Exploring the π–π Interaction in lattice QCD[a]

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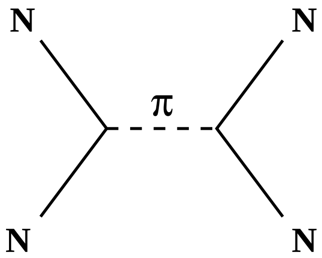
[a] This material is based upon work supported by the National Science Foundation under Grant No. 9700502.

Nuclear forces

Boson exchange model

Yukawa, 1935

N–N, range $R = \hbar c/mc^2$



$R \approx 2\text{fm} \quad \longrightarrow m \approx 100\text{MeV} \quad \longrightarrow$ pion π

$$(\partial^2 + m^2)\varphi(x) = g\bar{\psi}\psi \qquad (\text{Klein} - \text{Gordon})$$

- replace $\bar{\psi}\psi \to \delta(\vec{r})$ static point source
- seek static solution (Green's function)

$$(-\Delta^2 + m^2)\varphi_\pi(x) = g\delta(\vec{r})$$

$\Longrightarrow$ Yukawa Potential

$$\varphi_\pi(\vec{r}) = \frac{g}{4\pi}\frac{e^{-mr}}{r}$$

Phenomenology

short range	intermediate	long range
($r < 0.5$fm)	($r \approx 1.0$fm)	($r > 1.5$fm)
repulsion	attraction	attraction

Refinements

∘ more mesons $\varphi = \pi, \rho, \omega, \sigma \ldots$

$$\mathcal{L}_{\bar{\psi}\varphi\psi} = \sum_{\varphi} \sum_{\gamma\partial} \quad \cdots$$

∘ couplings $\gamma\partial =$ pseudoscalar, vector, tensor $\ldots$

∘ vertex form factors, cutoff parameters

∘ relativistic effects

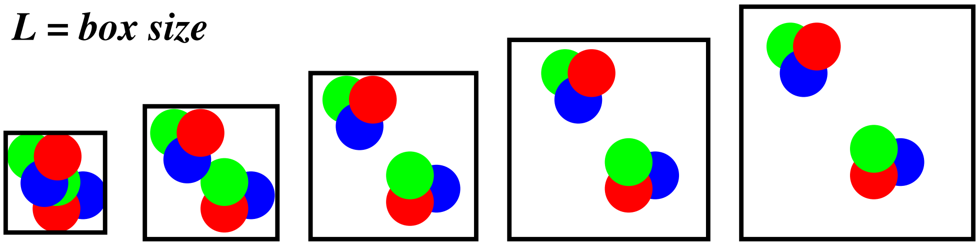
$\Longrightarrow$ *Bonn* Potential, and friends ...

$\Longrightarrow$ Chiral effective Lagrangians ...

- Status = highly developed, extensively applied
- Physics = QCD-based understanding desirable

Lattice explorations

Lüscher's formula - I



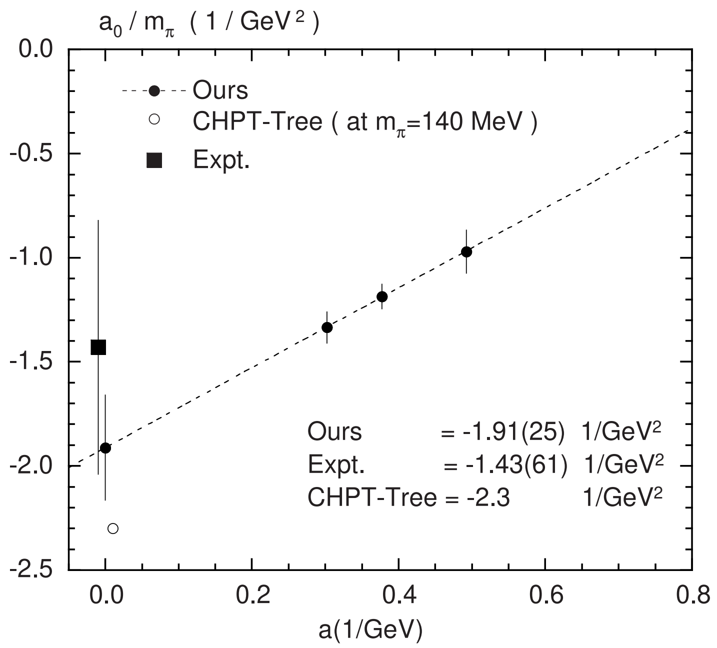
lattice 2-hadron ground-state energy

$$E_0 = 2m - \frac{4\pi a_0}{mL^3}[1 - 2.8\frac{a_0}{L} + 6.4\frac{a_0^2}{L^2} + \ldots]$$

