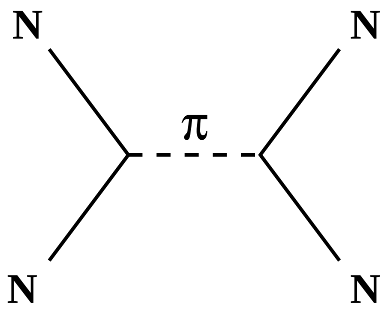
Interactions of Lattice Hadrons [a]

[a] This material is based upon work supported by the National Science Foundation under Grant No. 0073362 and upon resources provided by the Lattice Hadron Physics Collaboration LHPC through the SciDac program of the US Department of Energy.

Nuclear forces

Boson exchange model



Yukawa, 1935

N–N, range $R = \hbar c / mc^2$

$R \approx 2\text{fm} \quad \longrightarrow m \approx 100\text{MeV} \quad \longrightarrow$ pion π

$$(\partial^2 + m^2)\varphi(x) = g\bar{\psi}\psi \qquad (\text{Klein} - \text{Gordon})$$

• replace $\bar{\psi}\psi \to \delta(\vec{r})$, static solution (Green's function)

$\Longrightarrow$ Yukawa Potential

$$\varphi_\pi(\vec{r}) = \frac{g}{4\pi}\frac{e^{-mr}}{r}$$

Phenomenology

short range	intermediate	long range
$(r < 0.5\text{fm})$	$(r \approx 1.0\text{fm})$	$(r > 1.5\text{fm})$
repulsion	attraction	attraction

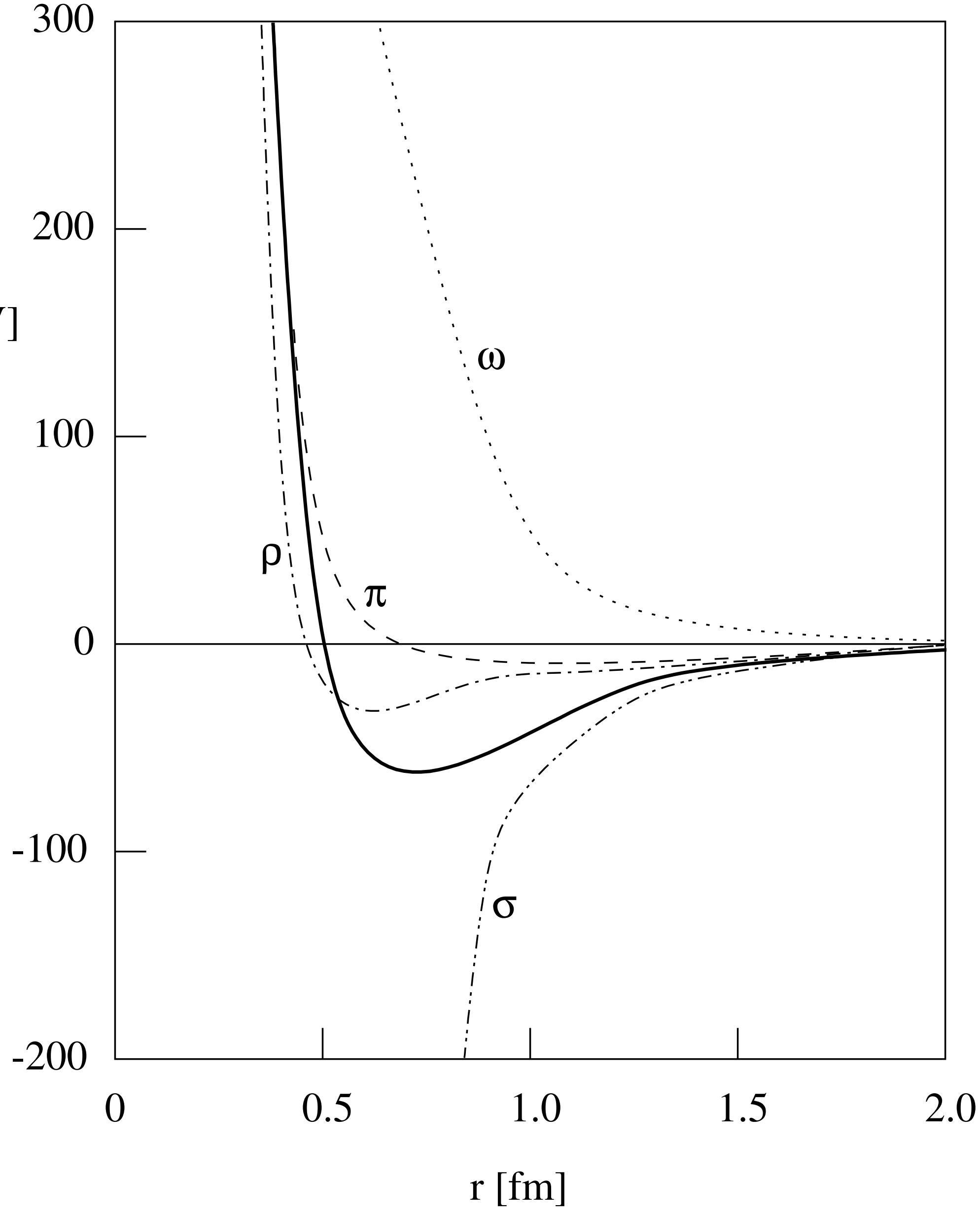
Refinements

- more mesons $\varphi = \pi, \rho, \omega, \sigma \ldots$

$$\mathcal{L}_{\bar{\psi}\varphi\psi} = \sum_{\varphi} \sum_{\gamma\partial} \quad \cdots$$

- couplings $\gamma\partial =$ pseudoscalar, vector, tensor $\ldots$
- vertex form factors, cutoff parameters
- relativistic effects

$\Longrightarrow$ *Bonn* Potential, and friends ...



300
200
100
0
-100
-200
V [MeV]
ω
ρ
π
σ
0
0.5
1.0
1.5
2.0
r [fm]

Lattice approaches

Technology[a]

- finite-size methods
- spectral methods
- ◦ geometric approaches
- heavy-light systems

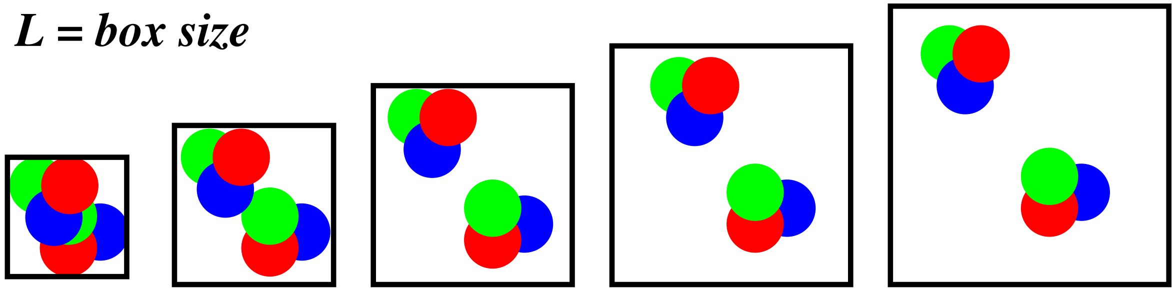
What can we expect to achieve?

- quantitative results in QCD? . . . very few, some scattering lengths
- qualitative insight, coupling constants? . . . some of that
- physics of interaction mechanism? . . . would be most desirable!

[a] LSZ approach obscured in euclidean space, 'no-go theorem' [L Maiani and M Testa, Phys Lett B 245 (1990) 585]

Lattice explorations

Lüscher's formula - I



lattice 2-hadron ground-state energy

$$E_0 = 2m - \frac{4\pi a_0}{mL^3}[1 - 2.8\frac{a_0}{L} + 6.4\frac{a_0^2}{L^2} + \ldots]$$

- fit L-dependence

$\Longrightarrow$ S-wave scattering length a_0

Selected s-wave scattering lengths a_0 from [Fukugita et al 1995]. Wilson fermions, $12^3 \times 20$ lattice, $\beta = 5.7$, $m_\pi/m_\rho = 0.74$ and $m_N/m_\rho = 1.57$.

		LAT	EXP	CUA
π–π	$I = 0$	$+3.02(17)$	$+0.37(7)$	$+0.222$
	$I = 2$	$-0.924(40)$	$-0.040(17)$	-0.0635
π–N	$I = 1/2$	$+3.04(66)$	$+0.245(4)$	$+0.221$
	$I = 3/2$	$-1.10(20)$	$-0.143(6)$	-0.111
N–N	${}^3S_1$	$+10.8(9)$	$-5.432(5)$	
	${}^1S_0$	$+8.0(1.1)$	$+20.1(4)$	

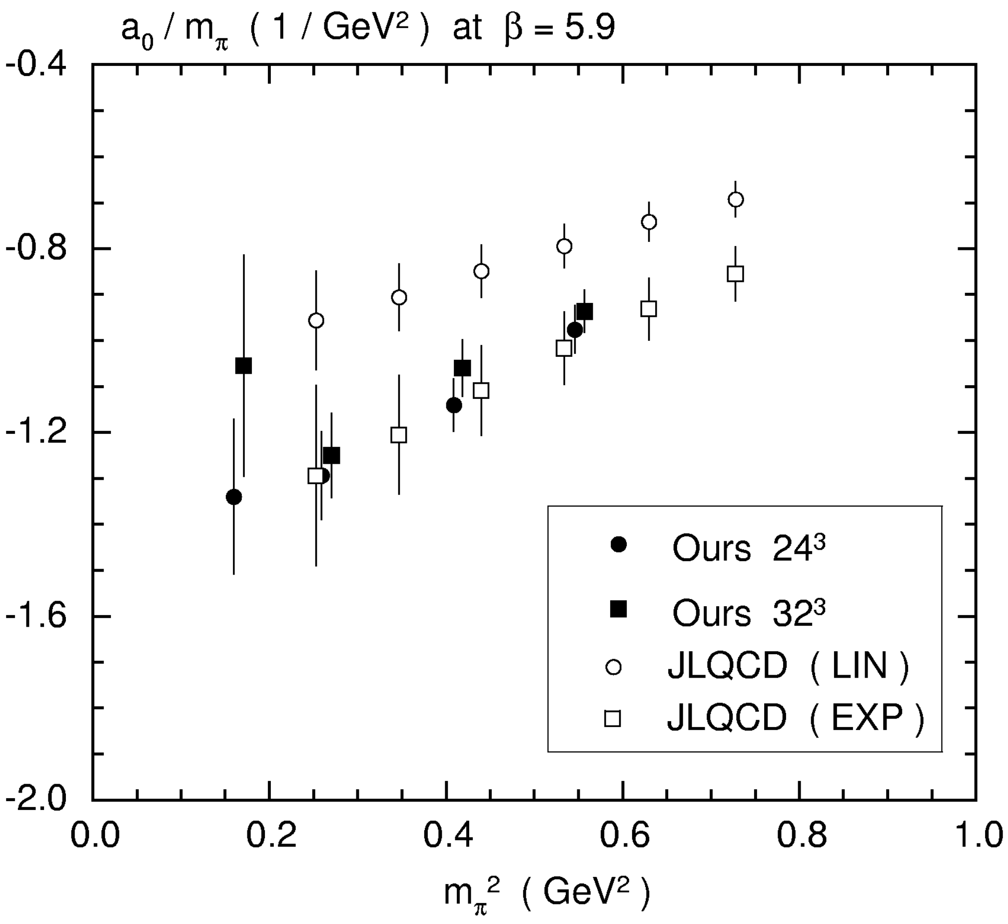
∘ quenched approximation . . .

