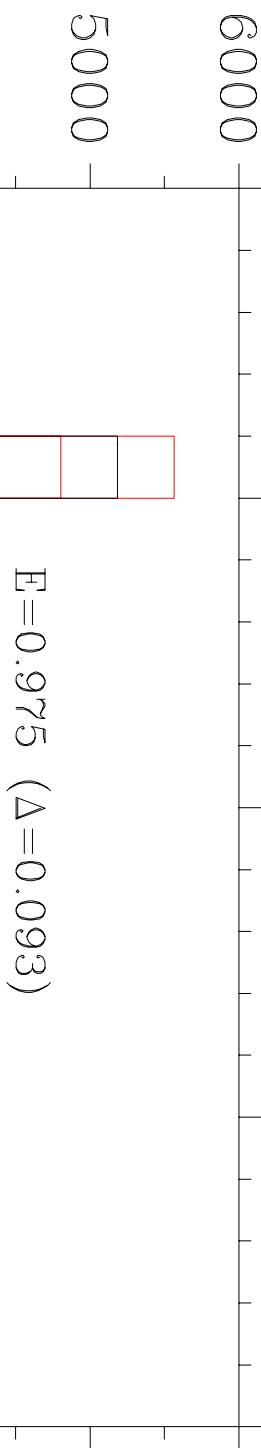
$$Z = \int [d\rho] e^{-\beta W[\rho]}$$

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$\mathcal{S}_{\text{val}}$



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## Spectral observables

loosely define

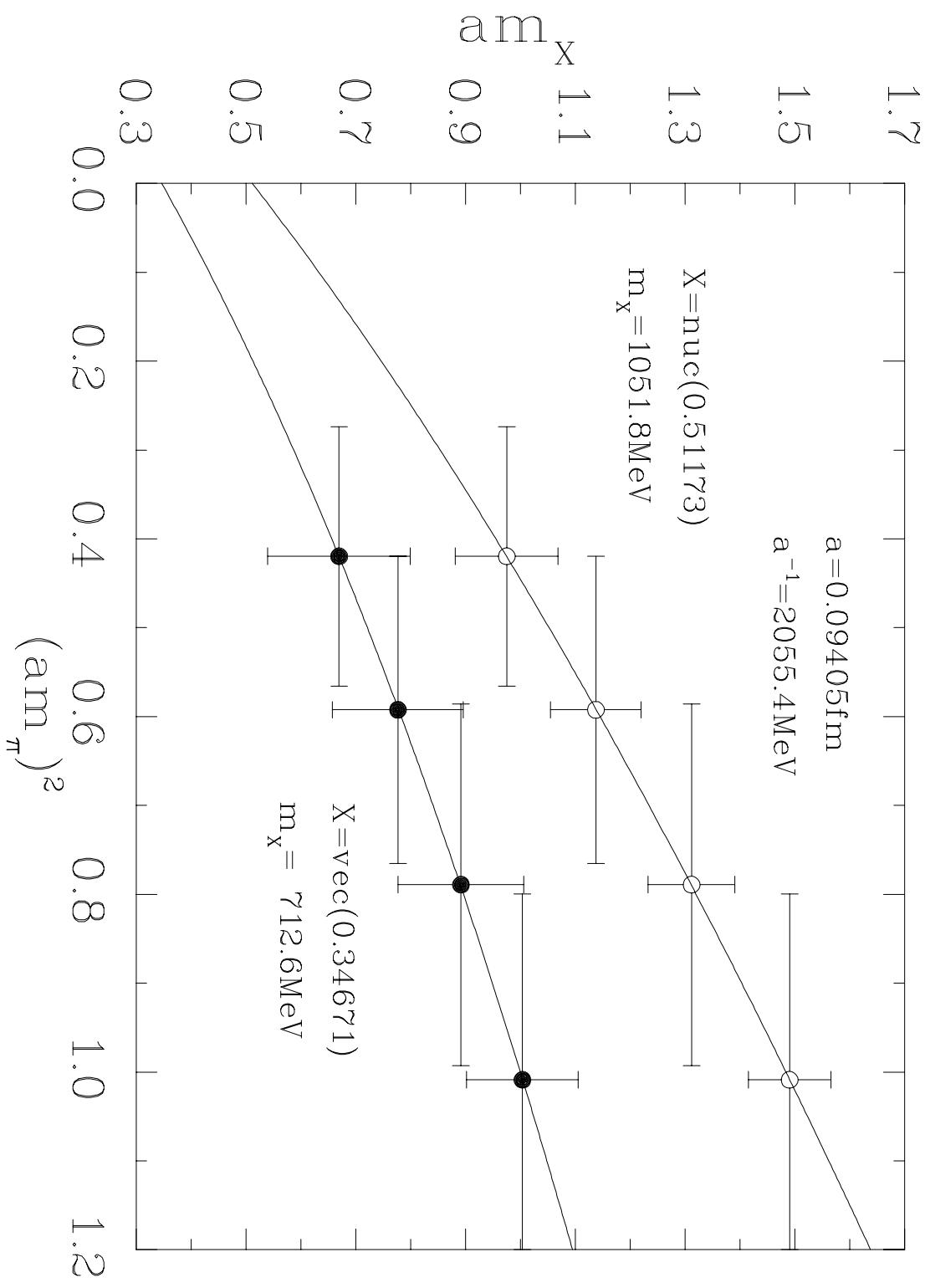
$$\delta_n = \{\omega : \omega \in \text{peak } \#n\} \quad n = 1, 2, \dots$$

then, for each peak  $n$

$$\begin{aligned} \text{volume} & \quad Z_n = \int_{\delta_n} d\omega \rho(\omega) \cong |\langle n | \hat{\Psi}(t_0) | 0 \rangle|^2 \\ \text{mass} & \quad E_n = Z_n^{-1} \int_{\delta_n} d\omega \rho(\omega) \omega \\ \text{width} & \quad \Delta_n^2 = Z_n^{-1} \int_{\delta_n} d\omega \rho(\omega) (\omega - E_n)^2 \end{aligned}$$

- moments  $\langle \omega^p \rangle_\rho$  low  $p = 0, 1, 2 \Leftarrow$  extractable INFORMATION
- smoothing  $\int \dots d\omega$  micro structure ...
- annealing average over random  $\rho$  starts
- tuning  $[m]$  and  $\alpha$ , insensitive over many orders of magnitude

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## Setting the scale

$\kappa$	$am_\pi$	$(a\Delta_\pi)$	$am_\rho$	$(a\Delta_\rho)$	$am_N$	$(a\Delta_N)$
0.128	1.004	(0.104)	1.004	(0.102)	1.491	(0.075)
0.132	0.888	(0.115)	0.892	(0.114)	1.312	(0.079)
0.136	0.770	(0.112)	0.777	(0.119)	1.138	(0.083)
0.140	0.648	(0.113)	0.670	(0.130)	0.975	(0.093)