- fit L-dependence

$\Longrightarrow$ S-wave scattering length a_0

Application: π-π, I=2, JLQCD hep-lat/9911025



Lüscher's method - II

Demo: 1-dim box L, Schrödinger eqn, interaction V
p.eriodic b.oundary c.onds $\psi(x) = \psi(x + L)$

(a)	V off		$k_0 L = 4.0 \times 2\pi$
(b)	V on,	ignore p.b.c.	$\delta = 1.2 \times 2\pi$
(c)	change $k \to$ restore p.b.c.		$kL = 1.6 \times 2\pi$

$$\Longrightarrow \qquad kL + 2\delta = k_0 L \qquad (1.6 + 2.4 = 4.0)$$

$$\Longrightarrow \qquad \exp 2i\delta(k) = \exp -ikL \qquad (*)$$

The plan

- find spectrum $E_l^2 = k_l^2 + m^2$
  on lattice, box, discrete $l = 1, 2 \ldots$
  solve $(*)$ $\Longrightarrow$ scattering phase shifts $\delta(k_l)$

- in 3-dim box

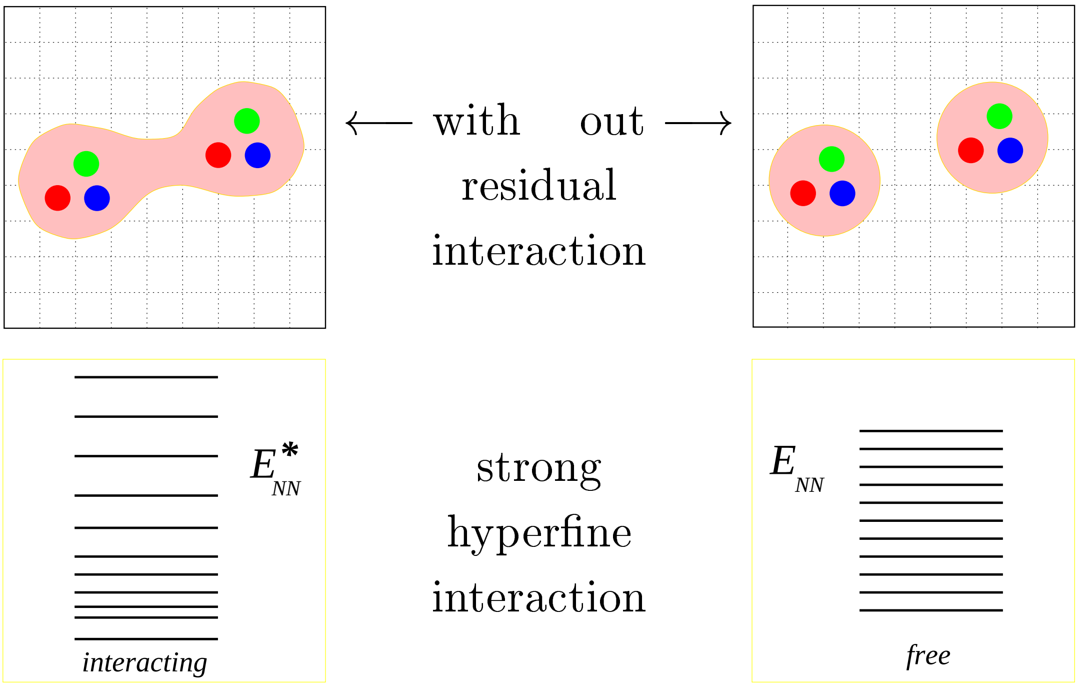
$$\det \left[e^{2i\delta(k)} - \frac{M(k) + i}{M(k) - i} \right] = 0 \qquad (*)$$

$n \times n$ matrix $M(k)$ $\quad \leftarrow$ Green's function in box
solve $(*)$ $\Longrightarrow$ degenerate $\delta(k_l)_{j=1\ldots n} \ni$ physical

Effective interaction

Two-hadron spectra

- In principle: finite-box spectral information
  $\Longrightarrow$ holds the physics of interaction/scattering[a]
- The plan: compare 2-hadron lattice propagators



effective $\mathcal{H}_I$

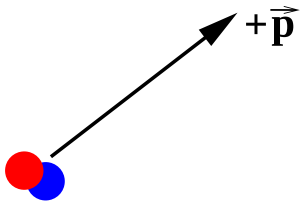
[a] LSZ approach obscured in euclidean space, 'no-go theorem' [L Maiani and M Testa, Phys Lett B 245 (1990) 585]

Theoretical framework

Interpolating fields

One-meson ($\pi^+ \sim \bar{d}u$)