∘ finite size effects . . .

∘ work continuing . . .

$\rightarrow$ still . . . a *quantitative* result

$\Longrightarrow$ π-π, I=2, CP-PACS hep-lat/0209124



Lüscher's method - II

Demo: 1-dim box L, Schrödinger eqn, interaction V
p.eriodic b.oundary c.onds $\psi(x) = \psi(x+L)$

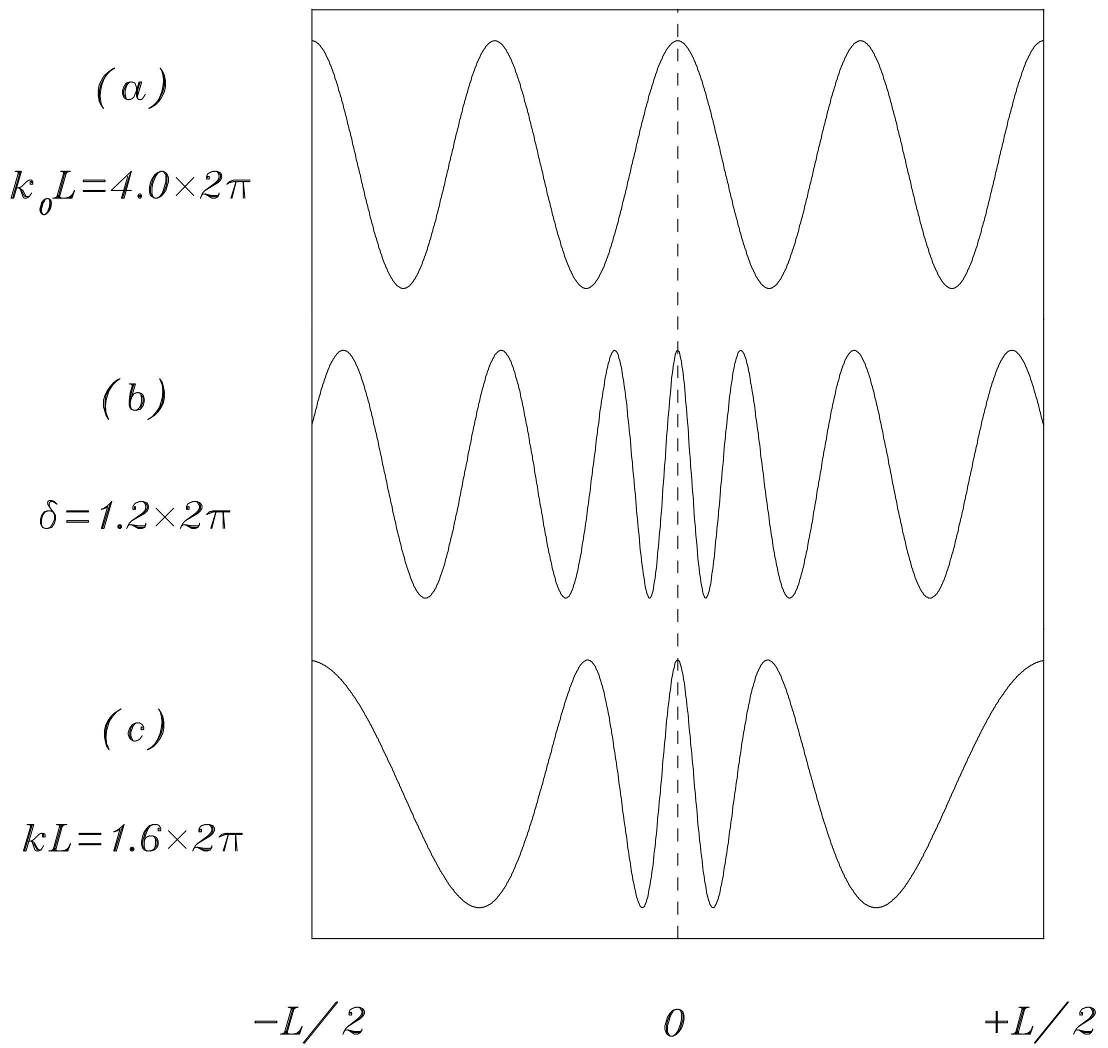
(a)	V off		$k_0 L = 4.0 \times 2\pi$
(b)	V on,	ignore p.b.c.	$\delta = 1.2 \times 2\pi$
(c)	change $k \to$ restore p.b.c.		$kL = 1.6 \times 2\pi$

$$\Longrightarrow \quad kL + 2\delta = k_0 L \qquad (1.6 + 2.4 = 4.0)$$

$$\Longrightarrow \quad \exp 2i\delta(k) = \exp -ikL \qquad (*)$$

The plan

- find spectrum $E_l^2 = k_l^2 + m^2$
  on lattice, box, discrete $l = 1, 2 \ldots$
  solve $(*)$ $\Longrightarrow$ scattering phase shifts $\delta(k_l)$



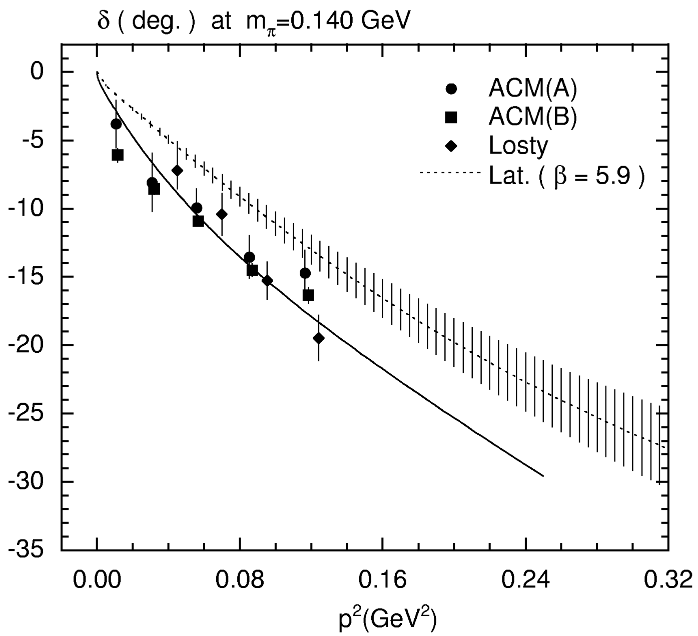
(a)
$k_0L=4.0\times 2\pi$
(b)
$\delta=1.2\times 2\pi$
(c)
$kL=1.6\times 2\pi$
$-L/2$
0
$+L/2$

• in 3-dim box $\Longrightarrow$

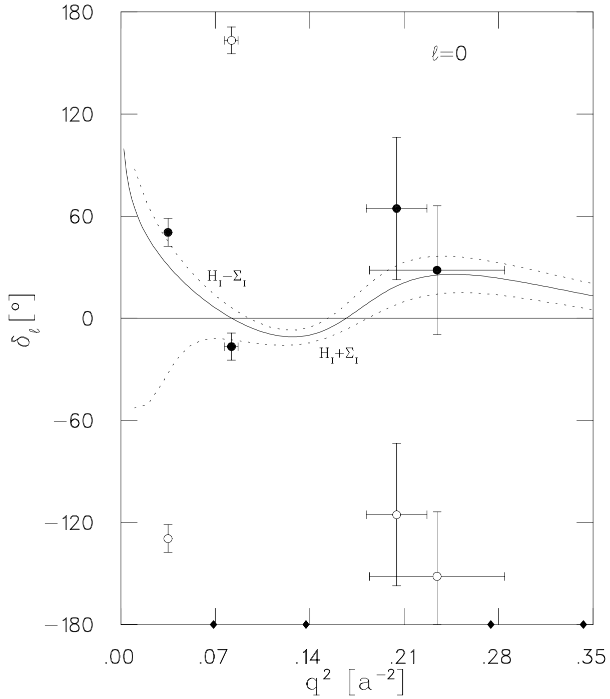
$$\det\left[e^{2i\delta(k)} - \frac{M(k)+i}{M(k)-i}\right] = 0 \qquad (*)$$

$n \times n$ matrix $M(k)$ $\quad\leftarrow$ Green's function in box, analytic

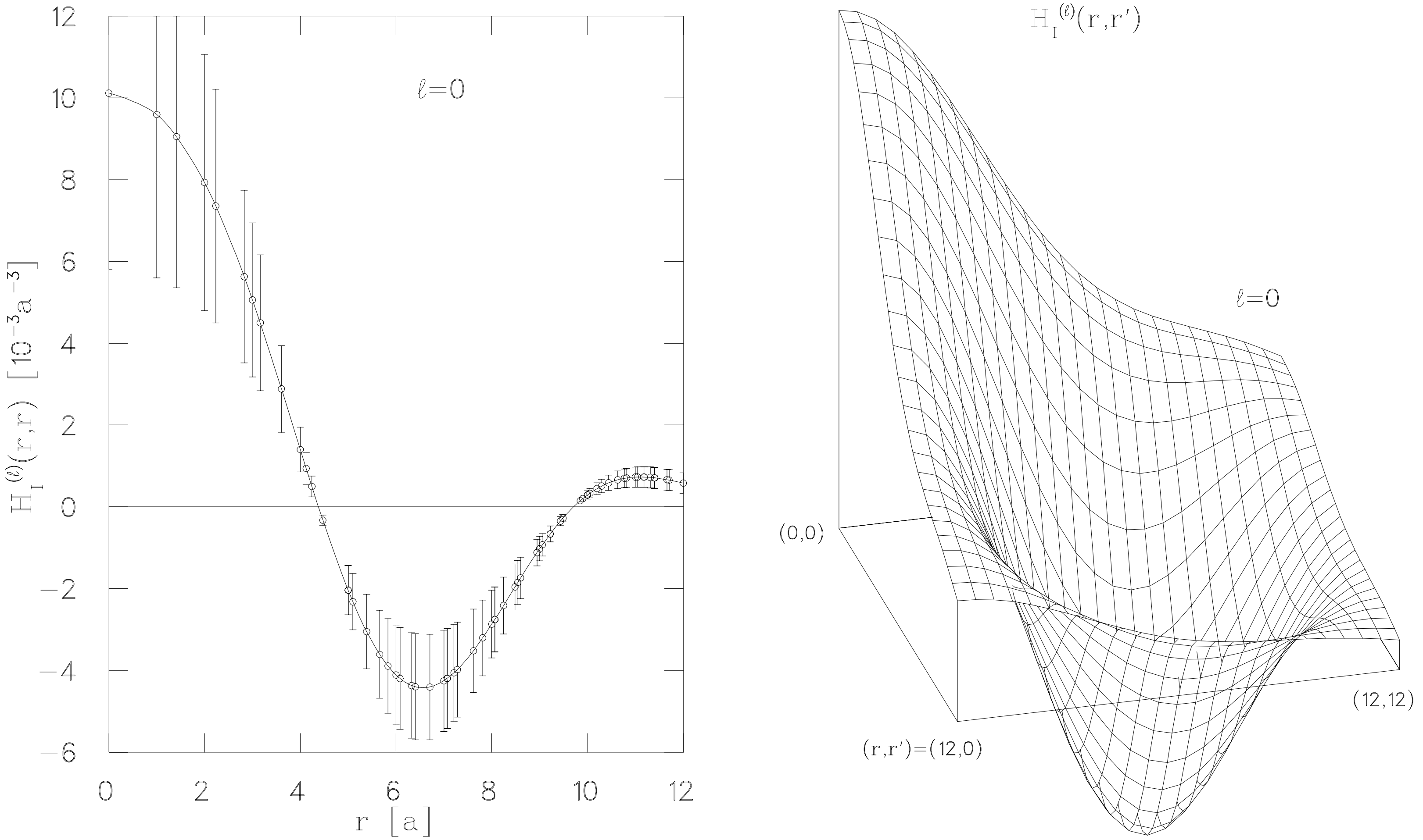
solve $(*)$ $\Longrightarrow$ multiple $\delta(k_l)_{j=1\ldots n} \ni$ physical