- extrapolate to  $am_\pi = 0$  with

$$y = am_x \quad x = (am_\pi)^2 \quad y = A + Bx + C \ln(1 + x)$$

- match to reduced mass of  $\rho$ - $N$  system

$$m_{\rho N} = 425 \text{ MeV} \quad \rightarrow \quad a = 0.096 \text{ fm} \quad a^{-1} = 2056 \text{ MeV}$$

$$m_\rho = 713(776) \text{ MeV}$$

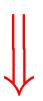
$$m_N = 1052(939) \text{ MeV}$$

$$\approx 8 - 12\%$$

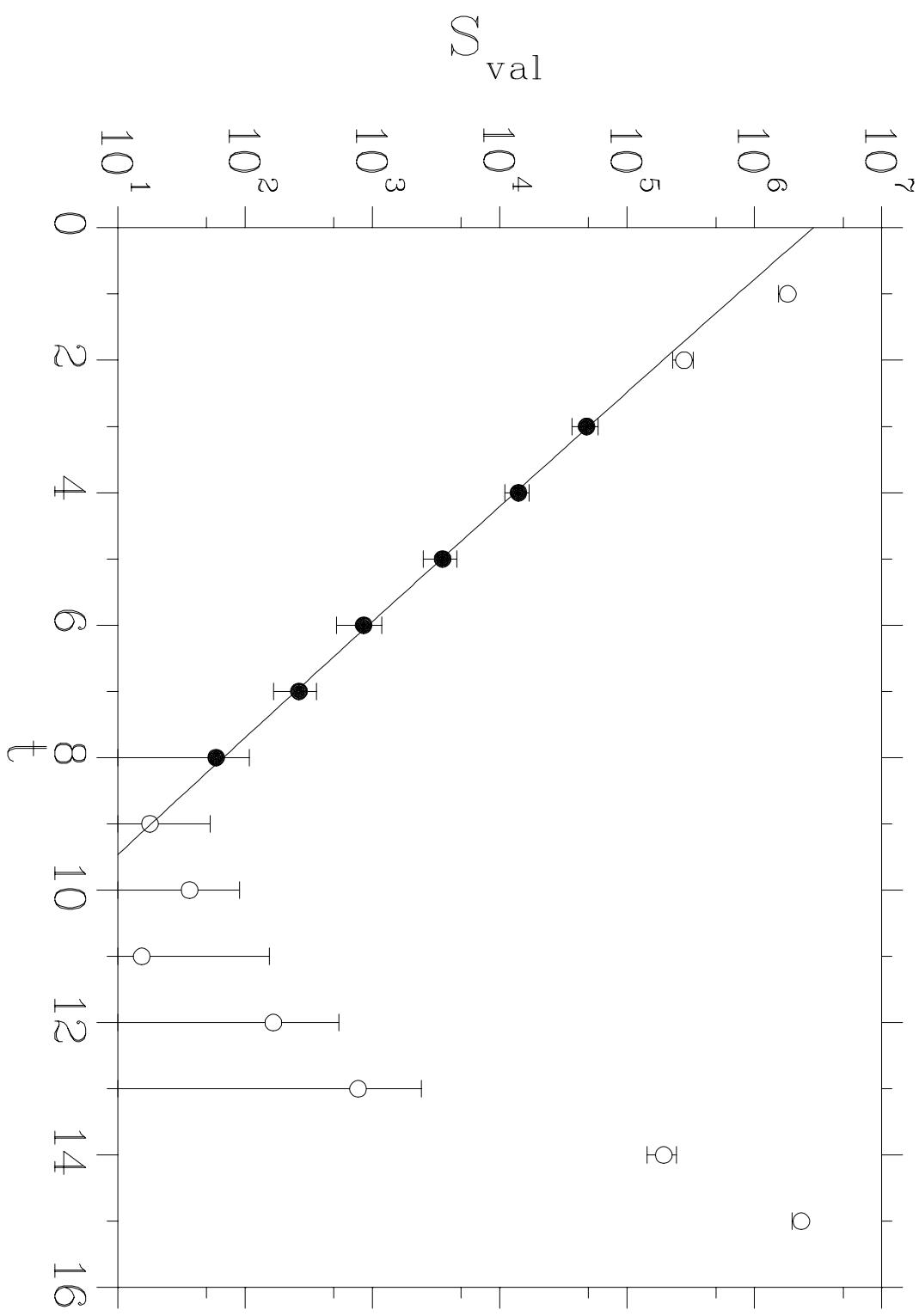
## Current status of K- $\Lambda$ analysis

- Only 90 configs used
- No  $am_\pi \rightarrow 0$  extrapolation done
- Available for  $\kappa = 0.140$  (highest  $am_\pi$ )
- SVD-diag on all time slices (employ  $t \rightarrow \infty$ )
- Plan is to use generalized SVD