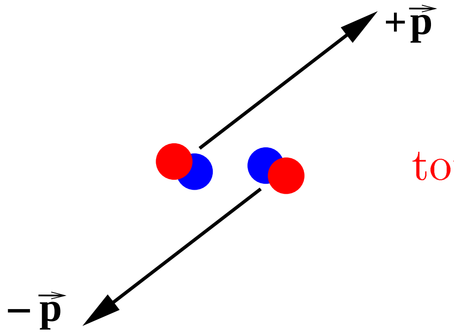
$$\phi_{\vec{p}}(t) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \bar{\psi}_d(\vec{x},t)\,\gamma_5\,\psi_u(\vec{x},t)$$



lat mom $\quad \vec{p} = \frac{2\pi}{L}\vec{k}$

Two-meson ($I = 2$)

$$\Phi_{\vec{p}}(t) = \phi_{-\vec{p}}(t)\,\phi_{+\vec{p}}(t)$$



total mom[a] $\quad \vec{P} = 0$

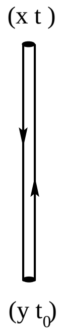
[a] reduced mass $= m/2$, relative mom $= \vec{p}$

Correlation matrices

2-point (one-meson)

$$\begin{aligned} C^{(2)}_{\vec{p}\,\vec{q}}(t,t_0) &= \langle \phi^{\dagger}_{\vec{p}}(t)\,\phi_{\vec{q}}(t_0)\rangle - \langle \quad \rangle\langle \quad \rangle \\ &= \langle \overset{21}{\phi^{\dagger}_{\vec{p}}} \overset{12}{\phi_{\vec{q}}} \rangle = \sim \langle \overset{2}{G^*} \overset{1}{G} \; \rangle = \\ &= \delta_{\vec{p}\,\vec{q}}\; c_{\vec{p}}(t,t_0) \qquad \text{diagonal} \end{aligned}$$

$$\overset{2}{G}{*}\overset{1}{G} \; =$$



- $\phi \sim \bar{\psi}\psi$, contractions $n = 1, 2,$

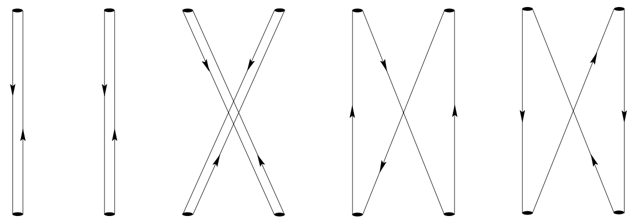
$$\overset{n}{\psi}(x)\;\overset{n}{\bar{\psi}}(y) = \overset{n}{G}(x,y) \quad \text{quark propagator}$$

- one-meson correlator

$$\begin{aligned} c_{\vec{p}}(t,t_0) &= \sum_{\vec{x},\vec{y}} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\langle |G_{A\mu,B\nu}(\vec{x}t,\vec{y}t_0)|^2\rangle \\ &= c^*_{-\vec{p}}(t,t_0) \end{aligned}$$

4-point (two-meson)

$$
\begin{aligned}
C^{(4)}_{\vec{p}\,\vec{q}}(t,t_0) &= \langle \Phi^{\dagger}_{\vec{p}}(t)\, \Phi_{\vec{q}}(t_0) \rangle - \langle \quad \rangle \langle \quad \rangle \\
&= \langle \phi^{\dagger}_{+\vec{p}} \phi^{\dagger}_{-\vec{p}} \phi_{-\vec{q}} \phi_{+\vec{q}} \rangle = \\
&= \langle \overset{43}{\phi^{\dagger}_{+\vec{p}}} \overset{21}{\phi^{\dagger}_{-\vec{p}}} \overset{12}{\phi_{-\vec{q}}} \overset{34}{\phi_{+\vec{q}}} \rangle + \ldots \\
&= \sim \langle \overset{4}{G^*} \overset{3}{G}\, \overset{2}{G^*} \overset{1}{G} \;\rangle + \ldots \\
&= C^{(4A)} + C^{(4B)} - C^{(4C)} - C^{(4D)}
\end{aligned}
$$



A B C D

- $\Phi \sim \phi\phi$, $\phi \sim \bar{\psi}\psi$, contractions $n = 1, 2, 3, 4$
- exchange:
  - whole-meson, Pauli
  - quark-antiquark pair, boson
  - correlated-gluon, gauge config ave $\langle \sim \rangle$

Uncorrelating the mesons

- *Free* propagation contained in

$$
\begin{aligned}
C^{(4A)} &= \langle \overset{43}{\phi^\dagger_{+\vec p}} \overset{21}{\phi^\dagger_{-\vec p}} \overset{12}{\phi_{-\vec q}} \overset{34}{\phi_{+\vec q}} \rangle \sim \langle \overset{4}{G^*} \overset{3}{G}\ \overset{2}{G^*} \overset{1}{G}\ \rangle \\
C^{(4B)} &= \text{“\textit{Xchange diagram, Bose symm}”}
\end{aligned}
$$