π-π, I=2
CP-PACS hep-lat/0209124

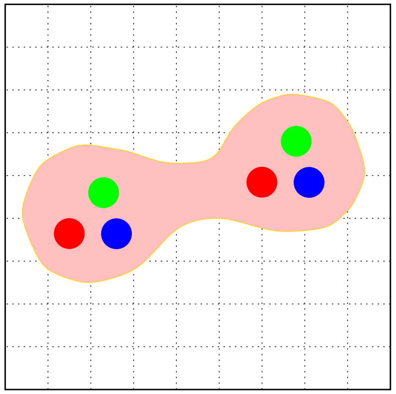


QED$_{2+1}$
spectrum $\rightarrow$ potential $\rightarrow$ $\delta(k)$



QED$_{2+1}$ Local and nonlocal effective potentials

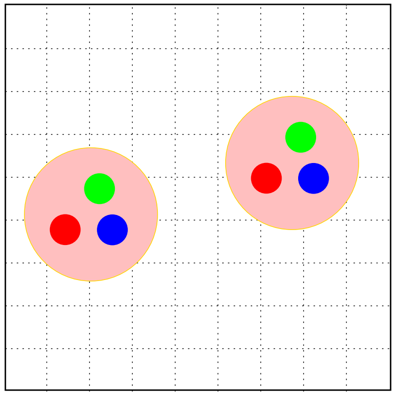
Two-hadron spectra

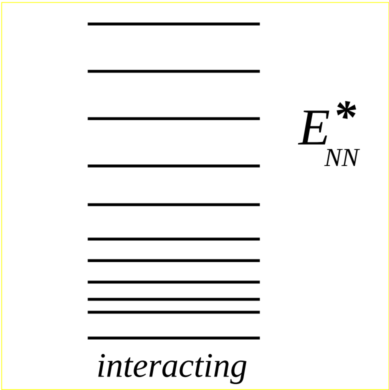


$\longleftarrow$ with out $\longrightarrow$

residual

interaction

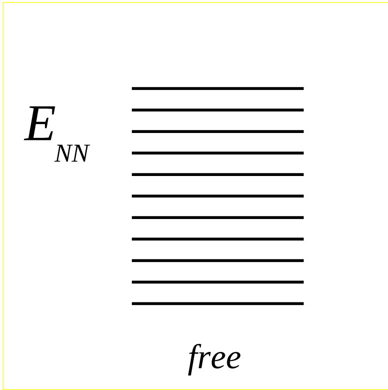




strong

hyperfine

interaction



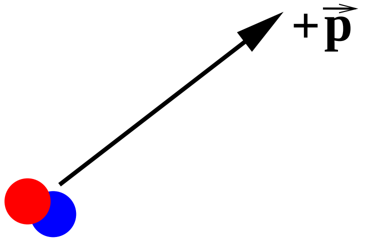
$\searrow$ $\swarrow$

effective $\mathcal{H}_I$

$\Longrightarrow$ finite-box spectral information holds the physics of interaction

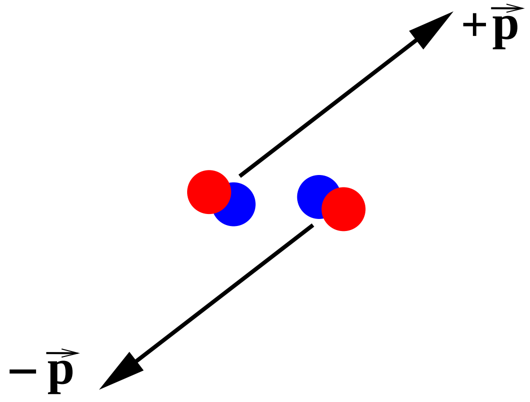
Interpolating fields

One-meson ($\pi^+ \sim \bar{d}u$) $\qquad \phi_{\vec{p}}(t) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \bar{\psi}_d(\vec{x},t)\,\gamma_5\,\psi_u(\vec{x},t)$



lat mom $\qquad \vec{p} = \frac{2\pi}{L}\vec{k}$

Two-meson ($I = 2$) $\qquad \Phi_{\vec{p}}(t) = \phi_{-\vec{p}}(t)\,\phi_{+\vec{p}}(t)$



total mom $= 0$

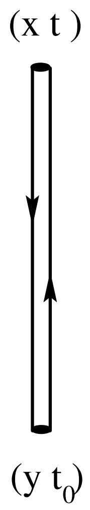
1-meson $C^{(2)}_{\vec{p}\,\vec{q}}(t,t_0) = \langle \phi^\dagger_{\vec{p}}(t)\,\phi_{\vec{q}}(t_0)\rangle - \langle\ \rangle\langle\ \rangle = \ldots$

- $\phi \sim \bar{\psi}\psi$, contractions $n = 1,2$

$$\overset{n}{\psi}\,(x)\,\overset{n}{\bar{\psi}}\,(y) = \overset{n}{G}\,(x,y) \quad \text{quark propagator}$$

$$\ldots = \langle \overset{21}{\phi^\dagger_{\vec{p}}}\,\overset{12}{\phi_{\vec{q}}}\rangle = \sim \langle \overset{2}{G}{}^*\overset{1}{G}\ \rangle = \delta_{\vec{p}\,\vec{q}}\; c_{\vec{p}}(t,t_0) \qquad \text{diagonal}$$

$$\overset{2}{G}{}^*\overset{1}{G} =$$



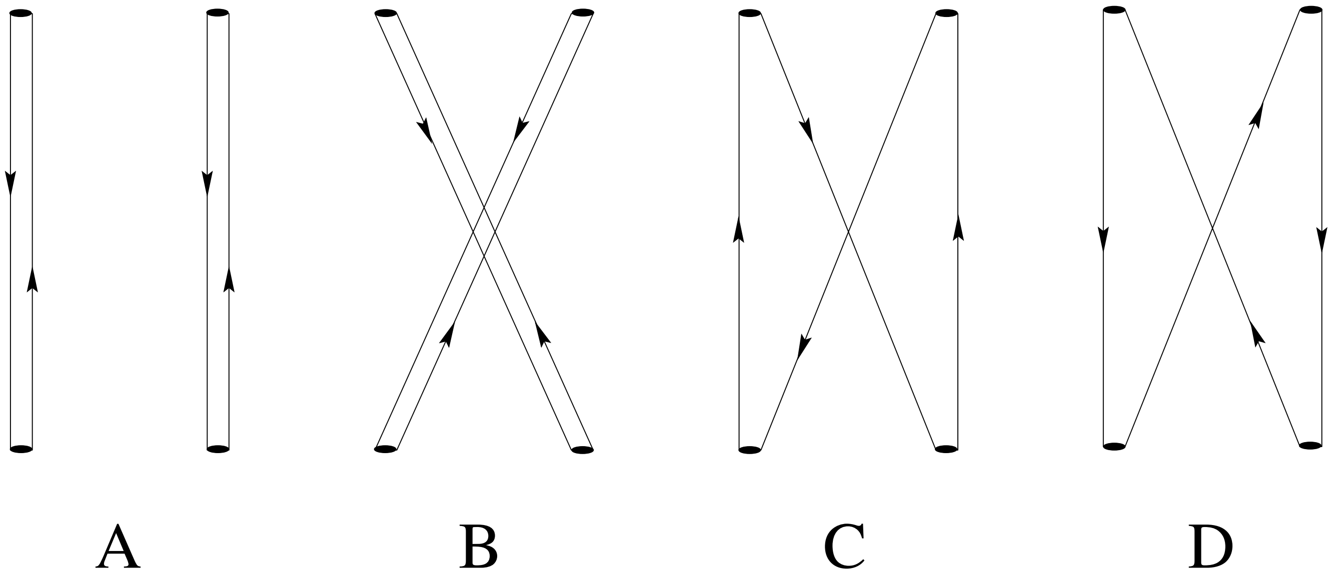
- $c_{\vec{p}}(t,t_0) = \sum_{\vec{x},\vec{y}} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\langle |G_{A\mu,B\nu}(\vec{x}t,\vec{y}t_0)|^2\rangle = c^*_{-\vec{p}}(t,t_0)$

2-meson $\quad C^{(4)}_{\vec{p}\,\vec{q}}(t,t_0) = \langle \Phi^\dagger_{\vec{p}}(t)\,\Phi_{\vec{q}}(t_0)\rangle - \langle\quad\rangle\langle\quad\rangle = \ldots$

- $\Phi \sim \phi\phi$, $\phi \sim \bar{\psi}\psi$, contractions $n = 1,2,3,4$

$$\ldots = \langle \phi^\dagger_{+\vec{p}}\phi^\dagger_{-\vec{p}}\phi_{-\vec{q}}\phi_{+\vec{q}}\rangle = \langle \overset{43}{\phi^\dagger_{+\vec{p}}}\overset{21}{\phi^\dagger_{-\vec{p}}}\overset{12}{\phi_{-\vec{q}}}\overset{34}{\phi_{+\vec{q}}}\rangle + \ldots =$$

$$= \sim \langle \overset{4}{G^*}\overset{3}{G}\,\overset{2}{G^*}\overset{1}{G}\ \rangle + \ldots = C^{(4A)} + C^{(4B)} - C^{(4C)} - C^{(4D)} =$$



- *Free* propagation contained in

$$
\begin{aligned}
C^{(4A)} &= \langle \overset{43}{\phi^\dagger_{+\vec{p}}}\overset{21}{\phi^\dagger_{-\vec{p}}}\overset{12}{\phi_{-\vec{q}}}\overset{34}{\phi_{+\vec{q}}}\rangle \sim \langle \overset{4}{G^*}\overset{3}{G}\ \overset{2}{G^*}\overset{1}{G}\ \rangle \\
C^{(4B)} &= \text{“\textit{Xchange diagram, Bose symm}”}
\end{aligned}
$$

- Gluon correlations $\langle\cdots\quad\cdots\rangle$ between mesons

$\langle\cdots\rangle\langle\cdots\rangle$ **off**

stochastic ave $\Rightarrow$

- Cumulant expansion, $\sum_{\text{all clusterings}}\langle\ldots\rangle\ldots\langle\ldots\rangle$

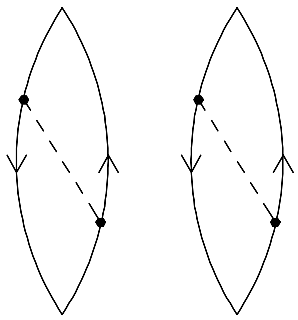
$$
\langle \overset{4}{G^*}\overset{3}{G}\ \overset{2}{G^*}\overset{1}{G}\ \rangle =
$$

$$
\langle \overset{4}{G^*}\overset{3}{G}\ \rangle\langle\overset{2}{G^*}\overset{1}{G}\ \rangle + \langle \overset{4}{G^*}\overset{1}{G}\ \rangle\langle\overset{2}{G^*}\overset{3}{G}\ \rangle + \langle \overset{4}{G^*}\overset{2}{G^*}\rangle\langle\overset{3}{G}\ \overset{1}{G}\ \rangle
$$