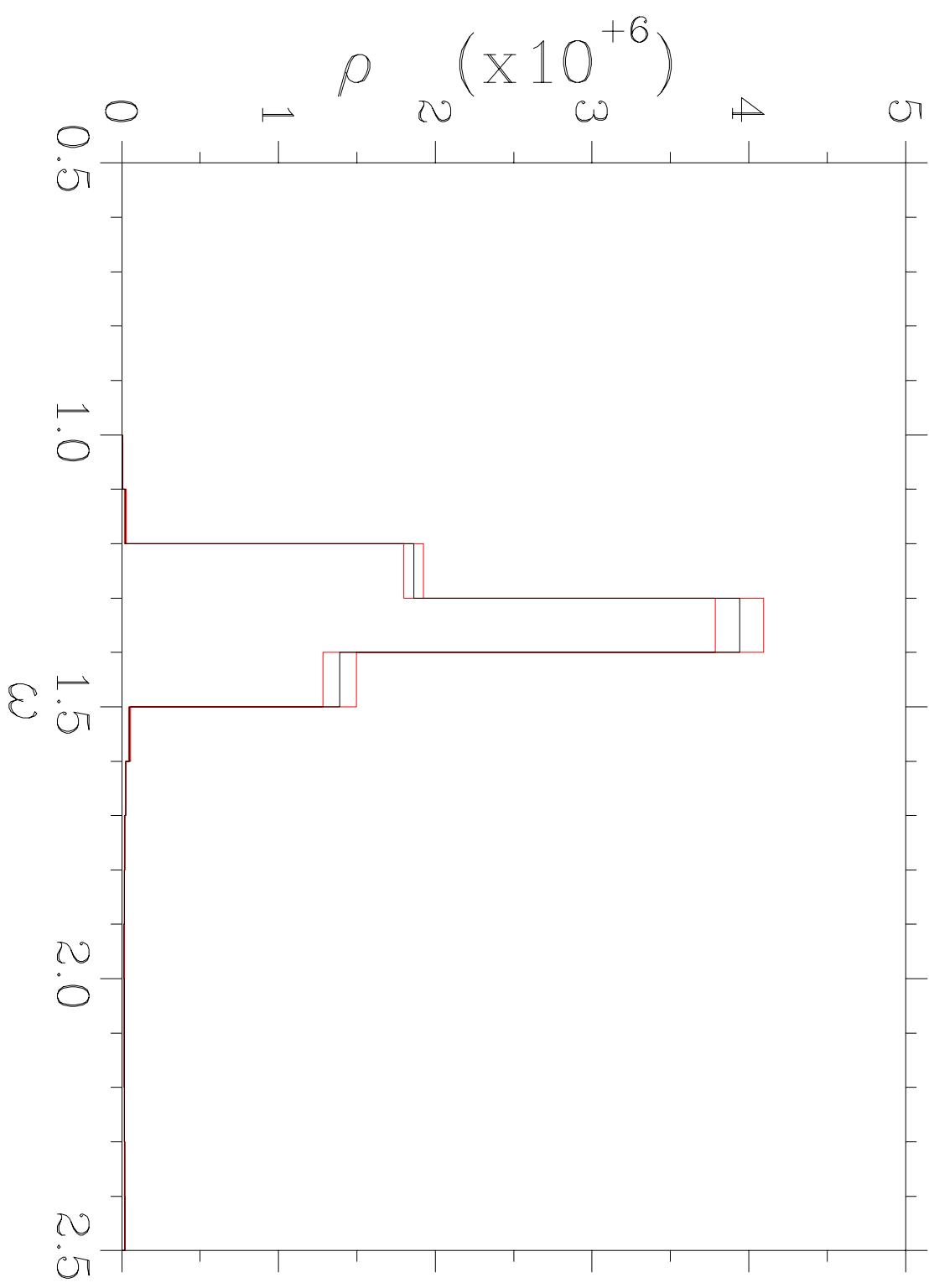
**EXPECT RESULTS TO CHANGE!**



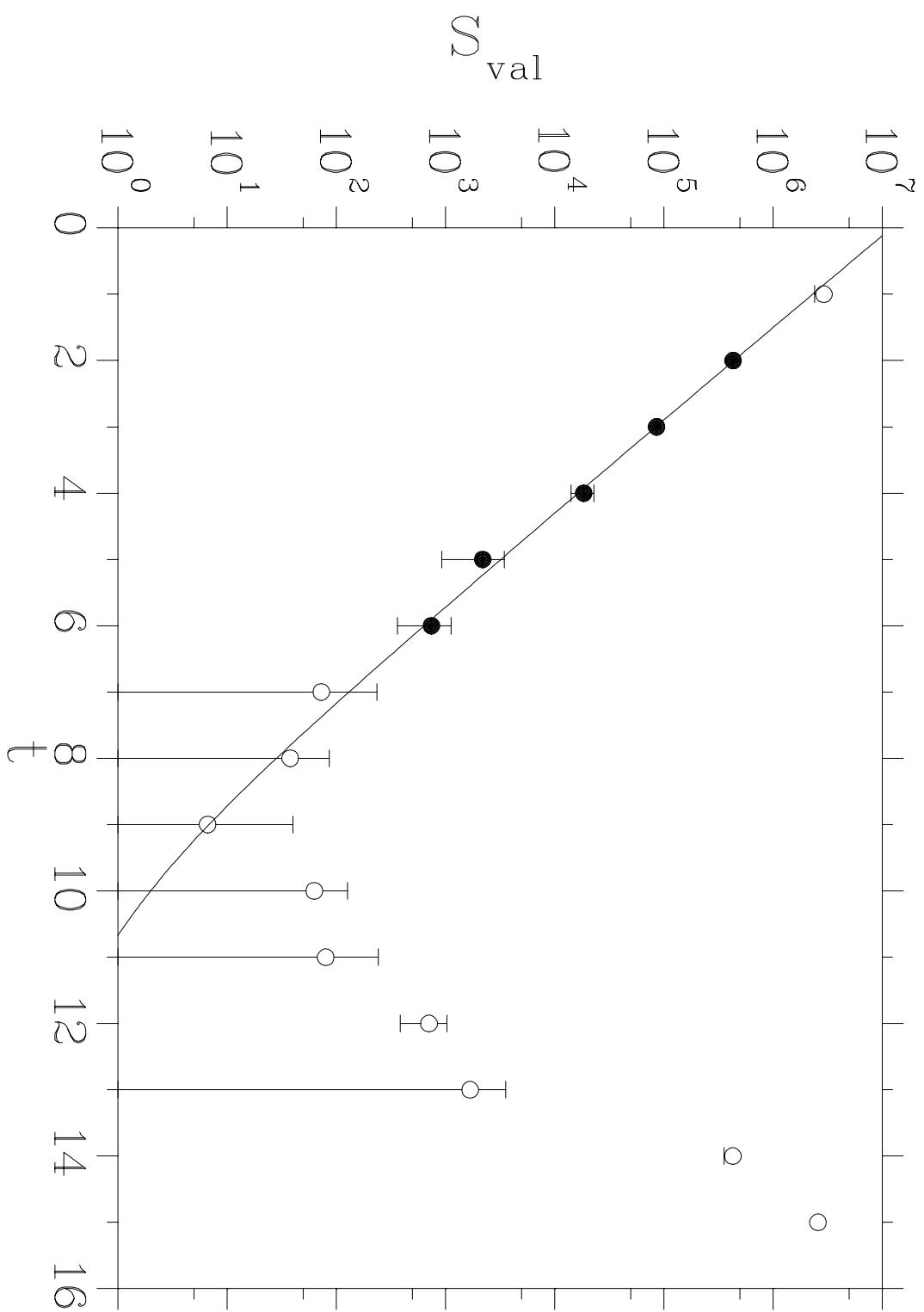
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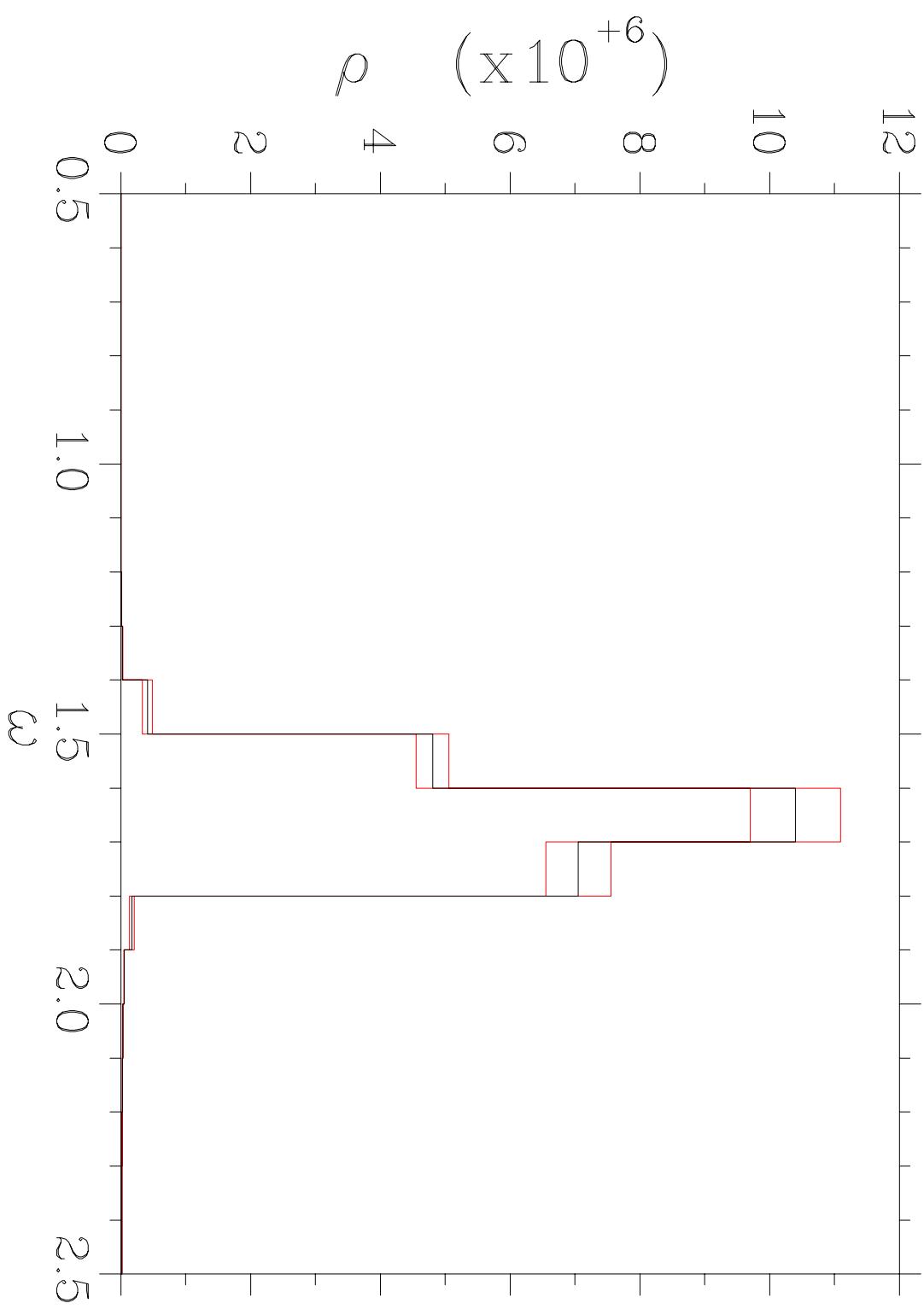