- Gluon correlations $\langle \cdots \quad \cdots \rangle$ between mesons
  $\langle \cdots \rangle \langle \cdots \rangle$ **off**
  stochastic ave $\Rightarrow$

- Cumulant expansion, $\sum_{\text{all clusterings } \langle \ldots \rangle \ldots \langle \ldots \rangle}$

$$
\begin{aligned}
&\langle \overset{4}{G^*} \overset{3}{G}\ \overset{2}{G^*} \overset{1}{G}\ \rangle = \\
&\quad \langle \overset{4}{G^*} \overset{3}{G}\ \rangle \langle \overset{2}{G^*} \overset{1}{G}\ \rangle + \langle \overset{4}{G^*} \overset{1}{G}\ \rangle \langle \overset{2}{G^*} \overset{3}{G}\ \rangle + \langle \overset{4}{G^*} \overset{2}{G^*} \rangle \langle \overset{3}{G}\ \overset{1}{G}\ \rangle \\
&\quad + \langle\langle \overset{4}{G^*} \overset{3}{G}\ \overset{2}{G^*} \overset{1}{G} \rangle\rangle \quad \leftarrow \text{cumulant}
\end{aligned}
$$

uncorrelated mesons
$\Longrightarrow$ 1-st separable term

$$\left.\begin{array}{ll} \texttt{direct} & \overline{C}^{(4A)} = \langle\phi^\dagger_{+\vec{p}}\phi_{+\vec{q}}\rangle\langle\phi^\dagger_{-\vec{p}}\phi_{-\vec{q}}\rangle \\ \texttt{xchang} & \overline{C}^{(4B)} = \langle\phi^\dagger_{-\vec{p}}\phi_{+\vec{q}}\rangle\langle\phi^\dagger_{+\vec{p}}\phi_{-\vec{q}}\rangle \end{array}\right\} \quad \text{add} \Longrightarrow$$

Free correlator matrix

$$\overline{C}^{(4)} = \overline{C}^{(4A)} + \overline{C}^{(4B)} = (\delta_{\vec{p},\vec{q}} + \delta_{-\vec{p},\vec{q}})\,|c_{\vec{p}}(t,t_0)|^2$$

Full correlator matrix

$$C^{(4)} = \overline{C}^{(4)} + \underset{\uparrow}{C_I^{(4)}} \quad = \textit{free} + \textit{interacting}$$

Interaction mechanisms

$\rightarrow$ quark-antiquark (boson) exchange

$\rightarrow$ correlated gluon (glueball?) exchange

$\rightarrow$ if unquenched, quark-antiquark loop effects

Future opportunities

? features of select mechanisms

- ? range: long intermediate short
- ? strength: attractive repulsive
- ? dependence: spin flavor, etc

? relation to effective models, etc

Residual effective $\mathcal{H}_I$

- *Off-the-CPU* exercise:
  - elementary Bose field $\hat{\phi}(x)$, $x \in$ lattice
  - $\hat{\mathcal{L}} = \hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_I \longrightarrow \hat{H} = \hat{H}_0 + \hat{H}_I$
  - canonical quantization

$$\begin{array}{lrcl} \textit{1-mes} & \hat{\phi}_{\vec{p}}(t) & = & \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \hat{\phi}(\vec{x}, t) \\ \textit{2-mes} & \hat{\Phi}_{\vec{p}}(t) & = & \hat{\phi}_{-\vec{p}}(t)\, \hat{\phi}_{+\vec{p}}(t) \end{array}$$

4-point correlator, perturbative expansion $H_I \to \lambda H_I$

$$\hat{C}^{(4)}_{\vec{p}\,\vec{q}}(t, t_0) = \langle 0 | \hat{\Phi}^{\dagger}_{\vec{p}}(t)\, \hat{\Phi}_{\vec{q}}(t_0) | 0 \rangle = \sum_{N=0}^{\infty} \lambda^N \, \hat{C}^{(4;N)}_{\vec{p}\,\vec{q}}(t, t_0)$$

Effective correlator matrix, define

$$\hat{\mathcal{C}}(t, t_0) = \hat{C}^{(4;N=0)}(t, t_0)^{-1/2}\, \hat{C}^{(4)}(t, t_0)\, \hat{C}^{(4;N=0)}(t, t_0)^{-1/2}$$

find

$$\begin{aligned} \hat{\mathcal{C}}(t, t_0) &= \mathbf{1} - \lambda \hat{H}_I (t - t_0) + \frac{1}{2}\left(-\lambda \hat{H}_I (t - t_0)\right)^2 + \mathcal{O}(\lambda^3) \\ &= e^{-\lambda \hat{H}_I (t - t_0)} + \mathcal{O}(\lambda^3) \end{aligned}$$