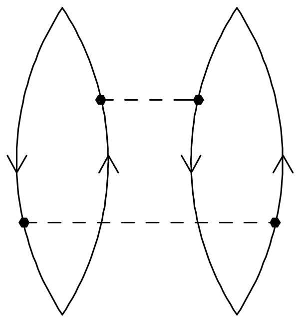
$$
+ \langle\!\langle \overset{4}{G^*}\overset{3}{G}\ \overset{2}{G^*}\overset{1}{G}\rangle\!\rangle \quad \leftarrow \text{cumulant}
$$

uncorrelated mesons $\Longrightarrow$ 1-st separable term

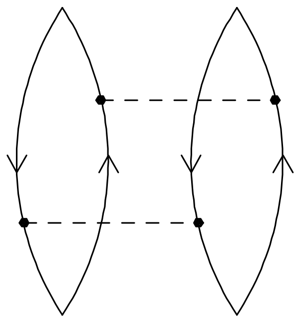
$$\langle \overset{4}{G}{}^{*}\overset{3}{G}\rangle\langle \overset{2}{G}{}^{*}\overset{1}{G}\rangle =$$



$$\langle \overset{4}{G}{}^{*}\overset{1}{G}\rangle\langle \overset{2}{G}{}^{*}\overset{3}{G}\rangle =$$



$$\langle \overset{4}{G}{}^{*}\overset{2}{G}{}^{*}\rangle\langle \overset{3}{G}\overset{1}{G}\rangle =$$



$$\left.\begin{array}{lll} \texttt{direct} & \overline{C}^{(4A)} = \langle \phi^\dagger_{+\vec{p}}\phi_{+\vec{q}}\rangle\langle \phi^\dagger_{-\vec{p}}\phi_{-\vec{q}}\rangle \\ \texttt{xchang} & \overline{C}^{(4B)} = \langle \phi^\dagger_{-\vec{p}}\phi_{+\vec{q}}\rangle\langle \phi^\dagger_{+\vec{p}}\phi_{-\vec{q}}\rangle \end{array}\right\} \quad \text{add} \Longrightarrow$$

Free correlator matrix

$$\overline{C}^{(4)} = \overline{C}^{(4A)} + \overline{C}^{(4B)} = (\delta_{\vec{p},\vec{q}} + \delta_{-\vec{p},\vec{q}})\, |c_{\vec{p}}(t,t_0)|^2$$

Full correlator matrix

$$C^{(4)} = \overline{C}^{(4)} + C_I^{(4)} = \textit{free} + \textit{interacting}$$

$\uparrow$

Interaction mechanisms

$\rightarrow$ quark-antiquark (boson) exchange

$\rightarrow$ correlated gluon (glueball?) exchange

$\rightarrow$ if unquenched, quark-antiquark loop effects

- guidance from: elementary Bose field
  canonical quantization
  perturbative interaction, 2nd order $\Longrightarrow$
- on the lattice:

Def: effective correlator matrix

$$\mathcal{C}(t,t_0) = \frac{\mathbf{1}}{\sqrt{\overline{C}^{(4)}(t,t_0)}} C^{(4)}(t,t_0) \frac{\mathbf{1}}{\sqrt{\overline{C}^{(4)}(t,t_0)}}$$

Def: effective interaction

$$\mathcal{H}_I = - \lim_{t\to\infty} \frac{\partial \ln \mathcal{C}(t,t_0)}{\partial t}$$

- contains
  – excitations of relative degrees of freedom

Lattice symmetry

- lattice $L^d \times T$, $d = 3$, discrete group $O(d, \mathbb{Z}) = \{$ rotations by 90^o etc, inversions $\}$

`irreps`	Γ:	A_1^+	$T_1^\pm$	E^+	$T_2^\pm$	A_2^-
`dimens`	N_Γ:	1	3	2	3	1

- angular momentum ℓ content

$$
\left.
\begin{array}{rcl}
\underline{\ell} & & \\
\underline{0} & = & A_1^+ \\
\underline{1} & = & T_1^- \\
\underline{2} & = & E^+ \oplus T_2^+ \\
\underline{3} & = & A_2^- \oplus T_1^- \oplus T_2^- \\
\underline{4} & = & A_1^+ \oplus E^+ \oplus T_1^+ \oplus T_2^+
\end{array}
\right\} \text{mod } 4
$$

Reduced matrix elements

full correlator

$$< (\Gamma, p)\epsilon | C^{(4)}(t, t_0) | (\Gamma', q)\epsilon' >= \delta_{\Gamma\Gamma'}\delta_{\epsilon\epsilon'}\, C^{(\Gamma)}_{p\,q}(t, t_0)$$

free correlator, diagonal

$$\overline{C}^{(\Gamma)}_{p\,q}(t, t_0) = \delta_{p\,q}\, \left| \bar{c}^{(\Gamma)}_{p}(t, t_0) \right|^2$$

effective correlator matrix

$$\mathcal{C}^{(\Gamma)}_{p\,q}(t, t_0) = \frac{C^{(\Gamma)}_{p\,q}(t, t_0)}{\left| \bar{c}^{(\Gamma)}_{p}(t, t_0) \right| \left| \bar{c}^{(\Gamma)}_{q}(t, t_0) \right|}$$

- numerically
  eigenvalues $\sim \exp(-E_n(t - t_0))$
  interaction $\mathcal{H}_I \sim -\partial \ln \mathcal{C}/\partial t$ indep of t $\Longrightarrow$ $\mathcal{H}^{(\Gamma)}_{I,p\,q}$

Simulation

lattice $L^3 \times T = 9^3 \times 13$, tree-level $O(a^2)$ tadpole improved

very coarse $a = 0.4\text{fm}$, $a^{-1} = 0.5\text{GeV}$, large $La = 3.6\text{fm}$

208 configurations, quenched

Propagators

from $8 \times (3 \times 4)$ random sources

multiple-mass solver

Wilson $\kappa^{-1} = 5.720 \ldots 6.140$ six

Operators

Gaussian smearing at sink

done with fuzzy link variables

on-axes momenta $\vec{p} = \frac{2\pi}{L}(k, 0, 0)$

Chiral extrapolation

fit $\mathcal{H}^{(\Gamma)}_{I,q\,q}(m^2)$ to model $h(m^2) = h'm^2 + h''m^3$

extrapolate $h(m^2 \to 0) =: \mathcal{H}^{(\Gamma)}_{I,q\,q}$

Potentials

$$< \vec{r}\,|\mathcal{H}_I|\,\vec{s} >= \quad \text{(relative coords } \vec{r}, \vec{s}\text{)}$$

$$= \quad L^{-3} \sum_{\vec{P}} \sum_{\vec{Q}} e^{i\vec{P}\cdot(\vec{r}-\vec{s})} e^{-i\vec{Q}\cdot(\vec{r}+\vec{s})} < \vec{P}-\vec{Q}\,|\mathcal{H}_I|\vec{P}+\vec{Q} >$$

split $\quad < \vec{P}-\vec{Q}\,|\mathcal{H}_I|\vec{P}+\vec{Q} >= < -\vec{Q}\,|\mathcal{H}_I|+\vec{Q} > +$ "the rest"

$\Longrightarrow \quad < \vec{r}\,|\mathcal{H}_I|\,\vec{s} >= \delta_{\vec{r}\,\vec{s}}\,\mathcal{V}(\vec{r}\,) + \mathcal{W}(\vec{r}, \vec{s}\,)$

local potential $\mathcal{V}(\vec{r}\,) = \sum_{\vec{Q}} e^{-i2\vec{Q}\cdot\vec{r}} < -\vec{Q}\,|\mathcal{H}_I|+\vec{Q} >$

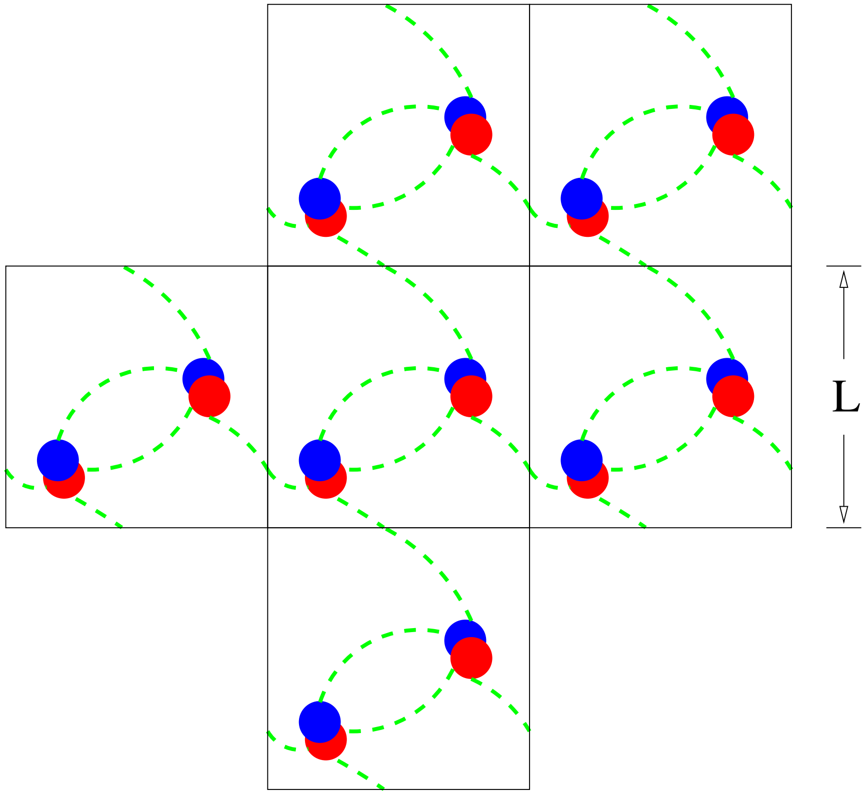
s-wave $(\ell = 0)$ projection

$\mathcal{V}_0(r) = \frac{1}{4\pi} \int d\Omega_{\vec{r}}\, \mathcal{V}(\vec{r}\,) = \sum_Q\, j_0(2Qr)\, \mathcal{H}_{I,QQ}^{(A_1^+)}$

- continuous $r \rightarrow$ interpolation between lattice sites

Parameterization

- periodic lattice, boundary conditions

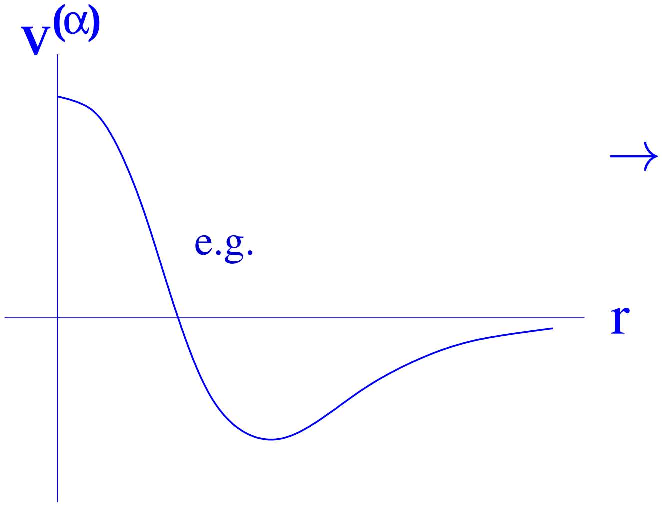


$$\Longrightarrow \quad \mathcal{V}(\vec{r}) = \mathcal{V}(\vec{r} + \vec{n}L), \quad \vec{n} \in \mathbb{Z}^3$$