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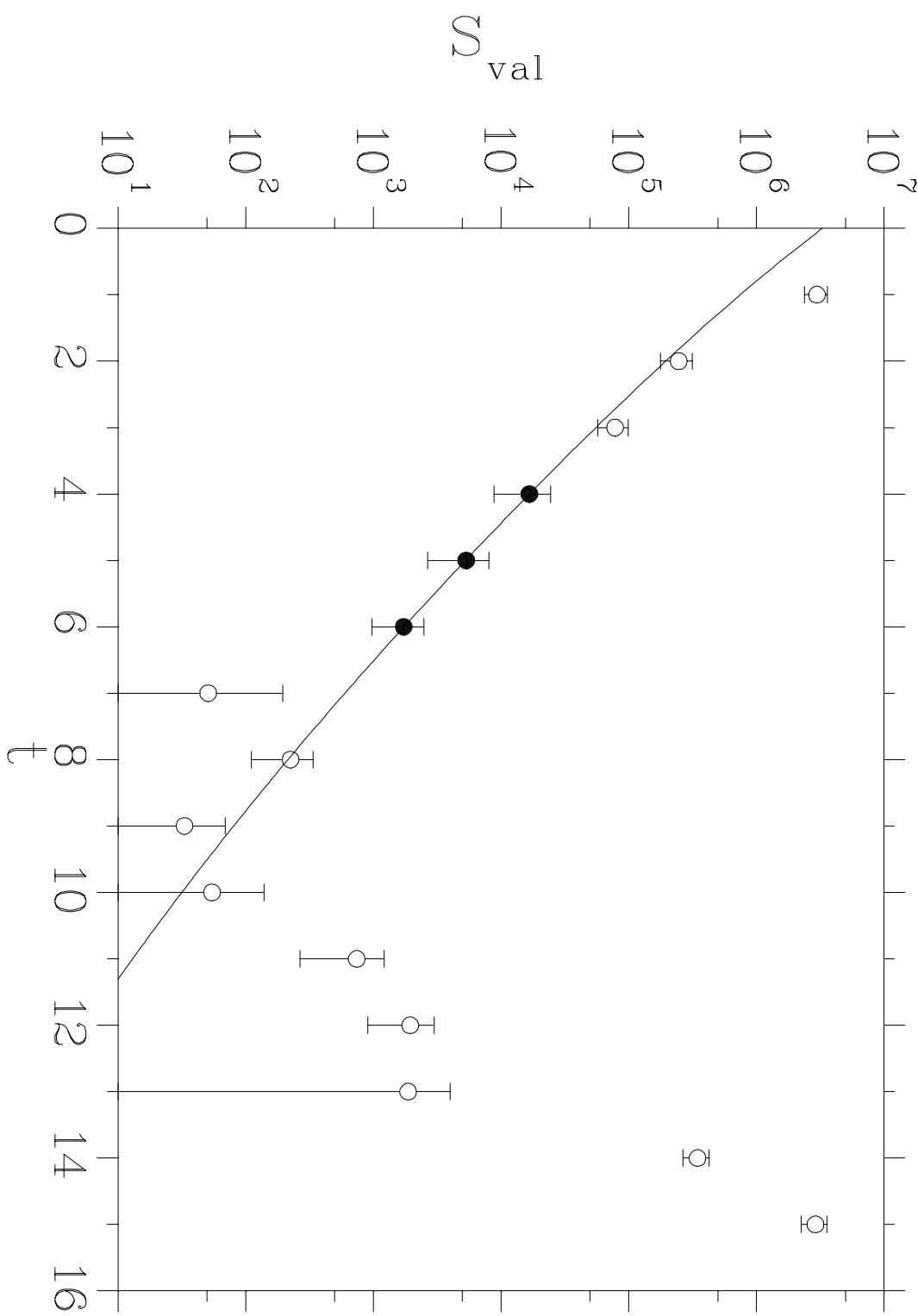
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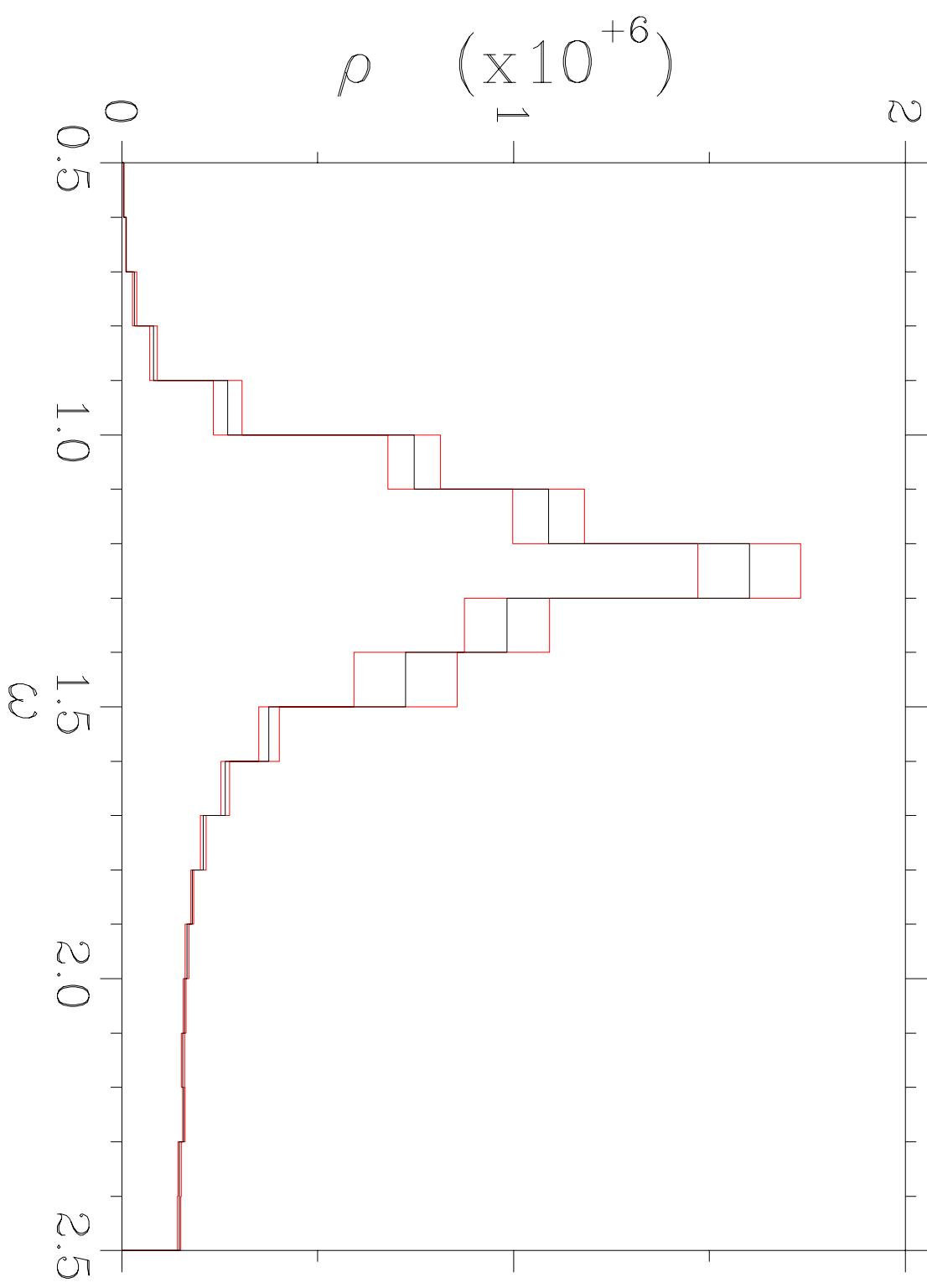
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