∘ typical use with $|t - t_0| \to$ “large”, perturbative $\hat{H}_I$

Reconstruct: $\hat{H}_I \Longleftarrow \hat{\mathcal{C}}(t, t_0)$

Effective residual interaction

- guidance from: elementary Bose field
  canonical quantization
  perturbative interaction, 2nd order $\Longrightarrow$
- on the lattice:

Def: effective correlator matrix

$$\mathcal{C}(t,t_0) = \frac{\mathbf{1}}{\sqrt{\overline{C}^{(4)}(t,t_0)}} C^{(4)}(t,t_0) \frac{\mathbf{1}}{\sqrt{\overline{C}^{(4)}(t,t_0)}}$$

Def: effective residual interaction

$$\mathcal{H}_I = -\lim_{t\to\infty} \frac{\partial \ln \mathcal{C}(t,t_0)}{\partial t}$$

- contains
  - excited degrees of freedom, relative motion
- consider
  - numerical diagonalization, variational
  - exp fits to eigenvalues
  - extract $\mathcal{H}_{I,\vec{p}\,\vec{q}}$

Lattice symmetry

- lattice $L^d \times T$, $d = 3$, discrete group $O(d, \mathbf{Z}) = \{$ rotations by 90^o etc, inversions $\}$

`irreps`	Γ:	A_1^+	$T_1^\pm$	E^+	$T_2^\pm$	A_2^-
`dimens`	N_Γ:	1	3	2	3	1

- angular momentum ℓ content

$$
\begin{array}{lcl}
\underline{\ell} & & \\
\underline{0} & = & A_1^+ \\
\underline{1} & = & T_1^- \\
\underline{2} & = & E^+ \oplus T_2^+ \\
\underline{3} & = & A_2^- \oplus T_1^- \oplus T_2^- \\
\underline{4} & = & A_1^+ \oplus E^+ \oplus T_1^+ \oplus T_2^+
\end{array}
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{mod } 4
$$

Reduced matrix elements

full correlator

$$< (\Gamma, p)\epsilon | C^{(4)}(t, t_0) | (\Gamma', q)\epsilon' > = \delta_{\Gamma\Gamma'}\delta_{\epsilon\epsilon'}\ C^{(\Gamma)}_{p\,q}(t, t_0)$$

free correlator, diagonal

$$\overline{C}^{(\Gamma)}_{p\,q}(t, t_0) = \delta_{p\,q}\ \left|\bar{c}^{(\Gamma)}_{p}(t, t_0)\right|^2$$

effective correlator matrix

$$\mathcal{C}^{(\Gamma)}_{p\,q}(t, t_0) = \frac{C^{(\Gamma)}_{p\,q}(t, t_0)}{\left|\bar{c}^{(\Gamma)}_{p}(t, t_0)\right| \left|\bar{c}^{(\Gamma)}_{q}(t, t_0)\right|}$$

- numerically
  eigenvalues $\sim \exp(-E_n(t - t_0))$
  interaction $\mathcal{H}_I \sim -\partial \ln \mathcal{C} / \partial t$ indep of t

$\Longrightarrow \quad \mathcal{H}^{(\Gamma)}_{I,p\,q}$

Numerical things

Lattice design parameters

Simulation

lattice $L^3 \times T = 9^3 \times 13$

tree-level $O(a^2)$ tadpole improved

very coarse $a = 0.4\text{fm}, a^{-1} = 0.5\text{GeV}$

but large $La = 3.6\text{fm}$

208 configurations, quenched

Propagators

from $8 \times (3 \times 4)$ random sources

multiple-mass solver

Wilson $\kappa^{-1} = 5.720 \ldots 6.140$ six

Operators

Gaussian smearing at sink

done with fuzzy link variables

on-axes momenta $\vec{p} = \frac{2\pi}{L}(k, 0, 0)$

Numerical limitations

- approx $C^{(\Gamma)} \to$ diag $C^{(\Gamma)}$
  small, noisy off-diag
  only $\Gamma = A_1^+$ $(\ell = 0)$
- large $E(p) \sim 2\sqrt{m^2 + p^2}$, mod(lat)
  $C_{p\,p}^{(\Gamma)} \sim e^{-E(p)(t-t_0)}$ steep dive into noise
  marginal numerical quality!
- source point $t_0 = 3$
  earlier time slices not used
  unphysical $O(a^2)$ improvement ghosts