- parametric function

$$V^{(\alpha)}(r) = \alpha_1 \frac{1-\alpha_2 r^{\alpha_5}}{1+\alpha_3 r^{\alpha_5+1} e^{\alpha_4 r}} + \alpha_0$$

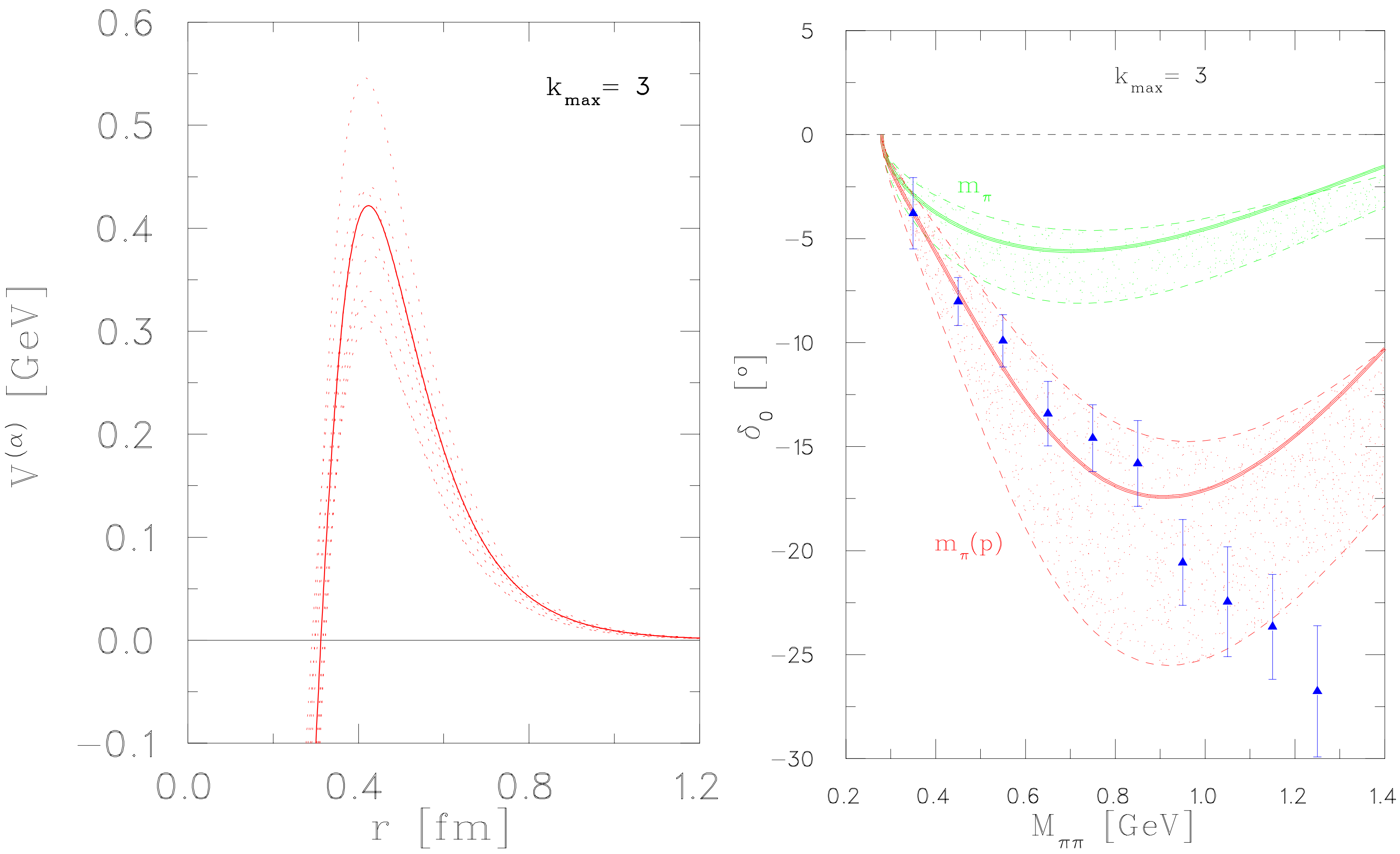
$$\rightarrow -\frac{\alpha_1\alpha_2}{\alpha_3}\frac{e^{-\alpha_4 r}}{r} \quad r \rightarrow \infty, \text{Yukawa}$$



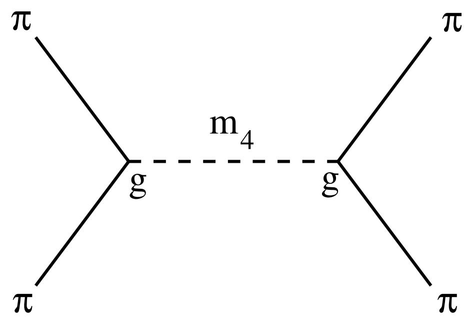
- $V_L^{(\alpha)}(r) = \frac{1}{4\pi}\int d\Omega_{\vec{r}} \sum_{\vec{n}} V^{\alpha}(|\vec{r}+\vec{n}L|)$ lattice'ized'

- fit to computed $\mathcal{V}_0(r)$

$$\int_0^{L/2} dr \left| V_L^{(\alpha)}(r) - \mathcal{V}_0(r) \right|^2 = \min(\alpha)$$



k max = 3
V(α) [GeV]
r [fm]
0.6
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
0.0
0.4
0.8
1.2
k max = 3
m π
m π (p)
δ 0 [°]
M ππ [GeV]
5
0
−5
−10
−15
−20
−25
−30
0.2
0.4
0.6
0.8
1.0
1.2
1.4



$$V^{(\alpha)}(r \to \infty) \to -\frac{\alpha_1\alpha_2}{\alpha_3}\frac{e^{-\alpha_4 r}}{r}$$

Xch-mass $\alpha_4 a^{-1} \quad \simeq m_4 = 1.0 - 1.4\text{GeV}$

Coupling $\alpha_1\alpha_2/\alpha_3 \simeq g^2/4\pi = 3.0 - 19.7$

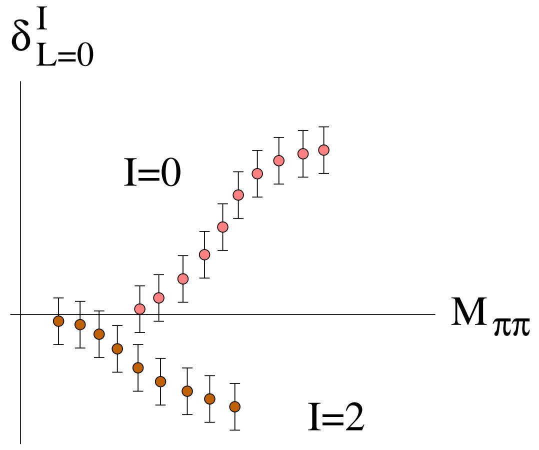
Scattering phase shifts $\delta^{\ell=0}_{I=2}$

- use $V^{(\alpha)}(r)$ in Schrödinger eqn, mass $m_\pi(p) = \sqrt{m_\pi^2 + p^2}$

experiment $\pi^+ p \longrightarrow \pi^+\pi^+ n$

CERN-Munich 1974 at 12.5 GeV

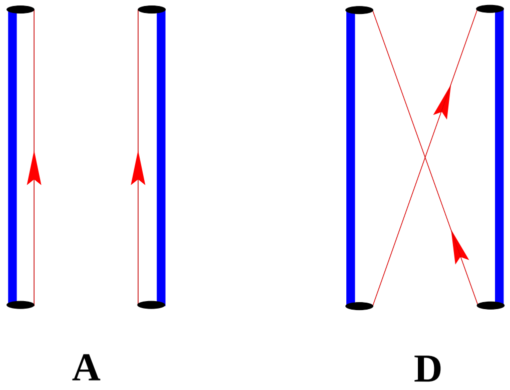
analysis: [Hoogland et al 1977]



Heavy-light systems

$$1-meson \quad \phi_{\vec{x}}(t) \quad = \quad \bar{Q}(\vec{x},t)\,\Gamma\, q(\vec{x},t) \qquad Q\text{=heavy}, \quad q\text{=light}$$

$$2-meson \quad \Phi_{\vec{r}}(t) \quad = \quad \sum_{\vec{x},\vec{y}} \phi_{\vec{x}}(t)\phi_{\vec{y}}(t)\,\delta_{\vec{x}-\vec{y},\,\vec{r}} \qquad \vec{r} = \text{relative dist}$$



- effective correlator $\longrightarrow$ diagonal

$$\mathcal{C}_{\vec{r}\vec{r}}(t,t_0) = \frac{C^{(4)}_{\vec{r}\vec{r}}(t,t_0)}{C^{(2)}(t,t_0)C^{(2)}(t,t_0)} \sim e^{-[E(r)-2m](t-t_0)}$$

$\Longrightarrow$ binding energy $[E(r)-2m] = V_{\text{adiabatic}}(r)$ potential

- Early work on heavy-light:

  D.G. Richards, Nucl. Phys. (Proc. Suppl.) B 9 (1989) 181

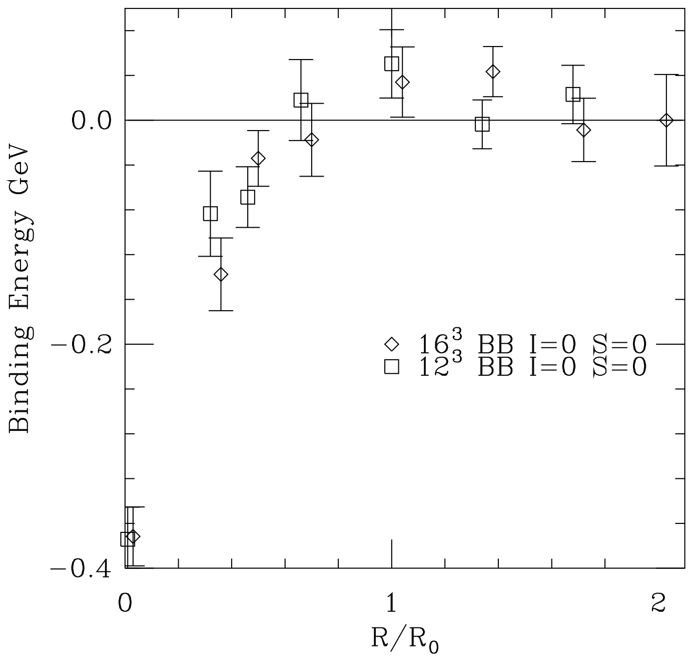
- Example:

  UKQCD,

  C. Michael,

  P. Pennanen,

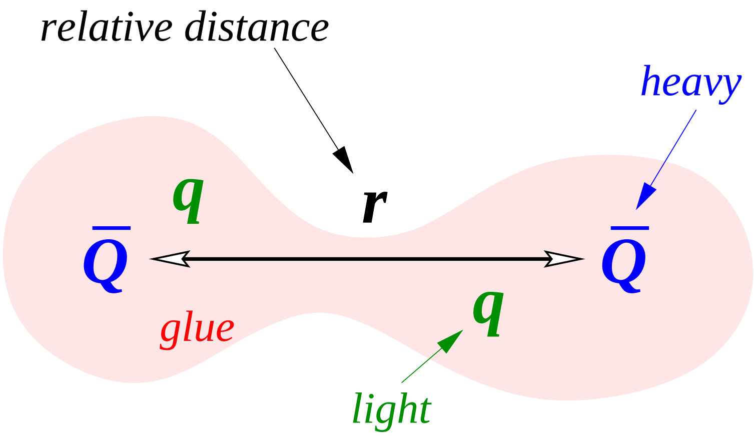
  hep-lat/9901007



◦ Geometric approach: *energy of quark configurations*

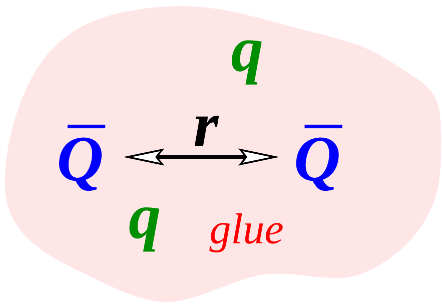
  T. Green et al, Helsinki group

Heavy-Light Meson-Meson Operators



? interaction. . .