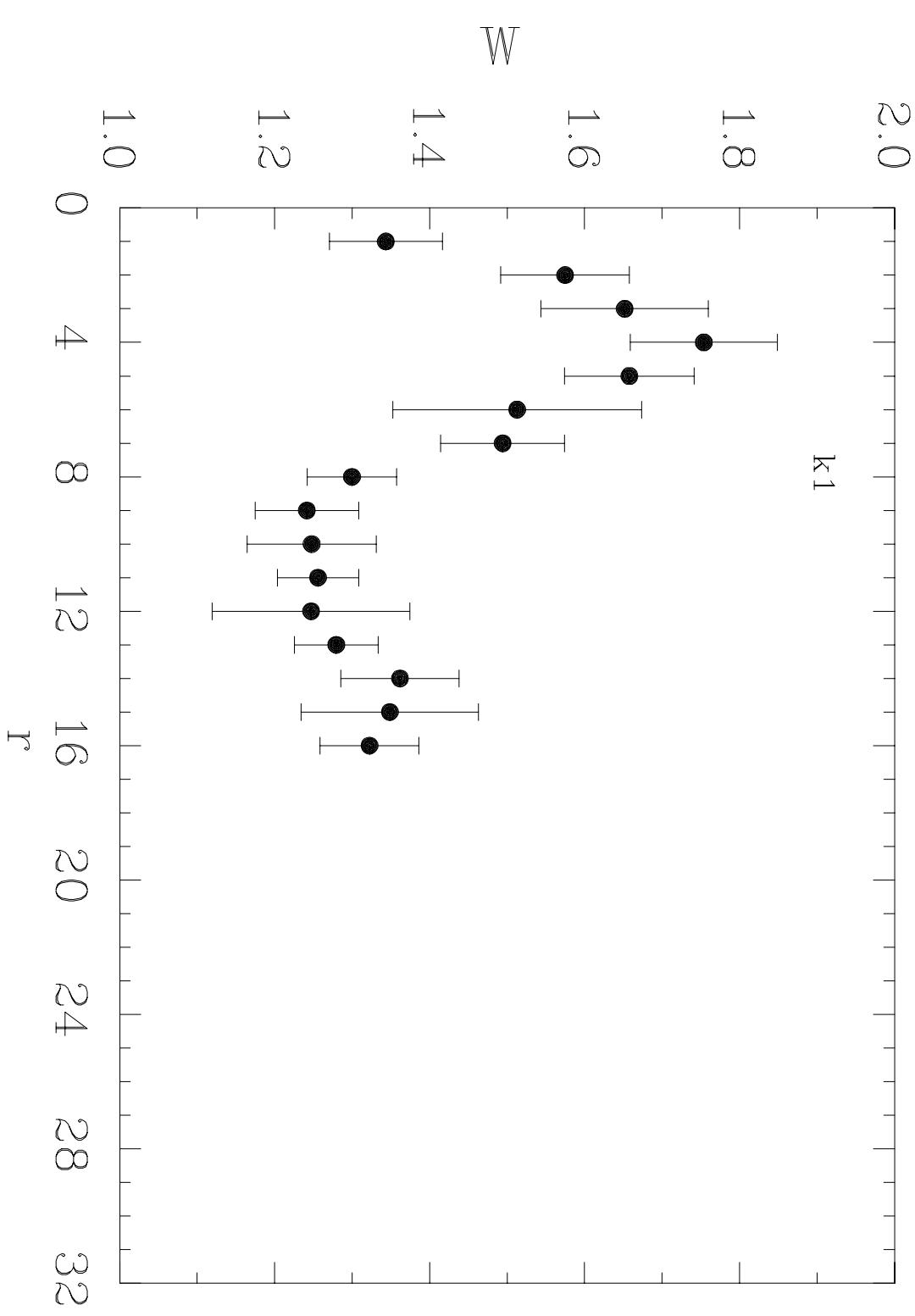
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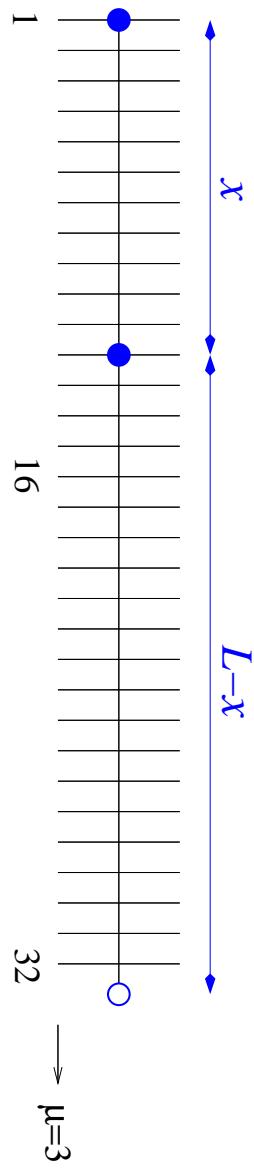
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## Adiabatic potential