Chiral extrapolation

- effective $\mathcal{H}_{I,q\,q}^{(\Gamma)}(m^2)$ depends on m^2 (Wilson κ^{-1})
  fit to χPT-motivated ansatz

$$h(m^2) = h' m^2 + h'' m^3$$

and extrapolate $\boxed{h(m^2 \to 0) =: \mathcal{H}_{I,q\,q}^{(\Gamma)}}$

Analysing the numbers

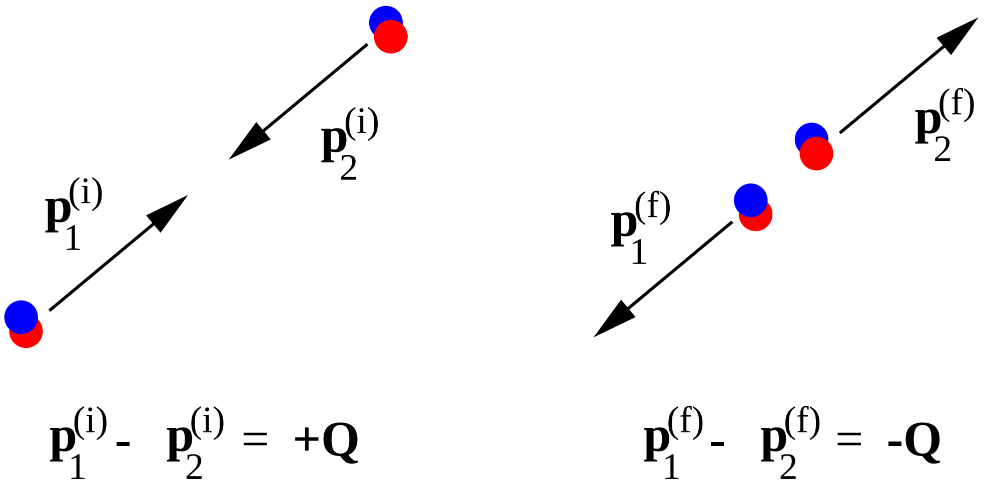
Potentials

- transformations

$$< \vec{p}|\mathcal{H}_I|\vec{q} >= \quad (\text{dominant } \Gamma = A_1^+)$$
$$= \sum_{\Gamma}\sum_{\epsilon} < \vec{p}|(\Gamma,p)\epsilon > \mathcal{H}_{I,pq}^{(\Gamma)} < (\Gamma,q)\epsilon|\vec{q} >$$

$$< \vec{r}|\mathcal{H}_I|\vec{s} >= \quad (\text{relative coords } \vec{r},\vec{s})$$
$$= L^{-3}\sum_{\vec{P}}\sum_{\vec{Q}} e^{i\vec{P}\cdot(\vec{r}-\vec{s})}e^{-i\vec{Q}\cdot(\vec{r}+\vec{s})} < \vec{P}-\vec{Q}\,|\mathcal{H}_I|\vec{P}+\vec{Q} >$$

- Breit frame



- split off $\vec{P} = 0$ contribution

$$< \vec{P}-\vec{Q}\,|\mathcal{H}_I|\vec{P}+\vec{Q} >= < -\vec{Q}\,|\mathcal{H}_I| + \vec{Q} > + \text{“difference”}$$

$\Longrightarrow$

$$< \vec{r}\,|\mathcal{H}_I|\vec{s}\,>= \delta_{\vec{r}\,\vec{s}}\,\mathcal{V}(\vec{r}\,) + \mathcal{W}(\vec{r},\vec{s}\,)$$

local potential

$$\mathcal{V}(\vec{r}\,) = \sum_{\vec{Q}} e^{-i2\vec{Q}\cdot\vec{r}} < -\vec{Q}\,|\mathcal{H}_I| + \vec{Q} >$$

partial wave ($\ell = 0$) projection

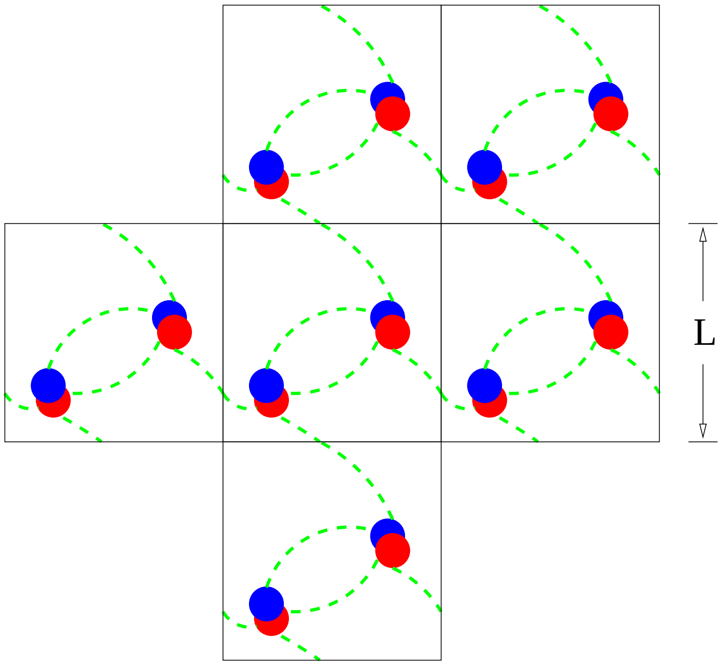
$$\begin{aligned} \mathcal{V}_0(r) &= \frac{1}{4\pi}\int d\Omega_{\vec{r}}\,\mathcal{V}(\vec{r}\,) \\ &= \sum_{\vec{Q}} j_0(2Qr)\, < -\vec{Q}\,|\mathcal{H}_I| + \vec{Q} > \\ &= \sum_{Q} j_0(2Qr)\,\mathcal{H}_{I,QQ}^{(A_1^+)} \end{aligned}$$