. . .mechanism ?



$$\Phi_1(t) = \sum_{\vec{x},\vec{y}} \delta_{\vec{r},\vec{x}-\vec{y}} \, \overline{Q}_A(\vec{x}t)\gamma_5 q_A(\vec{x}t) \, \overline{Q}_B(\vec{y}t)\gamma_5 q_B(\vec{y}t)$$

$$\Phi_2(t) = \sum_{\vec{x},\vec{y}} \delta_{\vec{r},\vec{x}-\vec{y}} \, U_{P;AA'}(\vec{x}t,\vec{y}t) \, U^{\dagger}_{P';B'B}(\vec{x}t,\vec{y}t) \, \overline{Q}_A(\vec{x}t)\gamma_5 q_B(\vec{x}t) \, \overline{Q}_{B'}(\vec{y}t)\gamma_5 q_{A'}(\vec{y}t)$$

- local vs non-local color singlets
- link products along spatial paths P, P'

- $2{\times}2$ matrix $(\alpha,\beta=1,2)$ $\hat{\Phi}(t)=\Phi(t)-\langle\Phi(t)\rangle$

$$C_{\alpha\beta}(t,t_0)=\langle\hat{\Phi}^\dagger_\alpha(t)\hat{\Phi}_\beta(t_0)\rangle \;=\; 2\delta^{(+)}_{\vec r,\vec r\,'}\langle\sum_{\vec x,\vec y}\delta^{(+)}_{\vec r\,,\vec x-\vec y}$$
$$\mathcal{U}_\alpha(t,\vec x\vec y)\,\mathcal{U}_\beta(t_0,\vec x\vec y)$$
$$H^*(\vec y t,\vec y t_0)H^*(\vec x t,\vec x t_0)$$
$$[G(\vec y t,\vec y t_0)G(\vec x t,\vec x t_0)-G(\vec y t,\vec x t_0)G(\vec x t,\vec y t_0)]\rangle$$

- light-quark propagator $\rightarrow$

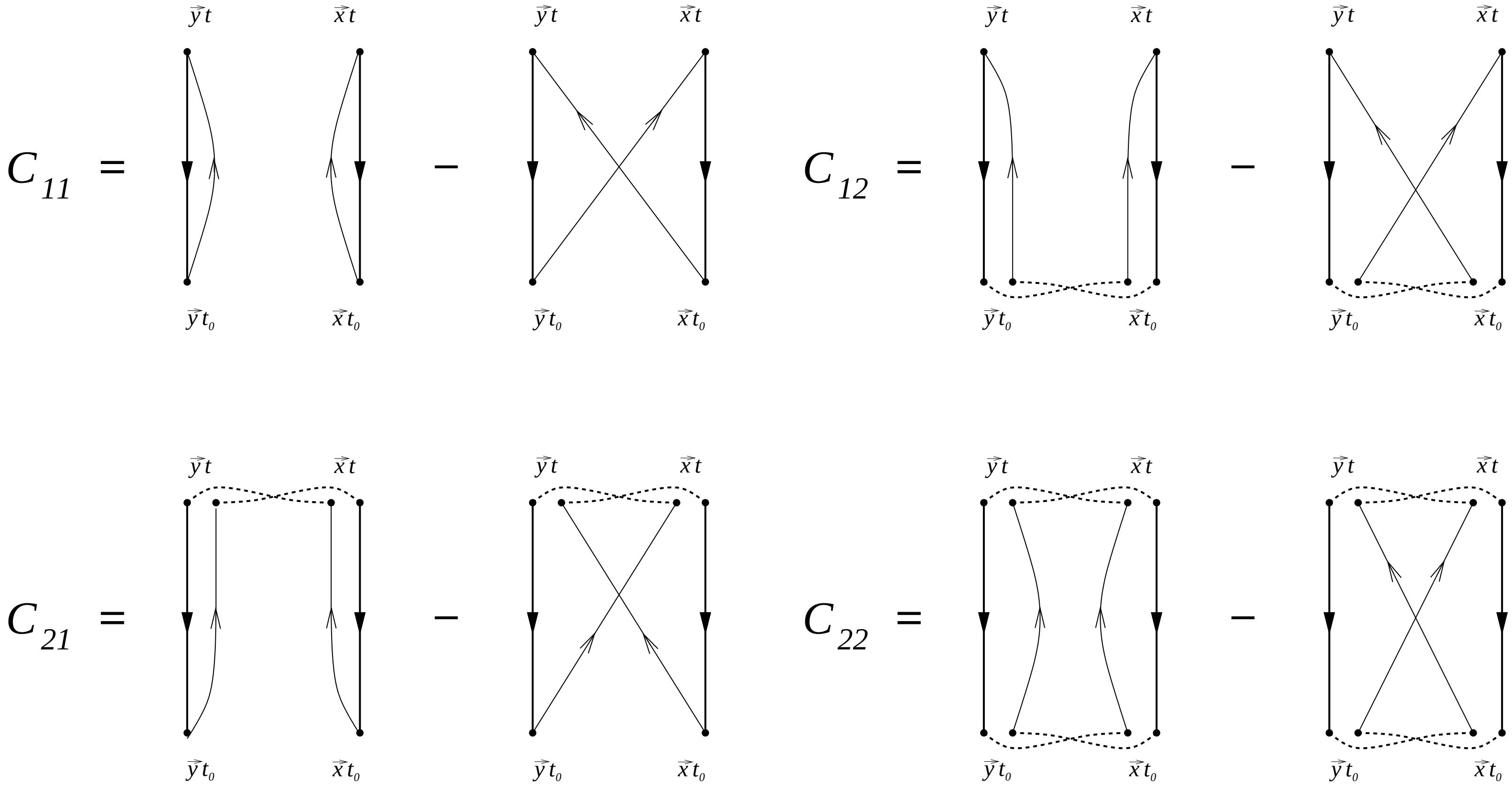
$$G(\vec y t,\vec x t_0)=\ldots\textit{stochastic estimator, BiCGStab}$$

- heavy-(anti)quark propagator, static$\rightarrow$

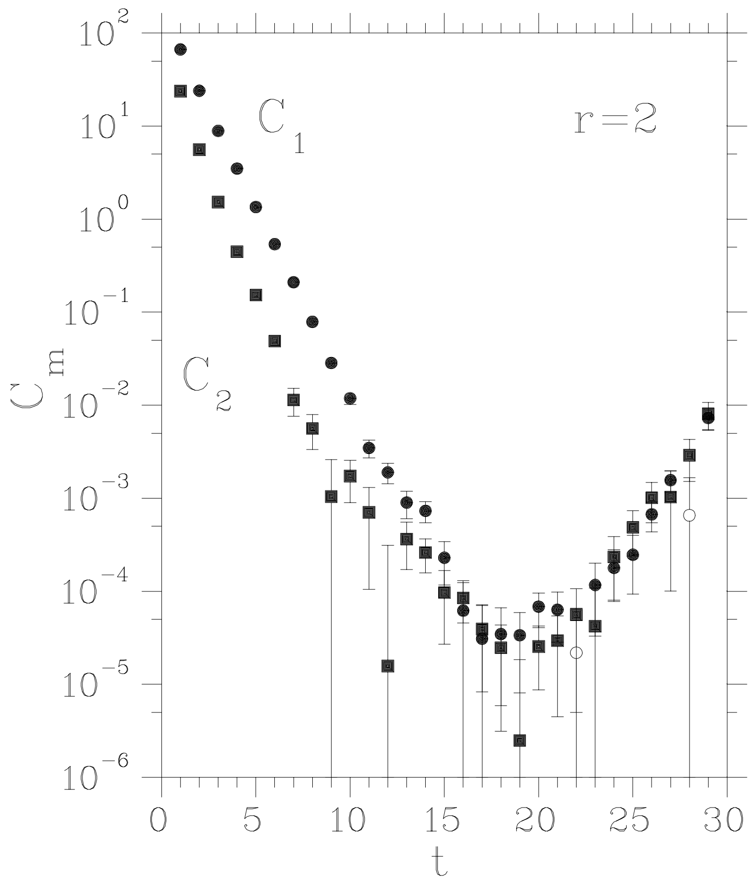
$$H^*(\vec x t,\vec y t_0)=\delta_{\vec x,\vec y}\,\frac{1}{2}(1+\gamma_4)U(\vec x t_0,\vec x t)$$

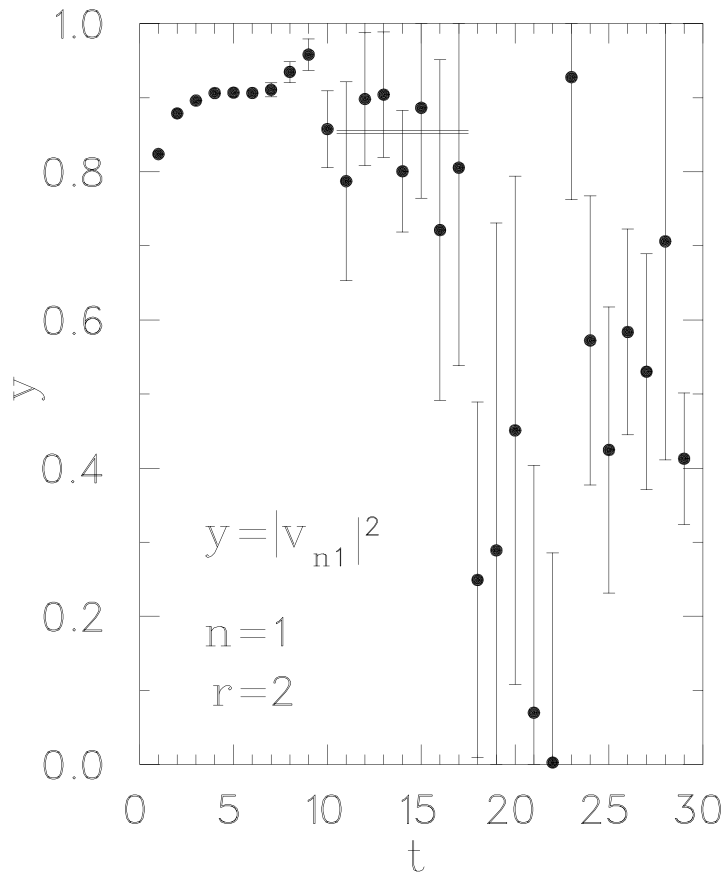
- nonlocality, spatial paths P,P' $\rightarrow$

$$\mathcal{U}_2(t,\vec x\vec y)=U_P(\vec x t,\vec y t)U^*_{P'}(\vec x t,\vec y t)+P\leftrightarrow P'$$