- parameterized model, periodic extension

$$V(x) = \exp(-\alpha_1 x^2) [\alpha_2 + \alpha_3 x^2 + \alpha_4 x^4] \quad x = r/a$$

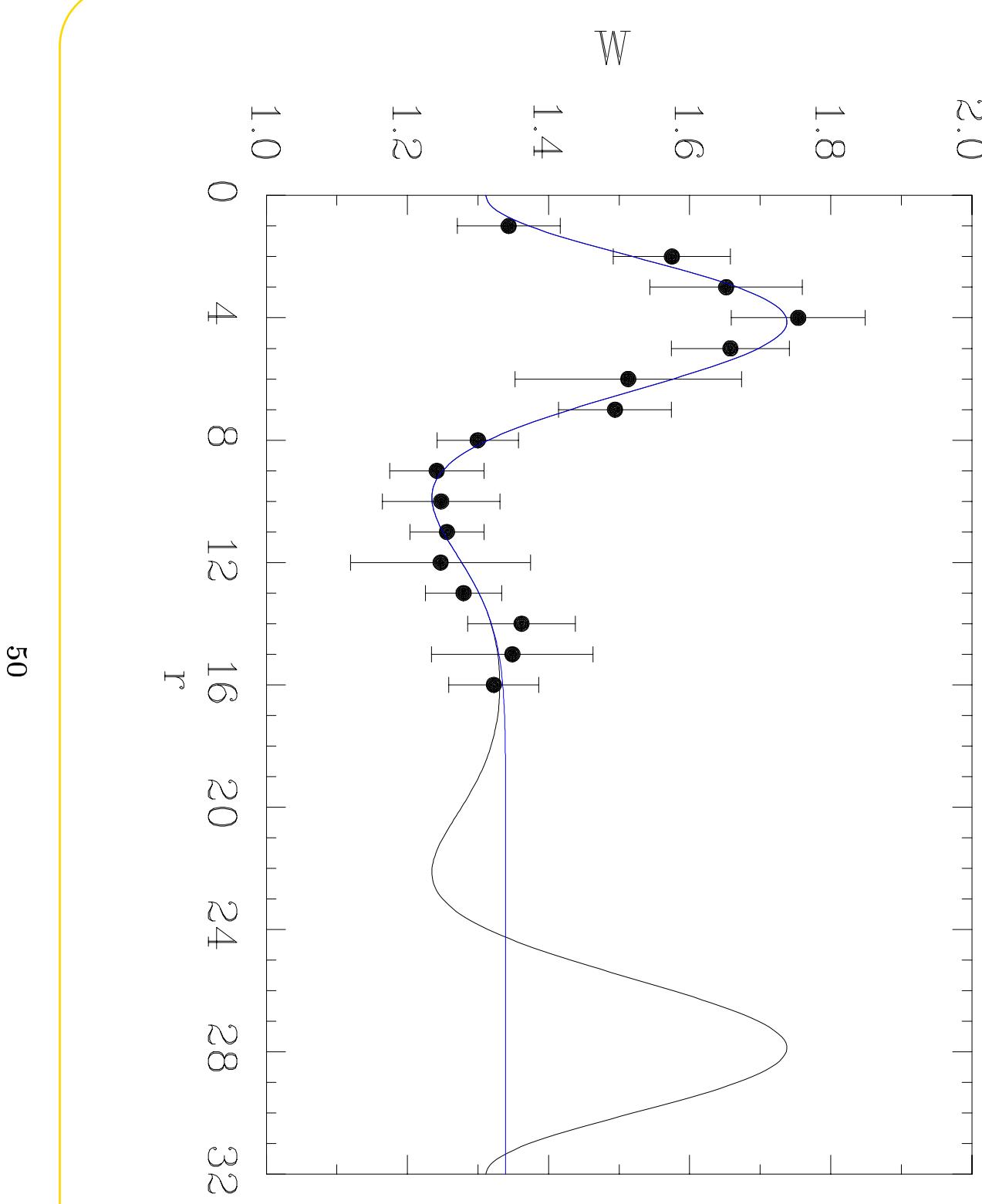


$$V_L(x) = \alpha_0 + V(x) + V(L-x), \quad L = 32$$

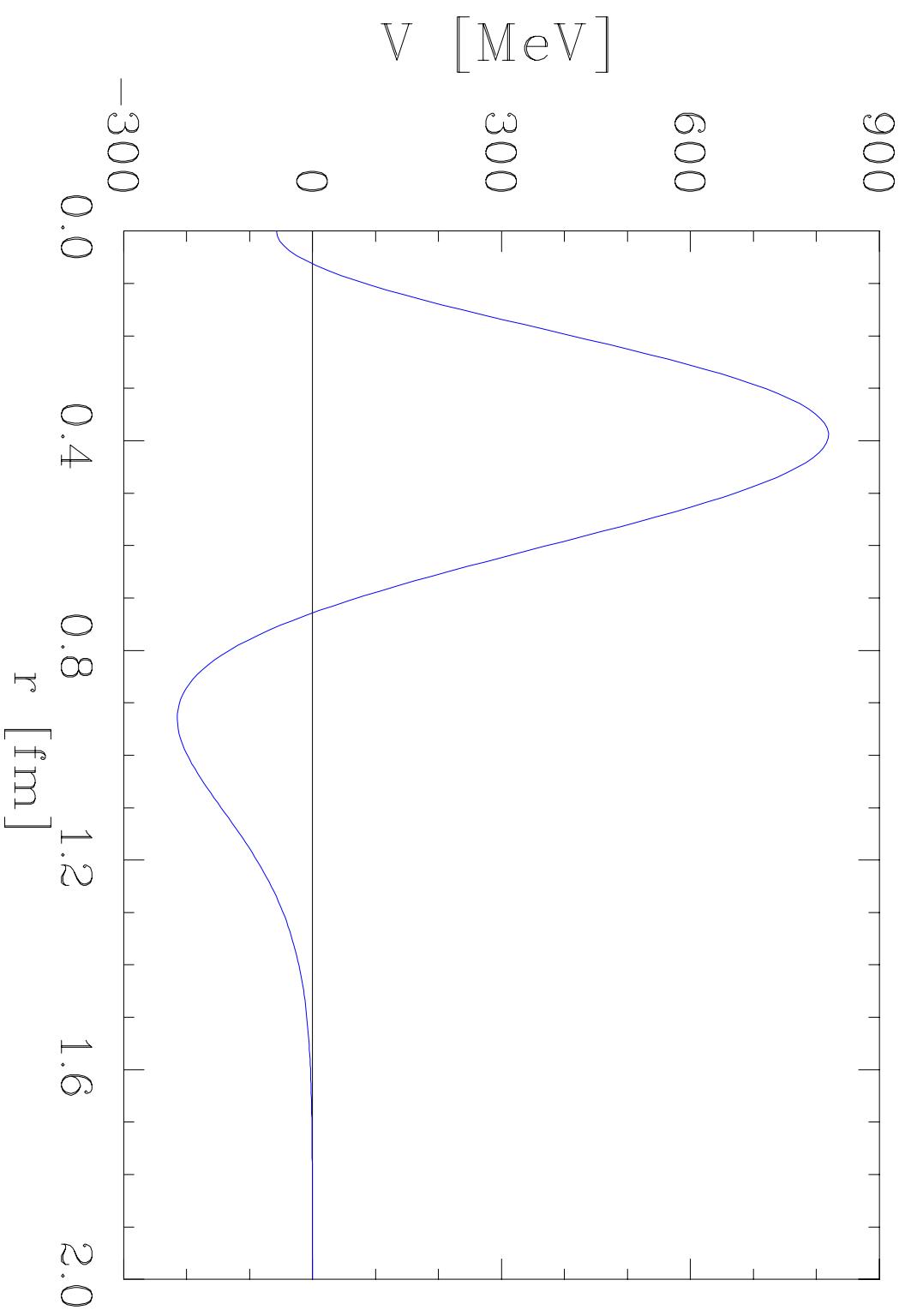
fit  $V_L(x)$  to lattice data

- S-wave Schrödinger eqn (Volterra  $\int$  eqn  $\rightarrow$  Jost functions)  