- discrete $\sum$ over $Q = 2\pi k/L$
- continuous r, $\rightarrow$ interpolation between lattice sites

Parameterization

- periodic lattice, boundary conditions

  $\Rightarrow \quad \mathcal{V}(\vec{r}) = \mathcal{V}(\vec{r} + \vec{n}L), \quad \vec{n} \in \mathbf{R}^3$



  interactions "around the world" $\sim \exp -L$ suppressed

- distinct Fourier "flavor"

$$\mathcal{V}_0(r) \quad = \quad \sum_{k=0}^{k_{\max}} j_0(2Q_k r)\, \mathcal{H}_{I,kk}^{(A_1^+)}$$

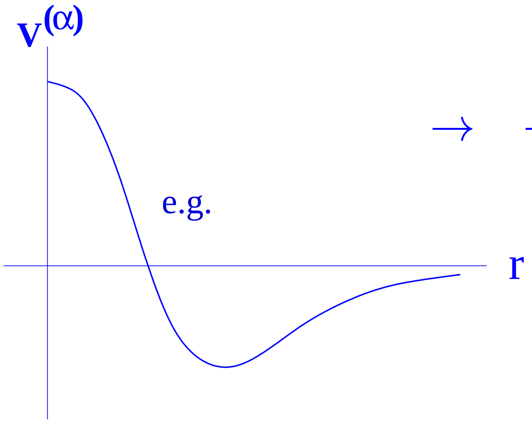
$$\text{e.g.} \qquad Q_k = \frac{2\pi}{L} k, \;\; k = 0, 1 \ldots k_{\max}$$

  truncation, $k_{\max}$ dependence

  oscillating, hard to interpret $\quad \Longrightarrow$

• parametric function

$$V^{(\alpha)}(r) \quad = \quad \alpha_1 \frac{1-\alpha_2 r^{\alpha_5}}{1+\alpha_3 r^{\alpha_5+1} e^{\alpha_4 r}} + \alpha_0$$



$$\rightarrow \quad -\frac{\alpha_1 \alpha_2}{\alpha_3} \frac{e^{-\alpha_4 r}}{r} \qquad r \rightarrow \infty, \text{ Yukawa}$$

• Latticized parametric function

$$V_L^{(\alpha)}(r) = \sum_{k=0}^{k_{\max}} j_0(2Q_k r)\, v_k^{(\alpha)} \qquad Q_k = \frac{2\pi}{L} k$$

require $V_L^{(\alpha)}(r_l) = V^{(\alpha)}(r_l)$
expansion coefficients $v_k^{(\alpha)}$ from
$j_0(2Q_k r_l) = B_{lk}$ inverse B^{-1} gives $\rightarrow v_k^{(\alpha)}$
support $r_0 = 0$, $r_l = \frac{L}{4(k_{\max}-l+1)}$, $l = 1 \ldots k_{\max}$

$$\Longrightarrow \quad \text{Span}\{V_L^{(\alpha)}(r)\} \subseteq \text{Span}\{\mathcal{V}_0(r)\}$$

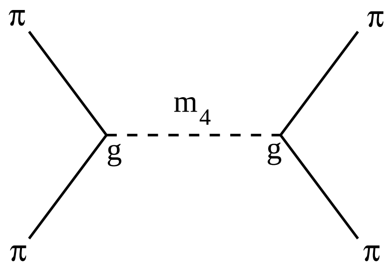
- fit Latticized $V_L^{(\alpha)}(r)$ to lattice-computed $\mathcal{V}_0(r)$

$$\int_0^{L/2} dr \left| V_L^{(\alpha)}(r) - \mathcal{V}_0(r) \right|^2 = \min(\alpha)$$

find $\alpha \longrightarrow$ <u>non-periodic</u> $V^{(\alpha)}(r), \quad 0 \leq r < \infty$

- present simulation

truncations:	$k_{\max} = 2, 3, 4$
fixed:	$\alpha_0 = 0, \alpha_5 = 2$
parameters:	four $\alpha_1, \alpha_2, \alpha_3, \alpha_4$



$$V^{(\alpha)}(r \to \infty) \to -\frac{\alpha_1 \alpha_2}{\alpha_3} \frac{e^{-\alpha_4 r}}{r}$$