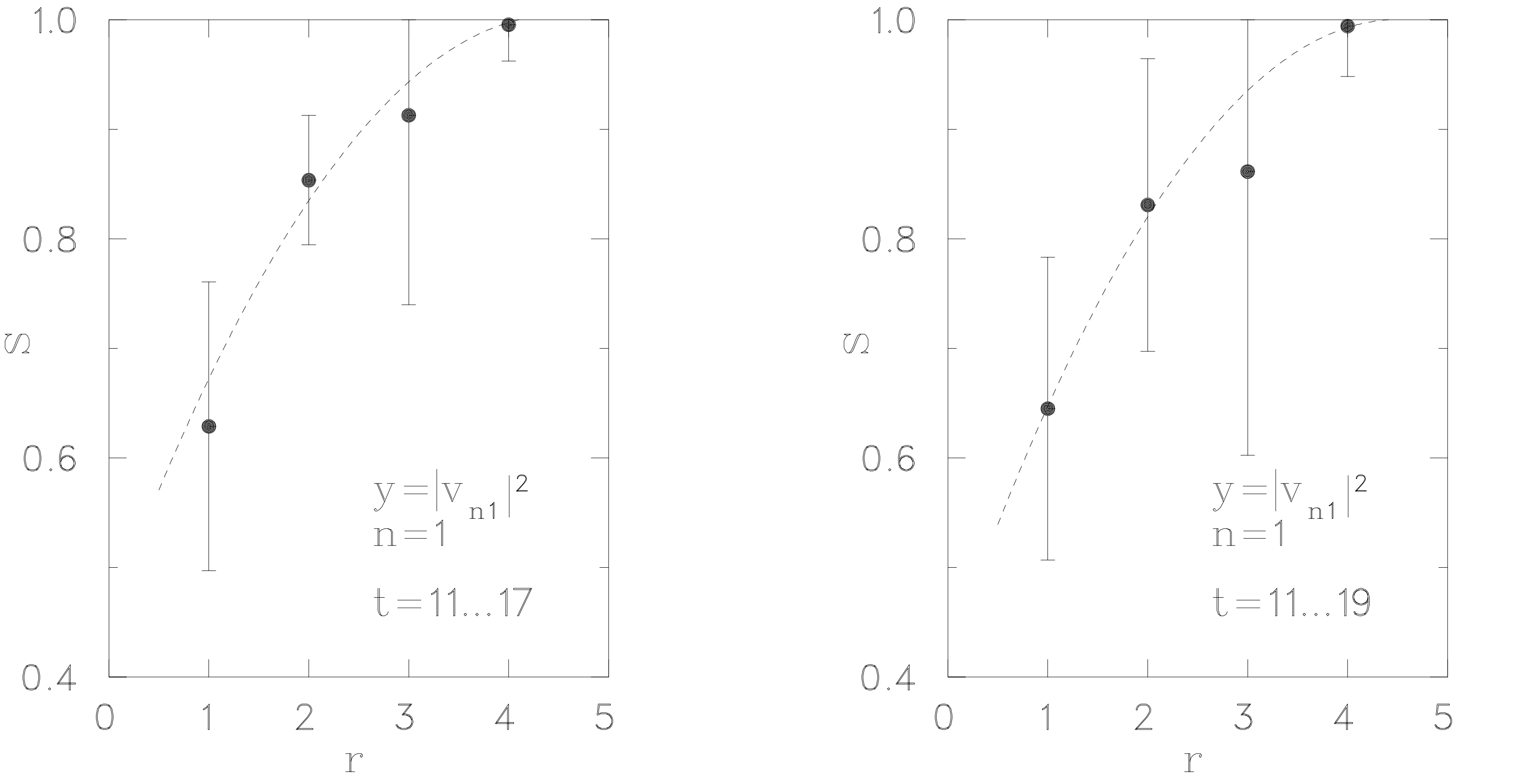

C_{11} =
−
C_{12} =
−
C_{21} =
−
C_{22} =
−
$\vec{y}\,t$
$\vec{x}\,t$
$\vec{y}\,t_0$
$\vec{x}\,t_0$

$$\begin{pmatrix} C_{11}(t,t_0) & C_{12}(t,t_0) \\ C_{21}(t,t_0) & C_{22}(t,t_0) \end{pmatrix} \begin{pmatrix} v_{m1}(t,t_0) \\ v_{m2}(t,t_0) \end{pmatrix} = C_m(t,t_0) \begin{pmatrix} v_{m1}(t,t_0) \\ v_{m2}(t,t_0) \end{pmatrix}$$





$$\bar{v}_1 = \begin{pmatrix} \sqrt{s} \\ \sqrt{1-s} \end{pmatrix} \quad \bar{v}_2 = \begin{pmatrix} \sqrt{1-s} \\ -\sqrt{s} \end{pmatrix}$$



- Use t-averaged $\overline{v}_m \Longrightarrow$ new operators, correlators

$$\Psi_m(t) \stackrel{\text{def}}{=} \sum_{i=1}^{M} \overline{v}_{mi} \Phi_i(t)$$

$$D_m(t, t_0) = \langle \hat{\Psi}_m^\dagger(t) \hat{\Psi}_m(t_0) \rangle = \sum_{n \neq 0} |\langle n | \hat{\Psi}_m(t_0) | 0 \rangle|^2 e^{-\omega_n (t - t_0)}$$

- Spectral model (analysis) $\Rightarrow$

$$F(\rho | t, t_0) = \int_{-\infty}^{+\infty} d\omega \, \rho(\omega) e^{-\omega(t - t_0)}$$

if exact $F(\rho | t, t_0) = D_m(t, t_0)$ then $\Rightarrow$

$$\rho(\omega) = \sum_{n \neq 0} \delta(\omega - \omega_n) \, |\langle n | \hat{\Psi}(t_0) | 0 \rangle|^2$$

- Discretization $\omega_k = \Delta\omega\, k, \quad k = K_1 \ldots K_2$

$$F(\rho|t, t_0) \simeq \sum_{k=K_1}^{K_2} \rho_k \, e^{-\omega_k (t-t_0)} \qquad \text{parameters} \mapsto \rho_k$$

# of parameters $(\sim 200) \gg$ # of lattice data (~ 30)

- Bayesian inference: given set of data D, fixed

$\rightarrow$ find conditional `pdf` of parameters ρ

$$\boxed{\begin{array}{l} \mathcal{P}(\rho \leftarrow D) \propto \mathcal{P}(D \leftarrow \rho) \times \mathcal{P}(\rho) \\ \text{posterior} \ \propto \ \text{likelihood} \times \text{prior} \end{array}}$$

$\rightarrow$ maximize posterior probability $\Rightarrow$ best "fit"

$\rightarrow$ more parameters than data? ... no problem!

Likelihood function

distance between . . .

$$\chi^2[D,\rho] = \sum_{t_1,t_2} \Big[\overset{\text{lattice data}}{\downarrow} D(t_1,t_0) - \overset{\text{spectral model}}{\downarrow} F(\rho|t_1,t_0)\Big]\, \Gamma^{-1}(t_1,t_2)\, \big[\,..\,t_1 \to t_2\,..\,\big]$$

covariance matrix

$$\Gamma(t_1,t_2) \propto \sum_{n=1}^{N_U} \Big[D(t_1,t_0) - \overset{\text{one gauge config}}{\downarrow} D(U_n|t_1,t_0)\Big] \big[\,..\,t_1 \to t_2\,..\,\big]$$

- given $[\rho]$, make (large number of) measurements of $[D]$, then (central limit theorem) $\Rightarrow$ Gaussian `pdf`

$$\mathcal{P}[D \leftarrow \rho] = e^{-\chi^2[D,\rho]/2}$$

Bayesian prior

if lack of info is maximal $\longrightarrow$ choose entropic prior

$$\mathcal{P}[\rho] = e^{\alpha \mathcal{S}[\rho]} \quad \text{with} \quad \mathcal{S}[\rho] = \sum_k \left(\rho_k - m_k - \rho_k \ln \frac{\rho_k}{m_k} \right)$$

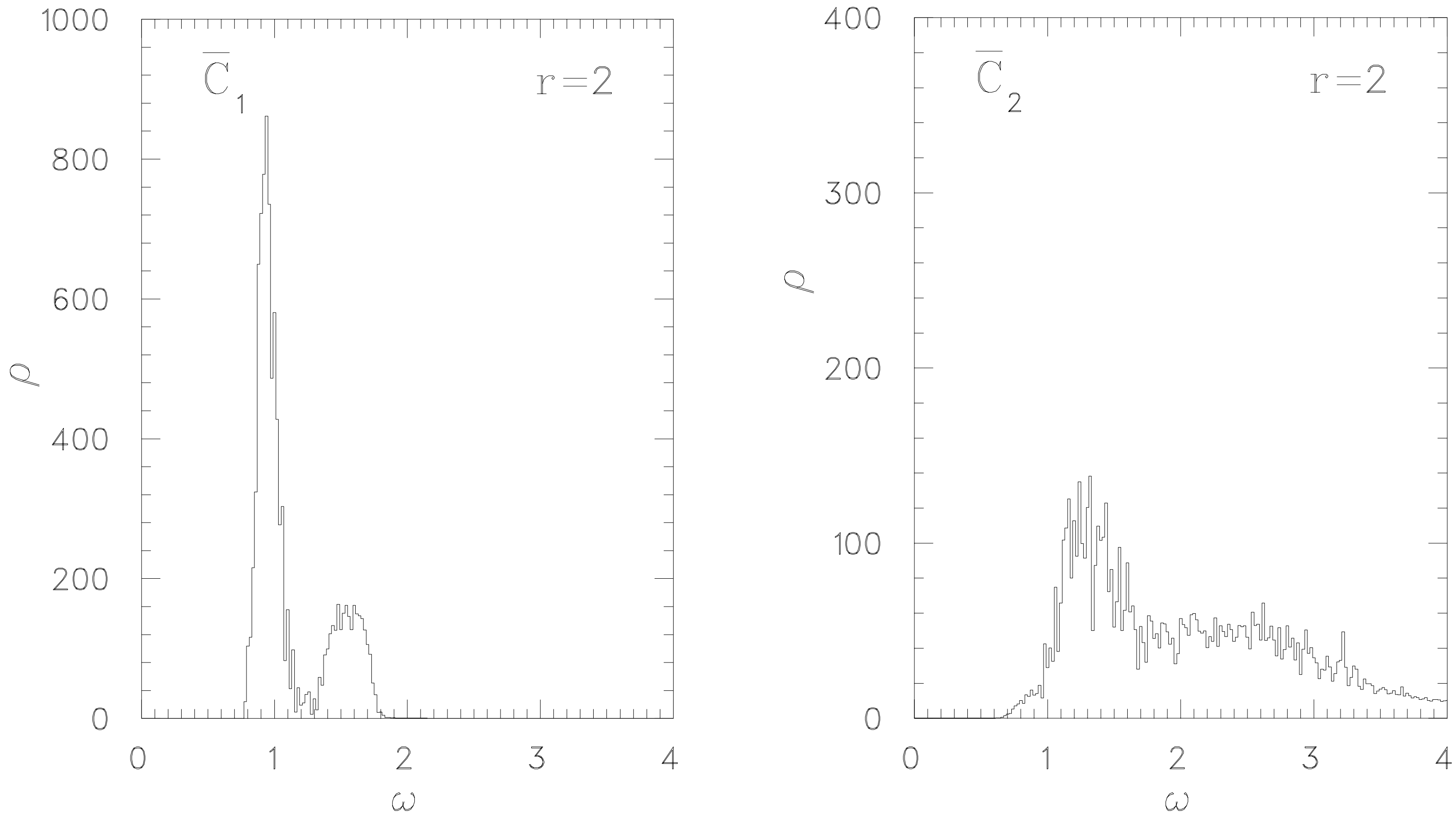
Shannon-Jaynes entropy $= \mathcal{S}$, default model $= [m]$, entropy weight $= \alpha$