with  $V(r/a)a^{-1}$  and  $m_{\text{reduced}}$  of  $\mathbf{K}-\Lambda$ ,  $\mathbf{D}-\Lambda_c$ ,  $\mathbf{B}-\Lambda_b$   
scattering phase shifts  $\delta_0(p)$

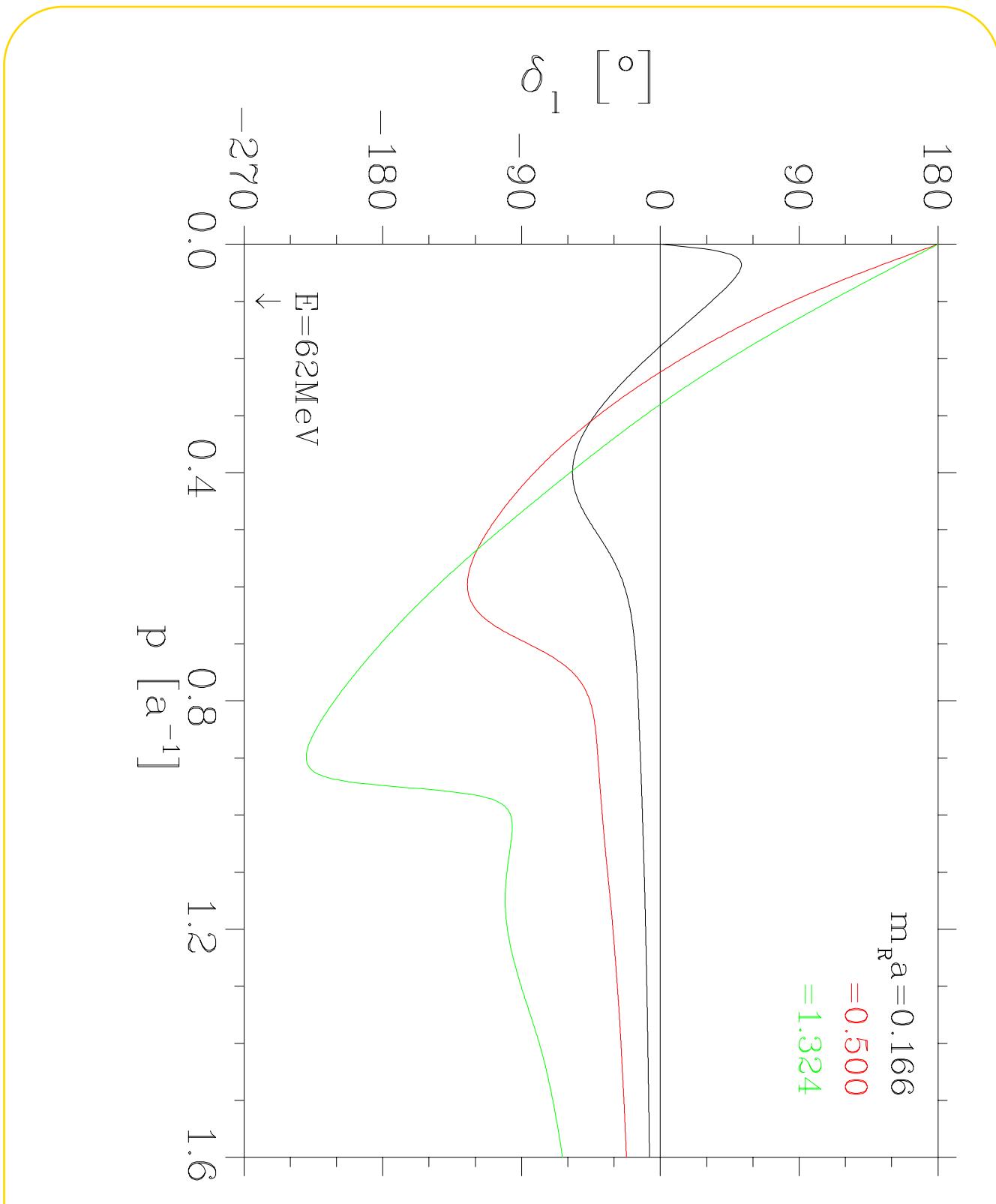
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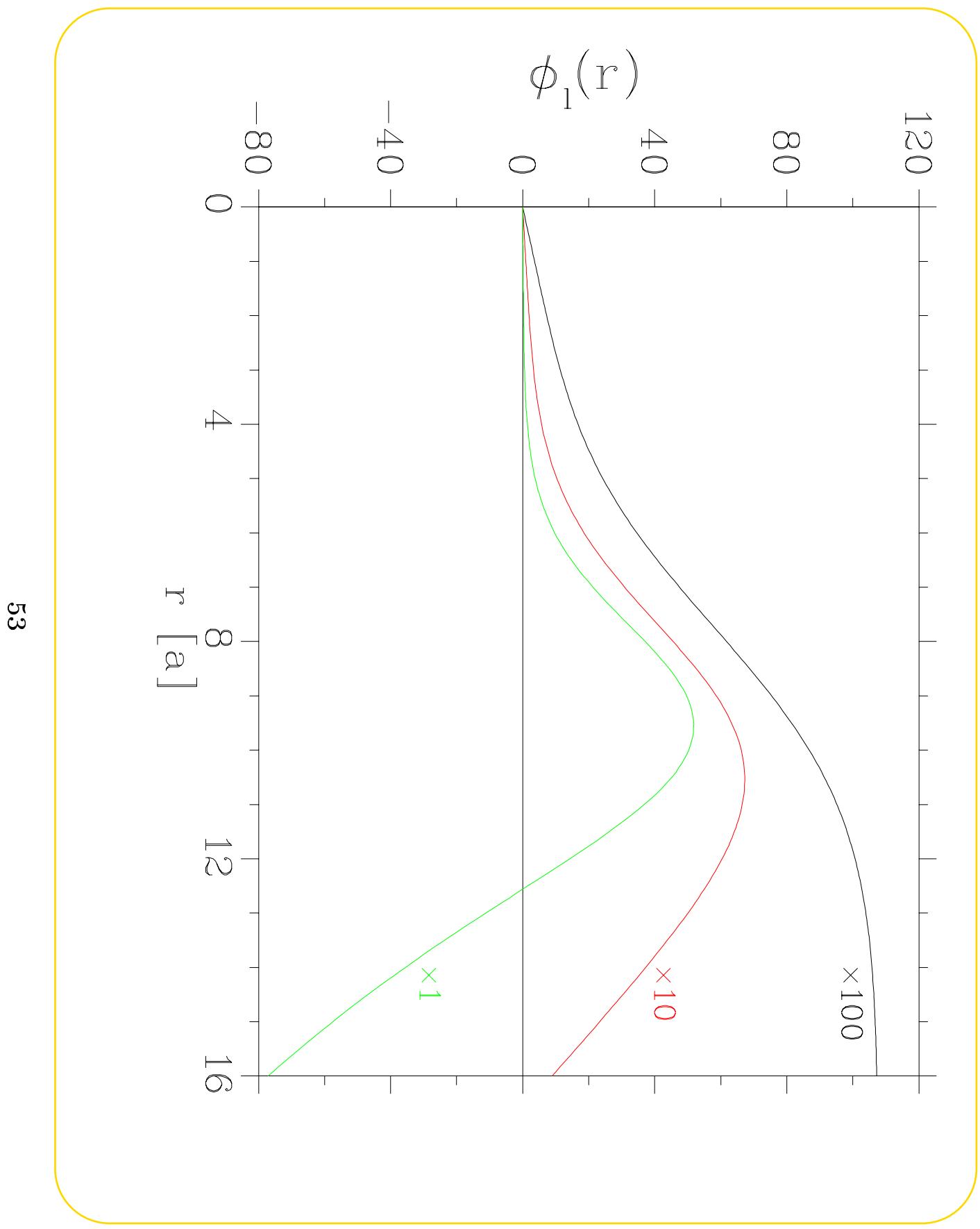