Xch-mass	$\alpha_4 a^{-1}$	$\simeq$	m_4	$=$	$1.0 - 1.4\,\mathrm{GeV}$
Coupling	$\alpha_1 \alpha_2 / \alpha_3$	$\simeq$	$g^2/4\pi$	$=$	$3.0 - 19.7$

Scattering phase shifts[a] $\delta^{\ell=0}_{I=2}$

$\rightarrow$ use $V^{(\alpha)}(r)$ in Schrödinger eqn, mass = $m_\pi > 0$
(non-local $W^{(\beta)}(r, r')$ not computed)

$\rightarrow$ low-p repulsive

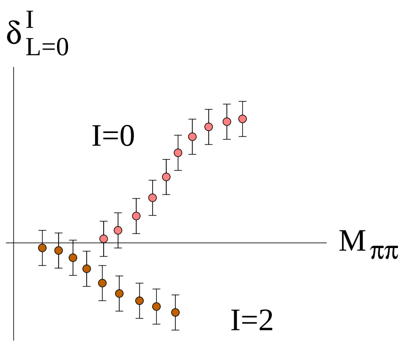
$\rightarrow$ relativistic effects, estimate
via $m_\pi \rightarrow m_\pi(p) = \sqrt{m_\pi^2 + p^2}$
large (as expected)

$\rightarrow$ experiment $\pi^+ p \longrightarrow \pi^+ \pi^+ n$
CERN-Munich 1974 at 12.5 GeV
analysis: [Hoogland et al, N.P. B126 (77) 109]

[a] Long write up: hep-lat/9906002, Few-Body Systems (Springer) special issue, summer 2000

Concluding remarks

- hadron-hadron physics from lattice QCD
  - a numerical challenge
  - an important problem for QCD
  - underdeveloped
- s-wave $I = 2$ π-π lattice results
  - long/interm - range repulsion
  - consistent with experiment
  - qualitative, exploratory
- indispensible technology, retool
  - anisotropic lattices, $a_{\rm time} < a_{\rm spce}$
  - all-to-all quark propagators
- exists resonance in $I = 0$ π-π channel
  - interesting case for lattice



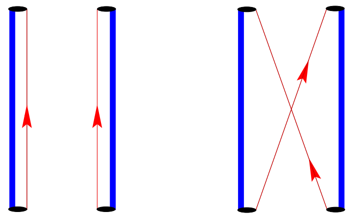
- heavy-light systems
  - less hard to do
  - learn about interaction mechanisms ← **The Future**

Heavy-light systems

1-meson = ϕ, 2-meson = Φ

$$\phi_{\vec{x}}(t) = \bar{\psi}_h(\vec{x},t)\,\Gamma\,\psi_l(\vec{x},t) \qquad h\text{=heavy},\ l\text{=light}$$

$$\Phi_{\vec{r}}(t) = \sum_{\vec{x},\vec{y}} \phi_{\vec{x}}(t)\phi_{\vec{y}}(t)\,\delta_{\vec{x}-\vec{y},\vec{r}} \qquad \vec{r} = \text{relative dist}$$

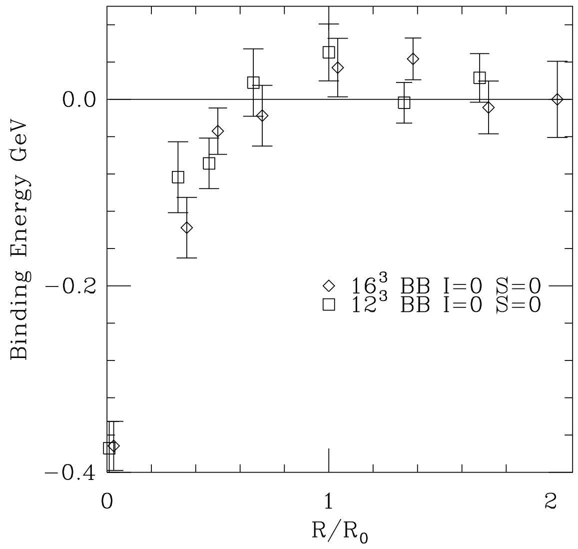


- effective correlator, diagonal

$$\mathcal{C}_{\vec{r}\,\vec{r}}(t,t_0) = \frac{C^{(4)}_{\vec{r}\,\vec{r}}(t,t_0)}{C^{(2)}(t,t_0)C^{(2)}(t,t_0)} \sim e^{-[E(r)-2m](t-t_0)}$$

$\Rightarrow$ binding energy $[E(r)-2m] = V_{\text{adiabatic}}(r)$ potential

- Example: UKQCD, C. Michael, P. Pennanen, hep-lat/9901007



- Early work on heavy-light:
  D.G. Richards, Nucl. Phys. (Proc. Suppl.) B 9 (1989) 181
- Geometric approach: *energy of quark configuratons*
  T. Green et al, Helsinki group