FIN-7845

Recurrence relation for central moments for the Binomial (n,p) distn:

$$\mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d\mu_r}{dp}\right]$$

$$\mu_r = E[x-np]^r = \sum_{x=0}^{n} (x-np)^r C_x^n p^x q^{(n-x)}$$

$$\frac{d\mu_r}{dp} = \left.\begin{aligned}&\sum_{x=0}^{n} C_x^n x p^{x-1} q^{n-x} (x-np)^r \\ &+ \sum_{x=0}^{n} C_x^n p^x (n-x) q^{n-x-1} (-1)(x-np)^r\end{aligned}\right\}$$

$$+ \sum C_x^n p^x q^{n-x} r (x-np)^{r-1} (-n) \Big\}$$

Note $\frac{dq}{dp} = -1$

$$= \sum_{x=0}^{n} C_x^n p^{x-1} q^{n-x-1} (x-np)^r \{xq - p(n-x)\}$$

$$+ \left\{\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} (x-np)^{r-1}\right\}(-nr)$$

$$= \sum_{x=0}^{n} C_x^n p^{x-1} q^{n-x-1} (x-np)^r \{x(q+p) - np\}$$

$$- nr\mu_{r-1}$$

$$pq\frac{d\mu_r}{dp} = \sum_{0}^{n} C_x^n p^x q^{n-x} (x-np)^r (x-np) - nr\mu_{r-1}pq$$

$$= \sum_{x=0}^{n} C_x^n p^x q^{n-x} (x-np)^{r+1} - nr\mu_{r-1}pq$$

$$= \mu_{r+1} - nr\mu_{r-1}pq$$

$$\Rightarrow \mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d\mu_r}{dp}\right]$$

for all real values of $t_1, t_2, \ldots, t_{k-1}$. Thus each one-variable marginal p.d.f. is binomial, each two-variable marginal p.d.f. is trinomial, and so on.

EXERCISES

3.1. If the m.g.f. of a random variable X is $(\frac{1}{3} + \frac{2}{3}e^t)^5$, find Pr $(X = 2$ or $3)$.

3.2. The m.g.f. of a random variable X is $(\frac{2}{3} + \frac{1}{3}e^t)^9$. Show that

$$\Pr(\mu - 2\sigma < X < \mu + 2\sigma) = \sum_{x=1}^{5} \binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x}.$$

3.3. If X is $b(n, p)$, show that

$$E\left(\frac{X}{n}\right) = p \qquad \text{and} \qquad E\left[\left(\frac{X}{n} - p\right)^2\right] = \frac{p(1-p)}{n}.$$

3.4. Let the independent random variables X_1, X_2, X_3 have the same p.d.f. $f(x) = 3x^2, 0 < x < 1$, zero elsewhere. Find the probability that exactly two of these three variables exceed $\frac{1}{2}$.

3.5. Let Y be the number of successes in n independent repetitions of a random experiment having the probability of success $p = \frac{2}{3}$. If $n = 3$, compute Pr $(2 \leq Y)$; if $n = 5$, compute Pr $(3 \leq Y)$.

3.6. Let Y be the number of successes throughout n independent repetitions of a random experiment having probability of success $p = \frac{1}{4}$. Determine the smallest value of n so that Pr $(1 \leq Y) \geq 0.70$.

3.7. Let the independent random variables X_1 and X_2 have binomial distributions with parameters $n_1 = 3, p_1 = \frac{2}{3}$ and $n_2 = 4, p_2 = \frac{1}{2}$, respectively. Compute Pr $(X_1 = X_2)$.

Hint: List the four mutually exclusive ways that $X_1 = X_2$ and compute the probability of each.

3.8. Toss two nickels and three dimes at random. Make appropriate assumptions and compute the probability that there are more heads showing on the nickels than on the dimes.

3.9. Let $X_1, X_2, \ldots, X_{k-1}$ have a multinomial distribution.

(a) Find the m.g.f. of $X_2, X_3, \ldots, X_{k-1}$.

(b) What is the p.d.f. of $X_2, X_3, \ldots, X_{k-1}$?

(c) Determine the conditional p.d.f. of X_1, given that

$$X_2 = x_2, \ldots, X_{k-1} = x_{k-1}.$$

(d) What is the conditional expectation $E(X_1 | x_2, \ldots, x_{k-1})$?

3.10. Let X be $b(2, p)$ and let Y be $b(4, p)$. If Pr $(X \geq 1) = \frac{5}{9}$, find Pr $(Y \geq 1)$.