Posterior probability

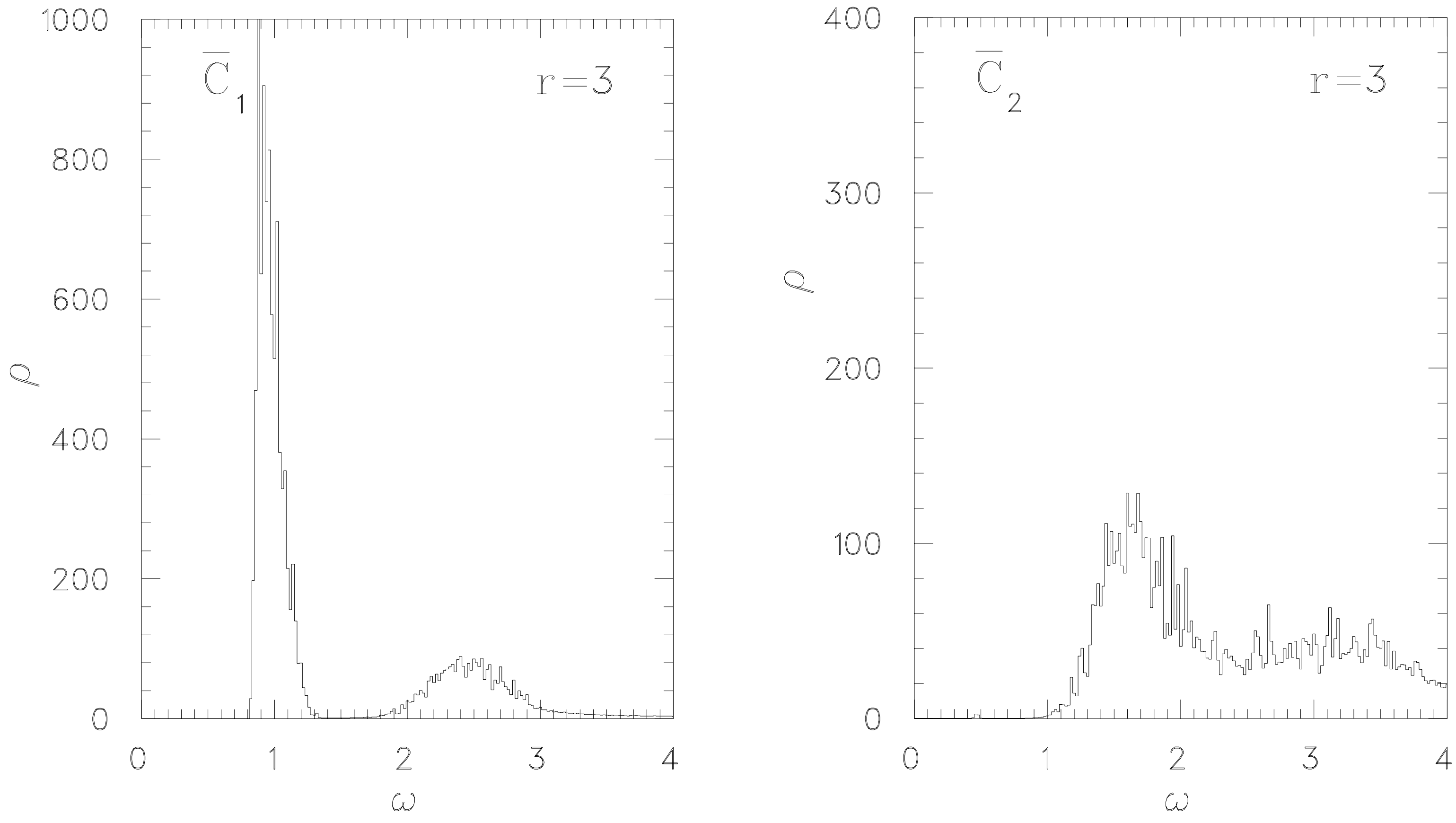
$$\begin{aligned} \mathcal{P}[\rho \leftarrow D] \quad &\propto \quad \mathcal{P}[D \leftarrow \rho]\mathcal{P}[\rho] \\ \mathcal{P}[\rho \leftarrow D] \quad &\propto \quad e^{-W[\rho]} \quad \text{with} \quad W[\rho] = \chi^2/2 - \alpha\,\mathcal{S} \end{aligned}$$

Solve $W[\rho] = \min$ by <u>Simulated Annealing</u>, cooling

$$Z = \int [d\rho]\, e^{-\beta_W W[\rho]} \qquad \beta_W^{-1} \longrightarrow 0 \quad \text{slowly}$$



$\overline{C}_1$
r=2
ρ
ω
$\overline{C}_2$
r=2
ρ
ω



1000
800
600
400
200
0
ρ
$\overline{C}_1$
r=3
0
1
2
3
4
ω
400
300
200
100
0
ρ
$\overline{C}_2$
r=3
0
1
2
3
4
ω

Spectral observables

loosely define

$$\delta_n = \{\omega : \omega \in \text{peak } \#n\} \quad n = 1, 2 \ldots .$$

then, for each peak n

$$\begin{array}{llcll}
\text{volume} & Z_n & = & \int_{\delta_n} d\omega \rho(\omega) & \cong \quad |\langle n|\hat{\Psi}(t_0)|0\rangle|^2 \\
\text{mass} & E_n & = & Z_n^{-1} \int_{\delta_n} d\omega \rho(\omega)\omega & \\
\text{width} & \Delta_n^2 & = & Z_n^{-1} \int_{\delta_n} d\omega \rho(\omega) \left(\omega - E_n\right)^2 &
\end{array}$$

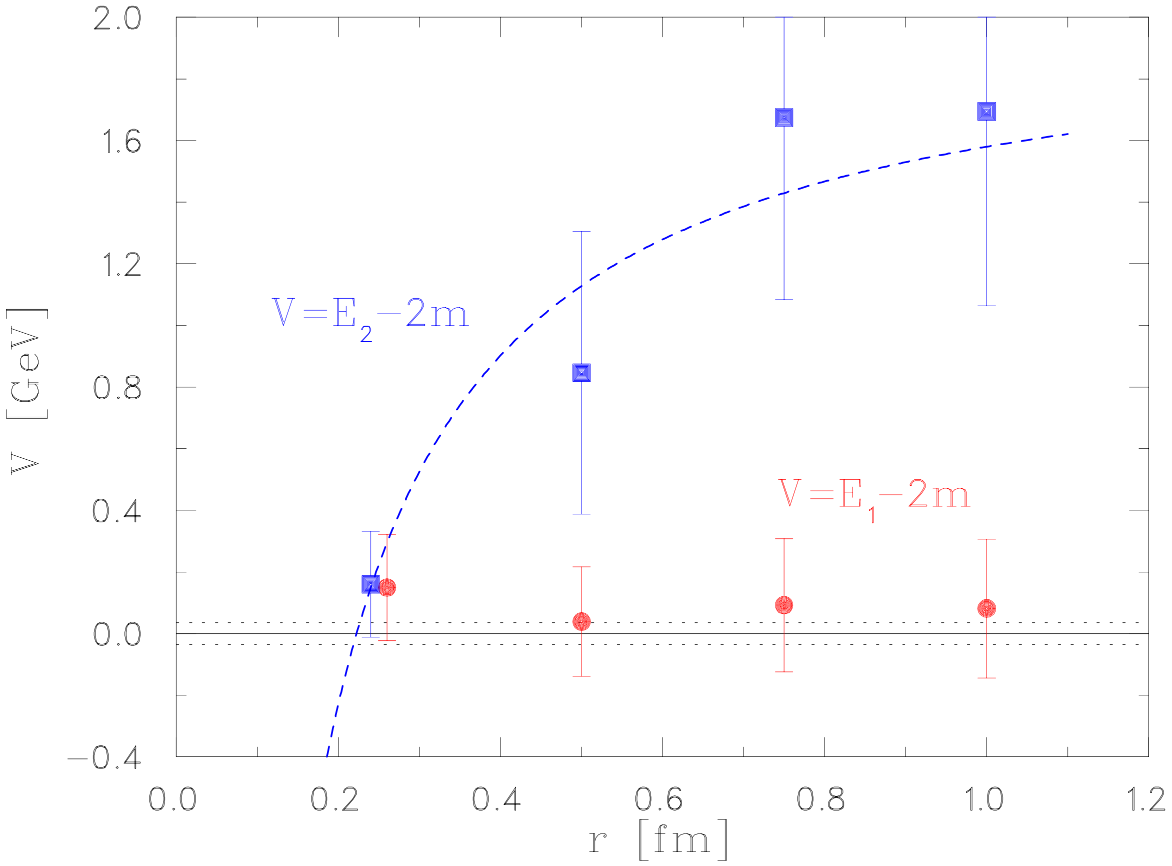
→ moments $\langle \omega^p \rangle_\rho$ low $p = 0, 1, 2 \Longleftarrow$ extractable INFORMATION

→ smoothing $\int \ldots d\omega$ micro structure ...

→ annealing average over random ρ starts

→ tuning m and α, insensitive over many orders of magnitude

r	$\bar{Z}_1$	$\bar{E}_1$	$\bar{\Delta}_1$	$\bar{Z}_2$	$\bar{E}_2$	$\bar{\Delta}_2$
1.0	107.(1)	0.997(3)	0.072(5)	67.2(9)	1.002(5)	0.072(9)
2.0	141.(2)	0.951(4)	0.074(7)	44.(1)	1.29(1)	0.191(8)
3.0	175.7(9)	0.974(3)	0.090(7)	60.(2)	1.63(1)	0.246(9)
4.0	162.6(6)	0.969(2)	0.094(3)	65.(1)	1.64(1)	0.263(9)



The error on masses: What is it?

measures of uncertainty on $E_n \rightarrow$

σ_n gauge configuration statistical error (jackknife analysis)

δ_n peak location statistical error (annealing starts)

Δ_n spectral peak width (second moments)

- typically find $\sigma_n \sim \delta_n < \Delta_n$

Summary

Lattice simulations aimed at hadronic interaction

→ mostly exploratory, so far

→ very few quantitative results

→ technically difficult, two-hadron excitation spectra

→ advanced analysis methods can help out

→ current heavy-light . . . needs more work

seek insight into physics of interaction