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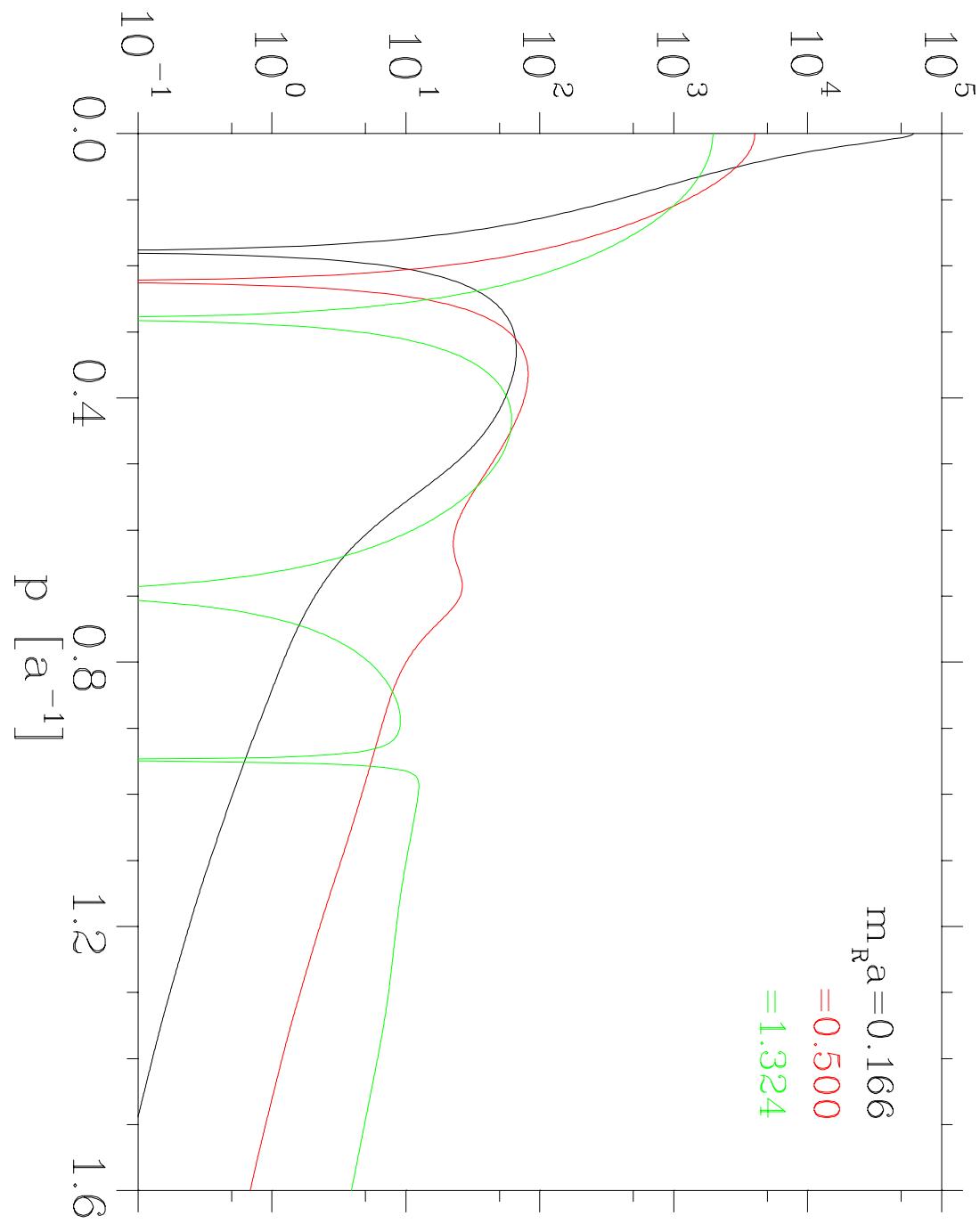


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$\sigma_1 [a^2]$



## Assessment

- Unresolved issues
  - at gauge 90 configs results are not yet stable
  - implement generalized SVD-diag at  $t_1 \gtrsim t_0$
  - error analysis
- Hint at possible physics
  - resonant behaviour of K- $\Lambda$  at  $50(\pm 50)$  MeV above threshold
  - may explain N(1650) as 5-quark K- $\Lambda$  molecule
  - the c- and b-quark like systems may be bound
- RESULTS MAY CHANGE, EVEN QUALITATIVELY !